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Trinification with a Bi-Adjoint Higgs Field

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Abstract

In this paper, we propose a novel extension of the Standard Model of particle physics, based on the trinification gauge group $SU(3)_C \times SU(3)_L \times SU(3)_R$. Symmetry breaking is achieved using a bi-adjoint Higgs field (transforming under the left- and right-handed subgroups) along with a more conventional bi-triplet to ensure the correct breaking and pattern of fermion masses. To preserve a discrete Z_2 symmetry (T-parity), we also introduce a right-handed triplet to completely break trinification symmetry to the Standard Model. The minimization conditions and conditions for the boundedness of the potential for this model are calculated. Additionally, the Standard Model quantum charges of the 64-component Higgs field are determined and mass matrices for the gauge bosons are constructed. The survival of T-parity ensures the stability of dark matter candidates in the model; the details of such candidates and the generation of neutrino masses are left to future work.

MACALESTER COLLEGE

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by

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in the

Department of Physics and Astronomy

Advisor: Saki Khan

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In this paper, we propose a novel extension of the Standard Model of particle physics, based on the trinification gauge group $SU(3)_C \times SU(3)_L \times SU(3)_R$. Symmetry breaking is achieved using a bi-adjoint Higgs field (transforming under the left- and right-handed subgroups) along with a more conventional bi-triplet to ensure the correct breaking and pattern of fermion masses. To preserve a discrete Z_2 symmetry (T-parity), we also introduce a right-handed triplet to completely break trinification symmetry to the Standard Model. The minimization conditions and conditions for the boundedness of the potential for this model are calculated. Additionally, the Standard Model quantum charges of the 64-component Higgs field are determined and mass matrices for the gauge bosons are constructed. The survival of T-parity ensures the stability of dark matter candidates in the model; the details of such candidates and the generation of neutrino masses are left to future work.

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CHAPTER 1: Introduction

The Standard Model of particle physics is a successful physical theory which allows highly precise calculations of physical quantities. However, it by no means offers a complete picture of particle physics. For example, neither the nature of dark matter on a quantum scale nor the presence of neutrino masses can be explained by the Standard Model [1]. To explain phenomena such as these, a higher-level theory is needed.

The Standard Model rests on two fundamental interactions: the strong interaction (with gauge group $SU(3)_C$) and the electroweak interaction (with gauge group $SU(2)_L \times U(1)_Y$). The overall gauge symmetry of the Standard Model is denoted by $SU(3)_C \times SU(2)_L \times U(1)_Y$. In the Standard Model, this group spontaneously breaks down to the gauge group $SU(3)_C \times U(1)_{EM}$ due to the vacuum expectation value (VEV) of a complex Higgs doublet [2]. This mechanism manages to explain the electromagnetic and weak interactions as the breakdown at low temperature of a unified electroweak interaction. It is natural, then, to suppose that the strong and electroweak forces might also be explained by a single unified force that breaks down at low energies. Theories that attempt to unify these interactions are called Grand Unified Theories (GUTs).

In this paper, we consider a particular GUT: trinification, based on the gauge group $SU(3)_C \times SU(3)_L \times SU(3)_R$. The framework of trinification was introduced in 1984 [3], and a discrete Z_3 group is often added to the gauge symmetry, written as $[SU(3)]^3 \times Z_3$, to ensure that the groups are indeed unified at high energies [4]. Although adding in Z_3 makes the model more elegant, it is not necessary. TeV-scale physics can be explored in the absence of this Z_3 symmetry, and as $SU(3) \times SU(3) \times SU(3)$ is a maximal subgroup of E_6 [5], trinification can be realized as a broken state of an E_6 gauge symmetry.

In this paper, we introduce a model of trinification with a Higgs sector that contains a field transforming as an octet under both the $SU(3)_L$ and $SU(3)_R$ subgroups. This field has 64 components and will be referred to as a bi-octet or bi-adjoint. One neutral component of the bi-octet can acquire a VEV that, combined with a triplet VEV and a bi-triplet VEV, break the symmetry of the system to that of the Standard Model. Three bi-triplets are needed to recreate the pattern of fermion masses seen in experiments, as described in Babu et al. [6]. As we want to preserve a discrete Z_2 symmetry (which we call T-parity, following [6]) naturally present in our Lagrangian, the bi-octet and bi-triplet alone are not enough to break the system down to the Standard Model. Hence, the presence of another Higgs field such as a triplet is necessary.

This paper begins with several chapters of background information. In Chapter 2, we briefly discuss elements of group theory as they relate to particle physics. In Chapter 3, we look at key elements of particle physics and basics of the Standard Model, and in Chapter 4 we describe spontaneous symmetry breaking and how it is applied in the Standard Model. In Chapter 5, the basic structure of trinification is given, along with an example of trinification symmetry breaking.

The main results of the model are contained in Chapter 6 and Chapter 7. The bi-octet's structure, charges and multiplets are described and listed in Appendix A and Appendix B, and the full Higgs potential is constructed. The trinification symmetry breaking path using a bi-octet, a triplet, and three bi-triplets is described, along with minimization conditions and potential boundedness conditions. Finally, the gauge boson mass matrices are constructed and are listed in Appendix C.

CHAPTER 2: Symmetry, Groups, and Representations

2.1 Physical Symmetries

Much of particle physics can be formulated by considering physical symmetries. There are main types of physical symmetries: spacetime symmetries and internal symmetries.

To begin with a simple example of spacetime symmetry, consider a collection of classical particles sitting in space and interacting gravitationally under the Newtonian force $F_M = G\frac{mM}{r^2}\hat{r}$. Now, rotate all vectors in the space by some angle θ. Because the gravitational force cares only about r^2, the square of the distance r between each pair of particles, nothing about the system's interaction has changed as a result of the rotation. Similarly, if we translate each vector in the space by the same vector α, nothing physical will have changed. These considerations seem banal, but in fact they relate to two extraordinarily important laws: invariance under rotations leads to conservation of angular momentum and invariance under translations leads to conservation of linear momentum. More generally, *every* continuous symmetry of a physical system (more specifically, its Lagrangian) leads to the conservation of a physical quantity. This is known as Noether's Theorem [2], and is one example of the utility of considering physical symmetries.

Internal symmetries are more subtle. Instead of being a symmetry of physical spacetime, internal symmetries relate to more abstract mathematical spaces. Consider the wavefunction in nonrelativistic quantum mechanics. As it turns out, the wavefunction is not unique: any complex phase of the form $e^{i\alpha}$ may be multiplied onto the wavefunction without changing any physical conclusions. This is an example of a global internal symmetry, where the wavefunction may be "rotated" in some abstract internal space, but the physics of the system remains the same. It is these internal symmetries that we will consider in more depth later.

2.2 Groups

Now that the significance of symmetries in physical theories is clear, we need to translate our observations about symmetry into mathematical language. We can do so by utilizing group theory.

A *group* is a set G of objects given an operation (here denoted by $\cdot$) between any two elements of the set such that (1) the group operation is associative: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$; (2) the set is closed under the group operation ($g_1 \cdot g_2 \in G$ for all $g_1, g_2 \in G$); (3) there is an identity element e such that $e \cdot g = g \cdot e = g$; and (4) every element g has an inverse such that g^{-1}: $gg^{-1} = g^{-1}g = e$ [1].

These four algebraic requirements for a set and an operation to be a group give what we are considering many powerful mathematical properties, some of which we will see below. Before moving on, we give some examples of groups below.

2.2.1 Z_2

One of the simplest groups, Z_2 has only two elements: the identity e and another element a. Three of the combinations are easy to work out $ee = e$, $ae = a$, and $ea = a$ (we will now omit the $\cdot$ for conciseness). Since the group must be closed under the group operation and every element must have an inverse, we know that the fourth combination must be $aa = e$. There are many ways of thinking about Z_2. In one realization of the group, its elements are the signs of numbers, with the group operation being the multiplication of two numbers. We can then identify e with the symbol $+$ and a with $-$. In another realization, the element e is labeled even and a odd, with the group operation of addition. Thus, we can think of the elements of Z_2 as positive and negative or as even and odd. We will use these interpretations interchangeably later when we discuss T-parity.

2.2.2 SO(n)

If we think about the set of all $n \times n$ real-valued matrices with determinant 1 ("special" matrices) such that $A^T A = I$ (orthogonal matrices, where I is the identity matrix), they form a group under matrix multiplication. This group is the Special Orthogonal Group of dimension n, called SO(n). The family of groups $SO(n)$ may be thought of as the group of rotations in n-dimensional space, preserving distances, volumes, and handedness of coordinate axes [1].

2.2.3 U(n)

If we now consider the set of all *complex*-valued matrices such that $A^\dagger A = I$, where $A^\dagger$ (called the Hermitian conjugate) is the complex conjugate of A^T, we similarly find that this set has a natural group structure under the operation of matrix multiplication. This group is $U(n)$, the Unitary Group of dimension n. For us, the most important example of $U(n)$ will be $U(1)$. This group can be identified with the unit circle in the complex plane, parameterized as $e^{i\alpha}$ for a continuous angle α. This identification clearly satisfies the unitary condition, since $A^\dagger A = e^{-i\alpha} e^{i\alpha} = e^0 = 1$. The internal symmetry of the quantum mechanical wavefunction described at the beginning of this section is an example of $U(1)$ symmetry.

2.2.4 SU(n)

Finally, the set of all *complex*-valued matrices with unit determinant such that $A^\dagger A = I$ forms a group under matrix multiplication, called the Special Unitary Group of dimension n, or $SU(n)$. Two groups in this family, $SU(2)$ and $SU(3)$, will be important later.

2.3 Representations and Lie Algebras

In group theory, we can represent group elements by matrices. A collection of matrices that satisfies the axioms of a particular group under the operation of matrix multiplication is called a *group representation*. More precisely, a representation is a mapping D of group elements of G onto linear operators such that (1) $D(e) = 1$ and (2) $D(g_1 g_2) = D(g_1)D(g_2)$. Theh second requirement simply ensures that the group operation is preserved in the space the linear operators act on. Representations allow us to connect groups with linear algebra and give us a way of translating the tools of group theory back into physical space. For example, if we can represent all the elements of a group as $n \times n$ matrices, then we can think of these matrices (or the group) as operating on physical space (itself represented by $n \times 1$ vectors).

Most of the groups above (the only exception being the discrete Z_2) are Lie groups, or groups whose elements can be organized by continuous parameters α_i. For example, the elements of $U(1)$ can be described by exponentiating a single continuous angle α. We can, in the case of a Lie group, write an element $g \in G$ as $g(\alpha_i)$.

From the definition of group representations, we know that D maps the identity element e to the identity matrix I, and we can write $D[g(0)] = D(e) = I$. For an infinitesimal variation $\delta\alpha$ in the parameters α_i, we can expand $D[g(0 + \delta\alpha]$ as a Taylor series to get

$$D[g(0 + \delta\alpha_i] = I + \delta\alpha_i \frac{\partial D[g(\alpha_i)]}{\partial \alpha_i}\bigg|_{\alpha_i = 0} + ...$$

where we sum over all parameters α_i. We can rewrite the derivatives in simpler form as

$$X_i = -i \frac{\partial D[g(\alpha_i)]}{\partial \alpha_i}\bigg|_{\alpha_i = 0}$$

The matrices X_i are called the *generators* of the Lie group; the number of generators is called the Lie group's dimension.

The Taylor expansion then becomes

$$D[g(0+\delta\alpha_i] = I + i\delta\alpha_i X_i + ...$$

We will approximate this infinite series by only considering terms linear in the infinitesimal $\delta\alpha_i$, cutting off the sum after the second term.

If we want instead to reach a *finite* group element instead of one an infinitesimal distance away from the identity, we need to multiply an infinite number of infinitesimal transformations, to get

$$\lim_{N\to\infty}(I + i\alpha_i X_i)^N = e^{i\alpha_i X_i}$$

where, again, the index i is summed over all parameters and generators. This section follows that given in [1]. The exponential of a matrix A is defined in terms of the Taylor series of e^x as $e^A = I + A + \frac{AA}{2!} + ...$

As it turns out, the generators of a Lie group form a basis for a vector space naturally connected to the group, called its Lie algebra. For matrices (and particularly generators of Lie groups), we define the commutator $[X_i, X_j] = X_i X_j - X_j X_i = if_{ijk}X_k$, where the repeated index k is summed over. The numbers f_{ijk} are called the structure constants of the group [1]. With the commutator as the bracket, the vector space spanned by the generators forms a Lie algebra which encodes much of the information in the group.

An important representation of $SU(n)$ is the fundamental representation. The fundamental representation of $SU(n)$, which has dimension n^2-1, consists of $n\times n$ matrices [1]. We will examine the fundamental representation of $SU(2)$ below.

The structure constants f_{ijk} can be used to construct another representation of a Lie algebra, called the adjoint representation [7]. An element T_a of the adjoint representation is defined by

$$[T_a]_{bc} = -if_{abc}$$

The T_a are $m \times m$ matrices, where m is the number of generators; as a representation of an m-dimensional Lie algebra, there are also m matrices T_a. For $SU(n)$, $m = n^2 - 1$.

2.3.1 Examples of Group Representations

It will be instructive (and important later) to see a couple explicit examples of Lie algebra representations.

First, we look at $SU(2)$'s fundamental representation. $SU(2)$ has three generators, and its fundamental representation can be written as

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

These are the Pauli matrices, seen frequently in studies of spin and angular momentum in quantum mechanics [1].

Although it appears natural to represent $SU(2)$ with 2×2 matrices, this is far from the only representation. For example, the adjoint representation of $SU(2)$ consists of 3×3 matrices, since there are three generators. The generators of $SU(2)$ in the adjoint representation are given by [1]

$$J'_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; J'_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \; J'_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We will later see examples of fields that transform under each of these representations.

Finally, we move to the fundamental representation of $SU(3)$. These eight 3×3 matrices are called the Gell-Mann matrices [1], and will be important for constructing the bi-octet later:

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; T_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; T_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \; T_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \; T_8 = \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & -2\frac{\sqrt{3}}{3} \end{pmatrix} \tag{2.2}$$

CHAPTER 3: Particle Physics

With basic concepts from group theory in hand, we now turn to fundamental ideas of particle physics. The basic objects of study in quantum field theory and particle physics are quantum fields. Loosely, a field is the assignment of a mathematical quantity at each point of a space. For example, a scalar field in three-dimensional Euclidean space assigns a scalar to each point in space and a vector field assigns a vector. The quantum fields we talk about here assign values to each point in four-dimensional Minkowski spacetime; we will see scalar quantum fields (Higgs bosons), vector quantum fields (gauge bosons), and spinor quantum fields (fermions).

Because we are dealing with a quantum theory, one might wonder what physical quantities need to be quantized. It turns out that in quantum field theory, it is field excitations (the allowed jumps above vacuum values) that are quantized. These excitations are what we normally think of as particles.

3.1 Fundamental Forces

We can organize much of particle physics in terms of four fundamental interactions: (1) the electromagnetic force, mediated by the photon and responsible for the classical electric and magnetic forces; (2) the strong force, which is mediated by gluons and holds protons, neutrons, and atomic nuclei together; (3) the weak force, which governs radioactive decay and is mediated by the W^+, W^-, and Z^0 bosons; and (4) gravity, a quantum theory of which is not yet fully known and which is so weak on the energy scales considered in this paper that we will ignore it. As far as experiments have been able to show, these forces are the only ways that particles can interact, and form the building blocks of reality.

As mentioned in the introduction, the Standard Model provides a framework for the unification of the weak and electromagnetic interactions at higher energies (or, in other words, very early in the history of the universe). The unification of the strong and electroweak forces at even higher energies is the ultimate goal of a Grand Unified Theory. The concept of unification (and much of particle physics) can be understood in terms of symmetry and symmetry breaking. The remainder of this section describes the first of these, symmetry, discussing gauge symmetries and the basics of the Standard Model gauge group. To do so, we first turn to a mathematical formulation of physics: the Lagrangian.

3.2 Lagrangians

A Lagrangian is a function that summarizes the physical properties of a system. To begin with, we can define the classical action $S = \int L dt$, the integral over time of some function L, called the Lagrangian. Varying the action to find its local extremum allows us to find the equations of motion for a system [8]. In particle physics, we will usually write the Lagrangian as the integral over all space of a Lagrangian density $\mathcal{L}$, which itself is a function $\mathcal{L}(\phi, \partial_\mu \phi)$ of a field ϕ and its first 4-vector derivative $\partial_\mu \phi$:

$$L = \int \mathcal{L}(\phi, \partial_\mu \phi) d^3x,$$

We need the 4-gradient operator ∂_μ here, as we want a special relativity-compatible theory. We will later see another 4-vector, written x^μ, which represents spacetime coordinates. The Lagrangian density is from this point simply referred to as the Lagrangian.

There are three kinds of terms in a Lagrangian: kinetic terms, interaction (or potential) terms, and mass terms. Kinetic terms tell us how particles (i.e., field excitations) move in space without interacting with each other. Potential terms are where the fundamental interactions come in; they describe how different particles interact with each other, changing particles' motion just like a classical force. Mass terms are a specific kind of potential term, where a field interacts with itself (or a complex conjugate of itself, or another field that has compatible quantum charges). Mass terms are always quadratic in fields (the terms contain only two total fields), and take the form $\frac{1}{2}m^2\phi^2$ for real-valued scalar fields and $m^2|\phi|^2$ for complex-valued scalar fields. The quantity m denotes a particle's mass. As an example, take the scalar ϕ^4 theory Lagrangian [8]:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \tag{3.1}$$

The time-dependent first term $\frac{1}{2}(\partial_\mu \phi)^2$ is our kinetic term, and tells us how the field will change as we move forward and backward in time. The second term is a mass term, and tells us that an excitation of the field will have mass given by the value of m. Finally, the third term is an interaction term; in this case, a quartic term where the field interacts with itself.

From a Lagrangian, we can extremize the action to find the Euler-Lagrange equations for a system, thereby determining the equations of motions for various fields. This process will leave us with a set of differential equations that tell us how fields will change over spacetime. Testable predictions can be made using the Lagrangian of a theory; for example, Feynman diagrams can be constructed out of the Lagrangian's interaction terms. Thus, the form of the Lagrangian for a field theory tells us, in principle, everything we need to know about a physical system.

3.3 Gauge Symmetries

A gauge theory is a physical theory with extraneous mathematical degrees of freedom that have no impact on the spacetime symmetries of the system. These degrees of freedom are associated with *internal* symmetries, and turn out to be extraordinarily significant. In particle physics, we end up with certain quantum fields, called gauge fields, which arise naturally when we convert global symmetries into local ones, called "gauging a symmetry." To see how this might arise, let us begin with a simple Lagrangian for a complex scalar field [2]:

$$\mathcal{L} = |\partial_\mu \phi|^2 + m\phi^*\phi \tag{3.2}$$

This Lagrangian is invariant under $U(1)$: If α is not a function of spacetime, the transformation $\phi \to e^{-i\alpha}\phi$ does not change the Lagrangian at all. Evidently this is a global symmetry of the Lagrangian. What happens if we try to make the symmetry local? That is, what if α becomes a function of spacetime, $\alpha(x^\mu)$? The mass term is indeed still invariant, but the kinetic term is not. For the kinetic term to be invariant, we would need $\partial_\mu \phi \rightarrow e^{-i\alpha(x^\mu)}\partial_\mu \phi$, as the $U(1)$ transformation can then cancel out when we multiple the new field by its complex conjugate. However, when the calculation is carried out, the partial derivative transforms as

$$\partial_\mu \phi \rightarrow e^{-i\alpha}\partial_\mu \phi - i(\partial_\mu \alpha)e^{-i\alpha}\phi$$

Because this is a local transformation, the second term is not constant over all spacetime, and so we cannot ignore it. We can fix this problem by adding in a new field [2] A^μ, which (adding a factor of e, which can later be identified as an electric charge), it transforms as $A^\mu \rightarrow A^\mu + \frac{1}{e}\partial_\mu \alpha$. We can then define a *covariant* derivative $D^\mu = \partial_\mu + ieA^\mu$, which does leave the kinetic term invariant:

$$\begin{aligned} D^\mu \phi = (\partial_\mu + ieA^\mu)\phi \rightarrow D^\mu e^{-i\alpha}\phi &= (\partial_\mu + ieA^\mu + i\partial_\mu \alpha)e^{-i\alpha}\phi \\ &= e^{-i\alpha}(\partial_\mu \phi + ieA^\mu \phi) + i\partial_\mu \alpha^{-i\alpha}\phi - i\partial_\mu \alpha^{-i\alpha}\phi \\ &= e^{-i\alpha}D^\mu \phi) \end{aligned}$$

From this example, we can see that gauging a symmetry requires the introduction of a new field; in this case A^μ. The new fields are called gauge fields, and their excitations are gauge bosons. The field A^μ in this example, taken from scalar quantum electrodynamics, is analogous to the photon field, which arises from the gauging of $U(1)$ symmetry. For this reason, it is sensible to organize the gauge bosons - and the interactions they facilitate - according to the symmetries from which they naturally arise. Electromagnetism in the Standard Model thus has the gauge group $U(1)_{EM}$. As we can see above, A^μ has no mass term. In fact, for any gauge symmetry to be preserved, the associated gauge fields cannot have mass – a mass term would be evidence that a symmetry has been broken. This is a powerful concept, and we will return to it later when examining spontaneous symmetry breaking.

3.4 The Standard Model

The Standard Model has the combined gauge group of $SU(3)_C \times SU(2)_L \times U(1)_Y$. The subscripts C (for color) and Y (for weak hypercharge) refer to the conserved charges associated with these groups by Noether's Theorem. The subscript L refers to left-handed particles, as these are the only ones that transform under $SU(2)_L$. Every term in a Lagrangian must be a Lorentz scalar and must be chargeless. $SU(3)_C$ is the gauge group of the strong force; its associated gauge fields can be identified as the eight gluons (each generator of a gauge group is associated with a single gauge field). $SU(2)_L \times U(1)_Y$ is the gauge group of the electroweak force, which breaks down to $U(1)_{EM}$ at low energies (see Chapter 4). These two $U(1)$ symmetries are not the same; $U(1)_{EM}$ is a linear combination of the $U(1)$ gauge field and one of the $SU(2)_L$ gauge fields [1].

The fermions (quarks and leptons) in the Standard Model are grouped into "multiplets," which are essentially vectors in the space on which a gauge group acts. As an example, let us consider the first generation of leptons. This generation consists of the electron and the electron neutrino, which are grouped into the same multiplet (a doublet) under $SU(2)_L$, written $L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$ [1]. When $SU(2)_L$ symmetry is unbroken, these fields are in a sense equivalent, since any linear combination of them will be gauge invariant. Once the symmetry breaks down, the fields are separated. $U(1)$ symmetries are typically denoted by their charge, as opposed to writing out a multiplet. We will use the convention

$$Q = T_3 + \frac{Y}{2} \tag{3.3}$$

where Q is the charge under $U(1)_{EM}$ (the electric charge), Y is the weak hypercharge, and T_3 is the weak isospin (a charge associated with $SU(2)_L$). The electron has $Q = -1$ and the electron neutrino has $Q = 0$; they both have a hypercharge of -1 [1]. The electron and neutrino fields are invariant under $SU(3)_C$, and as a result have no color charge. In contrast to the doublet L, the electron's antiparticle $e^c = e_R$, the positron, acts as a singlet under $SU(2)_L$ (i.e., it does not change at all under the symmetry).

Similar to the first generation of leptons, the first generation of quarks (the up quark u and down quark d) are organized in a doublet under $SU(2)_L$, written $Q = \begin{pmatrix} u \\ d \end{pmatrix}_L$ [1]. The up and down quarks both have a hypercharge of 1/3; using Eq. (3.3), the up quark has electric charge +2/3 and the down quark has electric charge −1/3. The anti-up and anti-down quarks are $SU(2)_L$ singlets; $u^c = u_R$ has hypercharge −4/3 and $d^c = d_R$ has hypercharge +2/3 [1].

The other generations can be constructed similarly, and have the same general multiplet structure as the first. We have ignored the $SU(3)_C$ multiplets here, as we are mainly interested in symmetry breaking in this paper and the color group is an unbroken gauge symmetry.

An important aspect of the Standard Model that we have not discussed so far is how particle mass is generated. It turns out that explicit mass terms for fermions cannot simply be written down and added to the Lagrangian, as they would not be gauge invariant [1]. To see this, note that the combination $e_L \nu_L$ does not work because the fields have different masses and the total charge is nonzero. On the other hand, the term $e_L e_R$ has total charge of zero and fields of the same mass, but its overall hypercharge is +1.

This is a major issue, as particles clearly do have mass, but we can fix it by introducing a new scalar field and utilizing the phenomenon of spontaneous symmetry breaking. It is these ideas we turn to in the next section.

CHAPTER 4: Spontaneous Symmetry Breaking and Higgs Fields

4.1 Symmetry Breaking

The intuition behind the idea of symmetry breaking is simple. We start with some symmetry of an object and, after some process, this symmetry breaks down to a smaller one. For example, consider a red triangle whose structure is invariant under 120° rotations. Say we color one edge green. Then a 120° rotation of the triangle will no longer leave the triangle looking exactly the same, as the green edge will have moved - the symmetry of the triangle has been broken.

The concept of spontaneous symmetry breaking in particle physics is more specific [2]: We have a Lagrangian that has a particular symmetry, but, after some phase transition, the ground state of the field no longer exhibits this symmetry. The symmetry is not exactly broken so much as it is hidden - since the Lagrangian is still invariant - but the name has stuck nonetheless. It is helpful here to work through an example. Consider a Lagrangian that looks similar to that in Equation 3.1, but it now contains complex fields:

$$\mathcal{L} = -\frac{1}{2}|\partial_\mu \phi|^2 + m^2|\phi|^2 - \frac{g^2}{2}|\phi|^4 \tag{4.1}$$

where $|\phi|^2 = \phi^*\phi$. As discussed before, this Lagrangian does not change under any $U(1)$ transformation $\phi \to e^{i\alpha}\phi$. The potential in this case is

$$V(\phi, \phi^*) = -m^2|\phi|^2 + \frac{g^2}{2}|\phi|^4 \tag{4.2}$$

If we plot V vs. $|\phi|$, we end up with a bowl with ground state at $|\phi|^2 = 0$ [1]. This ground state does respect the $U(1)$ symmetry; a 2-dimensional cross-section is shown in Figure 4.1.

Let us say some phase transition in the universe occurs, so that the ground state value $\langle\phi\rangle$ is shifted by some real value v, called a vacuum expectation value, or

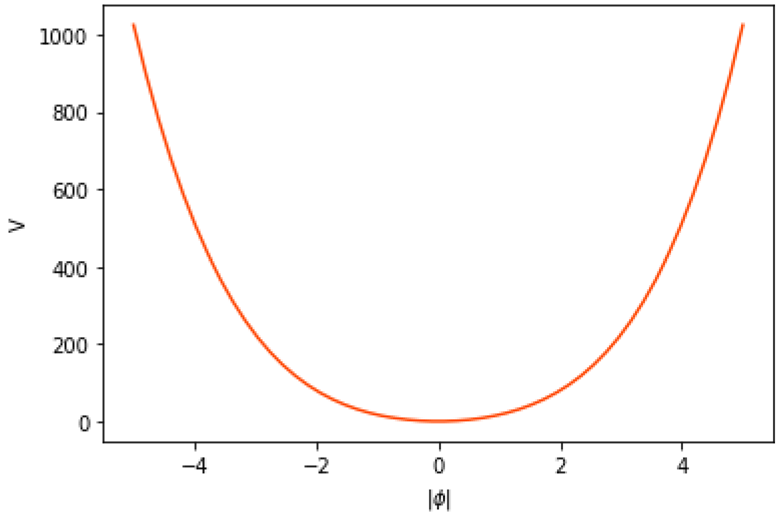

FIGURE 4.1: A cross-section of the potential V vs. the modulus of the field $|\phi|$. The unbroken $U(1)$ symmetry can be clearly seen, as the graph is symmetric about the ground state.

VEV. Making the change $\langle\phi\rangle = 0 \to \langle\phi\rangle = v$, we can expand ϕ around its new ground state, so that $\phi \to \phi + v$. Before we do that, however, it is helpful to find a formula for the VEV in terms of the coupling constants m and g. Since the ground state occurs at the minimum of the potential, we can set $\frac{\partial V}{\partial \phi} = 0$ to find such a formula. Taking the derivative with respect to ϕ and setting $\phi = \langle\phi\rangle$ gives us

$$-2m^2\langle\phi\rangle + 2g^2\langle\phi\rangle^3 = 0$$

Simplifying this equation gives us the minimization condition $\langle\phi\rangle^2 = \frac{m^2}{g^2}$. This equation has two roots, at $\langle\phi\rangle = +\frac{m}{g}$ and $\langle\phi\rangle = -\frac{m}{g}$. The field can now be plotted against V as before (Figure 4.2). However, we now see that at either of the two degenerate ground states, the $U(1)$ symmetry is no longer explicit: a rotation by $e^{i\alpha}$ does not leave the graph looking the same.

To see what is really happening mathematically, it helps to rewrite the complex field ϕ in terms of its real and complex components, with the $\sqrt{2}$ normalization chosen to get an even mixture of each field: $\phi = \frac{1}{\sqrt{2}}\varphi + \frac{i}{\sqrt{2}}\chi$. With this identification, we are now thinking about two independent fields φ and χ, which have the ground states $\langle\varphi\rangle = \sqrt{2}v$ and $\langle\chi\rangle = 0$. Changing from ϕ to φ and χ and expanding φ around its vacuum value $\sqrt{2}v$, the potential becomes

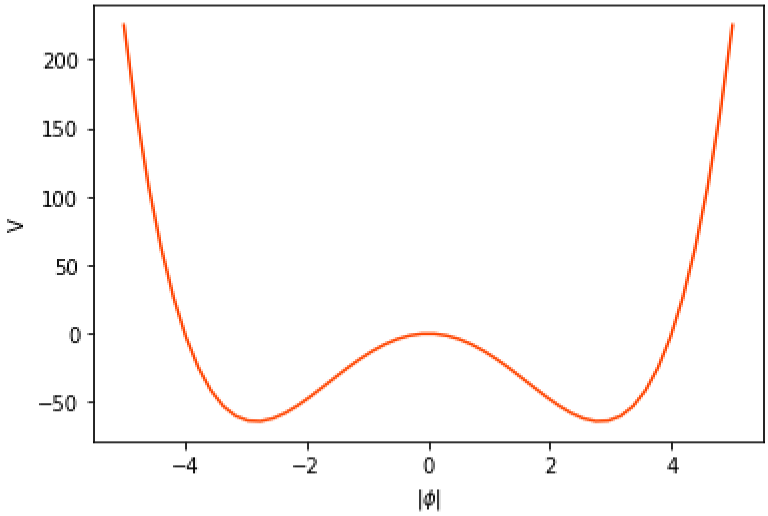

FIGURE 4.2: A cross-section of the potential V vs. the modulus of the field $|\phi|$. In this case, neither of the ground states are symmetric on each side, indicating that the $U(1)$ symmetry has been broken.

$$V(\phi,\phi^*) = -m^2\left(\frac{1}{\sqrt{2}}\varphi + v + \frac{i}{\sqrt{2}}\chi\right)\left(\frac{1}{\sqrt{2}}\varphi + v - \frac{i}{\sqrt{2}}\chi\right) + \frac{g^2}{2}\left[\left(\frac{1}{\sqrt{2}}\varphi + v + \frac{i}{\sqrt{2}}\chi\right)\left(\frac{1}{\sqrt{2}}\varphi + v - \frac{i}{\sqrt{2}}\chi\right)\right]^2$$

Simplifying the m^2 term, we get

$$-m^2\left(\frac{\varphi^2}{2} + \frac{\chi^2}{2} + \sqrt{2}v\varphi + v^2\right)$$

Meanwhile, the quartic term becomes

$$\frac{1}{2}g^2\left(v^4 + 3v^2\varphi^2 + \frac{\varphi^4}{4} + \frac{\chi^4}{4} + 2\sqrt{2}v^3\varphi + v^2\chi^2 + \sqrt{2}v\varphi\chi^2 + \sqrt{2}v\varphi^3 + \frac{1}{2}\varphi^2\chi^2\right)$$

Putting these equations together and substituting in the minimization condition $m^2 = g^2v^2$ gives us the potential

$$V(\varphi,\chi) = g^2v^2\varphi^2 - \frac{1}{2}g^2v^4 + \frac{1}{8}g^2\varphi^4 + \frac{1}{8}g^2\chi^4 + \frac{\sqrt{2}}{2}g^2v\varphi\chi^2 + \frac{\sqrt{2}}{2}g^2v\varphi^3 + \frac{1}{4}g^2\varphi^2\chi^2$$

We can see from this equation that the new potential has a mass term for φ, giving it a mass of $\sqrt{2}gv$, but no such term exists for χ. From χ's lack of a mass term, it is evident that the new potential is not explicitly invariant under $U(1)$ symmetry, since a rotation $e^{i\alpha}$ would change the physical mass of φ. The field χ is called a Goldstone boson, which can interact with other fields (as we can see in the last few terms in the potential) but does not itself have mass. Adding a VEV has spontaneously broken the $U(1)$ symmetry of the Lagrangian, resulting in one field becoming massless. The loss of mass that can occur as a result of spontaneous symmetry breaking is enormously important for the Standard Model, as we will see in the next section.

4.2 Higgs Fields

With an understanding of spontaneous symmetry breaking in hand, we can now look at the concept of a Higgs field. A Higgs field is a scalar field whose symmetry is broken (via the introduction of a VEV as described in the last section), which causes other particles (fermions and gauge bosons) to obtain mass. In this so-called Higgs mechanism, certain fields become massless (Goldstone bosons), allowing other particles to "eat" them and thereby gain mass. As fermion mass terms cannot be written into a Lagrangian, this Higgs mechanism is crucial to our understanding of particle physics and the success of the Standard Model.

4.2.1 Symmetry Breaking in the Standard Model

At high enough energies, the Lagrangian of the universe has 3 known gauge symmetries, conventionally written as $SU(3)_C \times SU(2)_L \times U(1)_Y$. Today, or equivalently at relatively low energies, this gauge symmetry breaks down to $SU(3)_C \times U(1)_{EM}$. $SU(3)_C$, the gauge group describing the strong interaction, remains unbroken, with eight massless gluons acting as messenger particles. The photon, written as A_μ (with the index running from 0 to 3, giving us a 4-dimensional field living in Minkowkski spacetime), is the gauge boson of $U(1)_{EM}$, and similarly remains massless. What remains to be seen is how the combined group $SU(2)_L \times U(1)_Y$ breaks down to $U(1)_{EM}$.

To see how this happens, we start by introducing a complex field ϕ, which has a weak hypercharge of $+1$ and is a doublet under $SU(2)_L$. We can write ϕ explicitly in terms of its real and imaginary components, so that it becomes [9]

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}$$

When no component of ϕ has a VEV, the Standard Model symmetry $SU(2)_L \times U(1)_Y$ remains in place. Let us see what happens when we give the second component of ϕ a real VEV v (equivalently, $\langle \varphi_3 \rangle = v$). We can then write the vacuum expectation value of the doublet ϕ as [8]

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

The general potential for ϕ is given by [9]

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

Minimizing this potential gives us the minimization condition [9] $\langle \phi \rangle^2 = v^2 = \frac{\mu^2}{2\lambda}$. Writing ϕ in terms of φ_1, φ_2, $\varphi_3 = v + h$, and φ_4 and substituting the minimization condition in the potential gives

$$\begin{aligned} V(\phi) &= -\mu^2 \big(\varphi_1^2 + \varphi_2^2 + (v+h)^2 + \varphi_4^2\big) + \lambda \big(\varphi_1^2 + \varphi_2^2 + (v+h)^2 + \varphi_4^2\big)^2 \\ &= -2\lambda v^2 \big(\varphi_1^2 + \varphi_2^2 + (v+h)^2 + \varphi_4^2\big) + \lambda \big(\varphi_1^2 + \varphi_2^2 + (v+h)^2 + \varphi_4^2\big)^2 \\ &= -2\lambda v^2 \varphi_1^2 - 2\lambda v^2 \varphi_2^2 - 2\lambda v^2 h^2 - 2\lambda v^2 \varphi_4^2 + 2\lambda v^2 \varphi_1^2 + \ 2\lambda v^2 \varphi_2^2 + 6\lambda v^2 h^2 \\ &\quad + 2\lambda v^2 \varphi_4^2 + \text{constant} + \text{cubic} + \text{quartic} \\ &= 4\lambda v^2 h^2 + \text{constant} + \text{cubic} + \text{quartic} \end{aligned}$$

Thus, φ_1, φ_2, and φ_4 are massless, whereas the field h now has a mass of $2\sqrt{2\lambda v^2}$. The introduction of a VEV has altered the masses of the fields in ϕ: one field still has mass, but three have become massless.

What happens to the gauge sector when we give a VEV to ϕ? Ignoring the $SU(3)_C$ symmetry but accounting for $SU(2)_L$ and $U(1)_Y$, we can write the covariant derivative for ϕ as [8]

$$D_\mu\phi = \left(\partial_\mu - igA^a_\mu\tau^a - i\frac{1}{2}g'B_\mu\right)\phi$$

Where B_μ is the gauge boson of $U(1)_Y$; A^a_μ (with a running from 1 to 3) are the three gauge bosons A^1_μ, A^2_μ, and A^3_μ of $SU(2)_L$; τ^a are the generators of $SU(2)_L$ (Equation 2.1); and g and g' are coupling constants. The covariant derivative term appearing in the Lagrangian is $|D_\mu\phi|^2$, and so the introduction of the VEV has altered the Lagrangian by a factor of [8] $\frac{1}{2}\frac{v^2}{4}\left[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g'B_\mu)^2\right]$. The physical (mass eigenstate) fields can be found by replacing A^1_μ, A^2_μ, A^3_μ, and B_μ by

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$$

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gA^3_\mu - g'B_\mu)$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'A^3_\mu + gB_\mu)$$

Making these substitutions allows us to see [8] that A_μ remains massless, whereas W^+_μ and W^-_μ obtain masses of $g\frac{v}{2}$ and Z^0_μ acquires a mass of $\sqrt{g^2 + g'^2}\frac{v}{2}$. Before the VEV was introduced, none of the gauge bosons had mass. With the VEV in place, three bosons have mass; these can be identified as the W^+, W^-, and Z^0 bosons observed by experiments today. The last field A_μ can be identified as the photon. The gauge group $SU(2)_L \times U(1)_Y$ has been broken. We can see evidence of this breaking in the fact that the fields now have a range of different masses and can no longer be freely rotated into each other by any arbitrary gauge transformation. Only one gauge symmetry of the Lagrangian remains: the $U(1)$ symmetry whose gauge boson is the photon A_μ. We call this symmetry $U(1)_{EM}$, the gauge symmetry corresponding to the electromagnetic interaction.

The above analysis demonstrates the utility of the Higgs mechanism. By introducing the Higgs field and giving it a VEV, we can break the electroweak interaction $SU(2)_L \times U(1)_Y$ down into the weak interaction (mediated by W^+, W^-, and Z^0) and the electromagnetic interaction (mediated by A_μ). One Higgs field remains massive, and one gauge field stays massless. This is an example of a general rule: every gauge boson that acquires a mass should correspond to a Higgs field losing mass. Furthermore, by introducing couplings between the Higgs field and the fermion fields, called Yukawa terms, we can see that fermion masses, like the gauge bosons above, arise from the spontaneous symmetry breaking of the Higgs field [1]. The fermion sector for the model we develop in Chapter 6 and Chapter 7 is left to future work; in this paper, we look mainly at symmetry breaking and the gauge sector.

CHAPTER 5: Trinification

The Standard Model of particle physics is a robust, effective theory with considerable explanatory power. Even so, it is incomplete. As mentioned in the introduction, the Standard Model is unable to describe the origins of dark matter and neutrino mass. For these outstanding question in particle physics, we need a new (or at least extended) theoretical structure.

The path to extending the Standard Model we take in this paper is called trinification. Trinification is a GUT based on the gauge group $SU(3)_C \times SU(3)_L \times SU(3)_R$. In trinification models, a Higgs field transforming as some multiplet of the gauge group is introduced so that the overall gauge symmetry breaks down to the Standard Model $SU(3)_C \times SU(3)_L \times U(1)_Y$, and then to $SU(3)_C \times U(1)_{EM}$.

As always, the relevant gauge bosons live in the adjoint representation [4], and so each gauge boson can be assigned to a generator of $SU(3)$. Since $SU(3)$ has eight generators Equation 2.3.1, there are eight gauge bosons for each copy of $SU(3)$ in the gauge symmetry. Eight of these bosons (the gluons) correspond to $SU(3)_C$, and will be ignored in the rest of this paper.

We can write the gauge bosons in a useful form by using the Gell-Mann matrices. Writing the gauge bosons of $SU(3)$ as W_i^μ (with μ denoting 4-vector Lorentz indices) and the Gell-Mann matrices Equation 2.3.1 as T_i, we can sum over all i to obtain the 3×3 matrix [6]

$$T_i \cdot W_i^\mu = \begin{pmatrix} W_3 + W_8/\sqrt{3} & \sqrt{2}W^+ & \sqrt{2}V^+ \\ \sqrt{2}W^- & -W_3 + W_8/\sqrt{3} & \sqrt{2}V^0 \\ \sqrt{2}V^- & \sqrt{2}V^{0*} & -2W_8/\sqrt{3} \end{pmatrix}^\mu \tag{5.1}$$

where we have used the substitutions

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$$

$$V_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^4 \mp iW_\mu^5)$$

$$V_\mu^{0(*)} = \frac{1}{\sqrt{2}}(W_\mu^6 \mp iW_\mu^7)$$

In this new notation, under the relevant $SU(2)$ subsymmetry of $SU(3)$, $\{W_\mu^+, W_{\mu 3}, W_\mu^-\}$ is a triplet of hypercharge 0, $\{V_\mu^+, V_\mu^0\}$ is a doublet of hypercharge +1 (and its conjugate $\{V_\mu^{0*}, V_\mu^-\}$ is a doublet of hypercharge −1), and $W_{\mu 8}$ is a neutral singlet.

In trinification (ignoring the color group), we will need two copies of Equation 5.1 to represent all 16 gauge bosons - one corresponding to $SU(3)_L$ and one to $SU(2)_R$.

A family of fermions in trinification belongs to a 27-plet [4], with each of the 27 fermion fields transforming as parts of triplets in two of the gauge groups and as singlets in the last. A 27 can be visualized as a cube with three layers, each of which is a 3 × 3 matrix (a "bi-triplet"). Each of these layers can be visualized as a 3 × 3 matrix. The first two, containing the quarks, are [6]

$$Q_L = \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \\ D_1 & D_2 & D_3 \end{pmatrix}_L \quad Q_R = \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \\ D_1 & D_2 & D_3 \end{pmatrix}_R \tag{5.2}$$

where u refers to up quarks, d to down quarks, and D to a new set of quarks. The subscript indices denote color (red, green, and blue), and the L and R subscripts outside the matrices label whether these are right- or left-handed bi-triplets. The final bi-triplet, which contains fields transforming as triplets under $SU(3)_R$ and $SU(3)_L$ and as singlets under $SU(3)_C$, contains the leptons and can be visualized as [6]

$$\xi_L = \begin{pmatrix} E^0 & E^- & e^- \\ E^+ & E^{c0} & \nu \\ e^+ & \nu^c & N \end{pmatrix} \tag{5.3}$$

where ν is the left-handed neutrino, e^- is the electron, e^+ is the positron, and E^0, E^{c0}, E^+, E^-, ν^c, and N are new fields not seen in the Standard Model. Once the symmetry breaks down to the Standard Model, the fermions break up into various singlets and doublets [6]: $\{E^+, E^0\}$ is a doublet of hypercharge +1; its conjugate $\{E^{0*}, E^-\}$ is a doublet of hypercharge −1; $\{\nu, e^-\}$ is a doublet of hypercharge −1; e^c is a singlet of hypercharge +2; and ν^c and N are neutral singlets. The hypercharge Y and weak isospin T_3 values are shown in Table 5.1. With the basic field content of trinification in hand, we look at a specific example of trinification symmetry breaking.

Table 5.1

Field	Y	T_3
E^0	+1	-1/2
E^+	+1	+1/2
E^-	-1	-1/2
E^{c0}	-1	+1/2
e^-	-1	-1/2
ν	-1	+1/2
e^+	+2	0
ν^c	0	0
N	0	0

5.1 Bi-Triplet Higgs Model

In this section, we introduce one of the simplest versions of trinification symmetry, as it is a good example and will be used in later calculations.

In what we will call the "Bi-Triplet Higgs Model," the trinification symmetry groups are broken to the Standard Model by a pair of bi-triplet Higgs fields. A bi-triplet Higgs field has a similar field content to the lepton field ξ_L described in Equation 5.3; that is, both transform in the same way under $SU(3)_C$, $SU(3)_L$, and $SU(3)_R$. Thus the scalar fields in the matrices below have the same hypercharges, weak isospins, and $SU(2)_L$ multiplets as those in Equation 5.3. We will, from now on, write a bi-triplet ϕ as ϕ_i^α, where each index runs from 1 to 3. Left-handed indices will be written as subscripts and have Latin letters; right-handed indices will be superscripts and have Greek letters.

The two bi-triplets in this model acquire two VEVs in order to break the gauge symmetry down to $SU(2)_L \times U(1)_Y$, which we will call V_1 and V_2. These VEVs are placed in neutral positions $\langle\phi^3_3\rangle_1 = V_1$ and $\langle\phi^3_2\rangle_2 = V_2$, or in matrix form [4]

$$\phi_1 = \begin{pmatrix} \hat{0} & 0 & 0 \\ 0 & \hat{0} & 0 \\ 0 & 0 & V_1 \end{pmatrix} \; \phi_2 = \begin{pmatrix} \hat{0} & 0 & 0 \\ 0 & \hat{0} & \hat{0} \\ 0 & V_2 & \hat{0} \end{pmatrix} \tag{5.4}$$

The bi-triplet components which are neutral with respect to Standard Model charges are able to obtain VEVs; these are denoted above with carets. The fields with carets gain smaller VEVs than V_1 and V_2, so that these components break the Standard Model symmetry $SU(2)_L \times U(1)$ down to $U(1)_{EM}$ at the electroweak scale. Due to the presence of a VEV in the position ϕ^3_2, this model breaks the T-parity symmetry naturally present in trinification; without this symmetry, a natural path towards stabilizing dark matter is not present in the model (see Chapter 6.3). However, because this model does break down to the Standard Model in a simple, clean way, in the next section we will use this pattern of VEVs to determine the charges and multiplets of the more complex bi-octet.

CHAPTER 6: Bi-Adjoint Model

The main result presented in this paper is a bi-octet model of trinification symmetry breaking. Previous authors have used various combinations of multiplets to achieve this symmetry breaking, such as a bi-triplet [4] (which on its own must break T-parity) or both a bi-triplet and a bi-sextet [6]. By contrast, we achieve breaking using three Higgs fields: a bi-triplet ϕ_n, a single triplet χ, and a bi-octet ψ. To maintain a consistent pattern of fermion masses, each of these models requires 3 copies of the bi-triplet.

6.1 General Structure of a Bi-Octet

The bi-octet ψ is a field introduced in this paper whose components transform as members of octets in the adjoint representation of $SU(3)$. As this is a new addition to models of trinification, we consider it in depth in this section. We first consider general properties of ψ which can be deduced from first principles, then move to a more exact description using group theory.

To perform future calculations, we need to find a component form for ψ. In ψ, each component of a right-handed octet is also a left-handed octet, so ψ must have 64 individual components. Thus, any tensor representation of ψ should have four indices: two to locate the left-handed octet and the component's place within it, and two more to do the same for the right-handed octet. To this end, we write the bi-octet as a four-index tensor $\psi_{ij}^{\alpha\beta}$, with each index running from 1 to 3. As before, Latin indices are left-handed; Greek indices are right-handed. Since it has four indices, ψ can be visualized as a matrix of matrices. First, we must choose a specific pattern of how to locate an element of $\psi_{ij}^{\alpha\beta}$ inside an array (as the actual choice is arbitrary); we choose here to let the outer matrices indicate left-handed indices and inner matrices indicate right-handed indices, giving us the array

$$\begin{bmatrix} \begin{pmatrix} \psi_{11}^{11} & \psi_{11}^{12} & \psi_{11}^{13} \\ \psi_{11}^{21} & \psi_{11}^{22} & \psi_{11}^{23} \\ \psi_{11}^{31} & \psi_{11}^{32} & \psi_{11}^{33} \end{pmatrix} & \begin{pmatrix} \psi_{12}^{11} & \psi_{12}^{12} & \psi_{12}^{13} \\ \psi_{12}^{21} & \psi_{12}^{22} & \psi_{12}^{23} \\ \psi_{12}^{31} & \psi_{12}^{32} & \psi_{12}^{33} \end{pmatrix} & \begin{pmatrix} \psi_{13}^{11} & \psi_{13}^{12} & \psi_{13}^{13} \\ \psi_{13}^{21} & \psi_{13}^{22} & \psi_{13}^{23} \\ \psi_{13}^{31} & \psi_{13}^{32} & \psi_{13}^{33} \end{pmatrix} \\ \begin{pmatrix} \psi_{21}^{11} & \psi_{21}^{12} & \psi_{21}^{13} \\ \psi_{21}^{21} & \psi_{21}^{22} & \psi_{21}^{23} \\ \psi_{21}^{31} & \psi_{21}^{32} & \psi_{21}^{33} \end{pmatrix} & \begin{pmatrix} \psi_{22}^{11} & \psi_{22}^{12} & \psi_{22}^{13} \\ \psi_{22}^{21} & \psi_{22}^{22} & \psi_{22}^{23} \\ \psi_{22}^{31} & \psi_{22}^{32} & \psi_{22}^{33} \end{pmatrix} & \begin{pmatrix} \psi_{23}^{11} & \psi_{23}^{12} & \psi_{23}^{13} \\ \psi_{23}^{21} & \psi_{23}^{22} & \psi_{23}^{23} \\ \psi_{23}^{31} & \psi_{23}^{32} & \psi_{23}^{33} \end{pmatrix} \\ \begin{pmatrix} \psi_{31}^{11} & \psi_{31}^{12} & \psi_{31}^{13} \\ \psi_{31}^{21} & \psi_{31}^{22} & \psi_{31}^{23} \\ \psi_{31}^{31} & \psi_{31}^{32} & \psi_{31}^{33} \end{pmatrix} & \begin{pmatrix} \psi_{32}^{11} & \psi_{32}^{12} & \psi_{32}^{13} \\ \psi_{32}^{21} & \psi_{32}^{22} & \psi_{32}^{23} \\ \psi_{32}^{31} & \psi_{32}^{32} & \psi_{32}^{33} \end{pmatrix} & \begin{pmatrix} \psi_{33}^{11} & \psi_{33}^{12} & \psi_{33}^{13} \\ \psi_{33}^{21} & \psi_{33}^{22} & \psi_{33}^{23} \\ \psi_{33}^{31} & \psi_{33}^{32} & \psi_{33}^{33} \end{pmatrix} \end{bmatrix}$$

Since ψ has four indices which can each take three values, ψ must have $3^4 = 81$ total entries. Additionally, since each entry of ψ can in theory be complex, our degrees of freedom double, and it would seem that there are 162 independent fields contained in the bi-octet. However, certain conditions limit the total number of fields to 64, as we expect for the joining of two octets (see the similar case of the bi-triplet in Equation 5.3).

First, the bi-octet has a "hermiticity" condition: $(\psi_{ij}^{\alpha\beta})^* = \psi_{ji}^{\beta\alpha}$. That is, although entries can be complex, the "opposite" entries in the bi-octet will be complex conjugates of each other, limiting the degrees of freedom to 81 (the same as if all entries were real). For example, $\psi_{13}^{\alpha\beta}$ is the Hermitian conjugate of $\psi_{31}^{\alpha\beta}$.

Second, 17 tracelessness conditions limit the number of independent fields from 81 to 64. These conditions stem from the fact that the generators of $SU(3)$ (which the adjoint representation is built from) are traceless themselves. In light of this, every right-handed submatrix should be traceless (the sum of its diagonal components must be 0). For example, summing over α, $\psi_{11}^{\alpha\alpha} = 0$. Since there are nine right-handed submatrices, these tracelessness conditions lower the overall degrees of freedom by nine. Similarly, every left-handed submatrix must be traceless (i.e, summing over i, $\psi_{ii}^{12} = 0$), which limits the degrees of freedom by another eight. It seems as if these conditions should remove 18 degrees of freedom (nine from the right-handed submatrices and nine from the left-handed ones), but the six conditions along the "main diagonal" (where $i = j$ and $\alpha = \beta$) turn out to contain only five independent equations. These 17 additional constraints indicate that only 64 independent fields exist, as we would expect for a bi-octet. The degrees of freedom and conditions are summarized in Table 6.1.

Table 6.1

Conditions	Degrees of Freedom
None	162
Hermiticity	-81
Tracelessness	-17
Total	64

The tensor ψ can be constructed using a different - and in some ways more useful - method based in group theory, which is quite similar to the strategy used to construct Equation 5.1. We start with 64 real fields W_k^γ, each of which can be identified as the field attached to a unique pairing of a right-handed generator T^γ and a left-handed generator T_k. Since T^γ and T_k are each 3×3 matrices Equation 2.3.1, we can write $T_k = (T_{ij})_k$ and $T^\gamma = T^{\alpha\beta})^\gamma$. With this notation in hand, we can find each component $\psi_{ij}^{\alpha\beta}$ of our bi-octet in terms of the fields W_k^γ by summing over the generators of $SU(3))_L$ and $SU(3)_R$:

$$\psi_{ij}^{\alpha\beta} = (T_{ij})_k \, W_k^\gamma \, (T^{\alpha\beta})^\gamma$$

In this form, the entire bi-octet is too large to be reproduced in-text, but some examples can be given. Carrying out the sum, the lower left submatrix is (holding left-handed indices constant) $\psi_{31}^{\alpha\beta}$:

$$\begin{pmatrix} W_4^3+\frac{\sqrt{3}W_4^8}{3}-iW_5^3-\frac{\sqrt{3}iW_5^8}{3} & W_4^1-iW_4^2-iW_5^1-W_5^2 & W_4^4-iW_4^5-iW_5^4-W_5^5 \\ W_4^1+iW_4^2-iW_5^1+W_5^2 & -W_4^3+\frac{\sqrt{3}W_4^8}{3}+iW_5^3-\frac{\sqrt{3}iW_5^8}{3} & W_4^6-iW_4^7-iW_5^6-W_5^7 \\ W_4^4+iW_4^5-iW_5^4+W_5^5 & W_4^6+iW_4^7-iW_5^6+W_5^7 & -\frac{2\sqrt{3}W_4^8}{3}+\frac{2\sqrt{3}iW_5^8}{3} \end{pmatrix}$$

The tracelessness of this submatrix can be seen visually by adding its diagonal components. The "hermiticity" condition can be seen by comparing $\psi_{31}^{\alpha\beta}$ to the upper right submatrix $\psi_{13}^{\alpha\beta}$ shown below. Looking at the entries of both, we can see that the opposite entries in each are indeed complex conjugates of each other.

$$\begin{pmatrix} W_4^3+\frac{\sqrt{3}W_4^8}{3}+iW_5^3+\frac{\sqrt{3}iW_5^8}{3} & W_4^1-iW_4^2+iW_5^1+W_5^2 & W_4^4-iW_4^5+iW_5^4+W_5^5 \\ W_4^1+iW_4^2+iW_5^1-W_5^2 & -W_4^3+\frac{\sqrt{3}W_4^8}{3}-iW_5^3+\frac{\sqrt{3}iW_5^8}{3} & W_4^6-iW_4^7+iW_5^6+W_5^7 \\ W_4^4+iW_4^5+iW_5^4-W_5^5 & W_4^6+iW_4^7+iW_5^6-W_5^7 & -\frac{2\sqrt{3}W_4^8}{3}-\frac{2\sqrt{3}iW_5^8}{3} \end{pmatrix}$$

6.2 Triplet Structure

Our symmetry breaking model requires not just a bi-octet and the bi-triplets discussed in Chapter 5, but also two single triplets χ_L and χ_R. Later on, we will give a VEV to the third component of χ_R, leaving χ_L out of the symmetry breaking process, but it is still necessary to include χ_L in our model to keep left-right symmetry in place. We can write these triplets in component form as

$$\chi_L = \begin{pmatrix} \chi_{L1} \\ \chi_{L2} \\ \chi_{L3} \end{pmatrix} \quad \chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ \chi_{R3} \end{pmatrix}$$

Table 6.2 shows the hypercharge and isospin values of these fields. In terms of multiplets, $\{\chi_{L1}, \chi_{L2}\}$ form a doublet of hypercharge $+1$, whereas the other four fields are all singlets. The charges were obtained from the gauge-invariant coupling $\chi_L^i \phi_i^\alpha \overline{\chi_R^\alpha}$.

Table 6.2

Field	Y	T_3
χ_{L1}	+1	+1/2
χ_{L2}	+1	-1/2
χ_{L3}	0	0
χ_{R1}	+2	0
χ_{R2}	0	0
χ_{R3}	0	0

6.3 T-Parity

Trinification naturally has a discrete Z_2 symmetry which, following [6], we call T-parity. This parity symmetry, under which particles are either "even" or "odd," allows us to stabilize any potential dark matter candidates.

The T-parity of a field (or fields) can be computed directly using the indices of a field. The T-parity of a field $Y^{\alpha\beta\ldots}_{ij\ldots}$ is given by the equation [6]

$$(-1)^{X_i+X_j+X_\alpha+X_\beta+\ldots+2S} \tag{6.1}$$

where S is the field's spin and

$$X_i = \begin{cases} +1 & \text{if } i = 1, 2 \\ -2 & \text{if } i = 3 \end{cases}$$

The Standard Model leptons all have even T-parity, as is easily verified from the bi-triplets in Equation 5.3. To be a reasonable dark matter candidate, a field should not have interactions where it decays into Standard Model particles. Our T-odd fields are then perfect candidates. Letting dark matter be our *lightest* T-odd field further stabilizes it against decay.

In order to preserve this symmetry after trinification symmetry breaking, the VEVs we give to the fields should have even T-parity. In the bi-triplet model described in Section 5.1, the ϕ_2 VEV has odd T-parity, making the model somewhat less viable. By contrast, the symmetry breaking model we develop in Chapter 7 does preserve T-parity, allowing us to stabilize dark matter using this symmetry in the future.

6.4 Higgs Potential

The bi-octet model requires three different multiplets to achieve consistent symmetry breaking while keeping the T-parity symmetry intact: the bi-octet ψ, the bi-triplets ϕ, and a right-handed triplet χ_R (as mentioned above, to keep left-right symmetry we also need a triplet χ_L, but since it does not obtain a VEV we ignore it for the rest of this paper and right $\chi = \chi_R$ for simplicity).

For this model, we first need to determine the potential terms in the Higgs sector. These terms can be used to find the necessary minimization conditions and the masses of the Higgs field once the symmetry is broken to the Standard Model. The total scalar potential can be broken down into five parts: the bi-triplet potential $V(\phi)$; the single triplet potential $V(\chi)$; the bi-octet potential terms $V(\psi)$; the bi-octet and bi-triplet potential $V(\psi, \chi)$; the triplet and bi-triplet potential $V(\phi, \chi)$. It is not necessary to consider a joint potential $V(\psi, \phi, \chi)$ here, as it would require χ_L, which does not obtain a VEV.

The first of these, $V(\phi)$, is found in [6]. Utilizing the Levi-Civita symbol and denoting Hermitian conjugates with an overline

$$V(\phi) = -m_\phi^2 \phi_i^\alpha \overline{\phi_i^\alpha} + (\mu_\phi \phi_i^\alpha \phi_j^\beta \phi_k^\gamma \epsilon_{ijk} \epsilon_{\alpha\beta\gamma} + h.c.) + \lambda_6 \phi_i^\alpha \overline{\phi_i^\alpha} \phi_j^\beta \overline{\phi_j^\beta} + \lambda_7 \phi_i^\alpha \overline{\phi_j^\alpha} \phi_j^\beta \overline{\phi_i^\beta} \tag{6.2}$$

Similarly, $V(\chi)$ was found by [10]. In index form

$$V(\chi) = -m_\chi^2 \chi^\alpha \overline{\chi^\alpha} + \lambda_{18}(\chi^\alpha \overline{\chi^\alpha})^2 \tag{6.3}$$

The most general potential for the bi-octet ψ must be determined using group theory. The relevant ways for two octets to contract are given by $8 \times 8 = 1_s + 8_s + 8_a$ [11]. There is therefore one path for two octets to contract to a singlet (m_ψ below), and two paths for four octets: $\psi_{ij}\psi_{ji}\psi_{kl}\psi_{lk}$ and $\psi_{ij}\psi_{jk}\psi_{kl}\psi_{li}$. Since we are dealing with a bi-octet, these two paths become four (λ_1 through λ_4). Finally, three octets appear to have two possible paths ($8_s \times 8$ and $8_a \times 8$), but we find only one independent term λ_5. The mystery surrounding the cubic terms is left to future work. In index form, we find the following contractions:

$$\begin{aligned} V(\psi) = &- m_\psi^2 \psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\alpha} + \lambda_1 \psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\alpha}\psi_{kl}^{\gamma\delta}\psi_{lk}^{\delta\gamma} + \lambda_2 \psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\gamma}\psi_{kl}^{\gamma\delta}\psi_{lk}^{\delta\alpha} \\ &+ \lambda_3 \psi_{ij}^{\alpha\beta}\psi_{jk}^{\beta\alpha}\psi_{kl}^{\gamma\delta}\psi_{li}^{\delta\gamma} + \lambda_4 \psi_{ij}^{\alpha\beta}\psi_{jk}^{\beta\gamma}\psi_{kl}^{\gamma\delta}\psi_{li}^{\delta\alpha} + \lambda_5 \psi_{ij}^{\alpha\beta}\psi_{ki}^{\gamma\alpha}\psi_{jk}^{\beta\gamma} \end{aligned} \tag{6.4}$$

The combined potential $V(\psi, \phi)$ was found by considering the paths for an octet and a triplet to contract. The relevant paths are $8 \times 3 = 3 + \overline{6}$ and $3 \times \overline{3} = 1 + 8$ [11]. Combining these paths, we find three possible ways for two octets and two triplets to contract: $\psi_{ij}\psi_{ji}\phi_k\overline{\phi_k}$, $\psi_{ij}\psi_{jk}\phi_i\overline{\phi_k}$, and $\psi_{ij}\psi_{kn}\phi_l\overline{\phi_m}\epsilon_{ikm}\epsilon_{jnl}$. Since we are dealing with bi-octets and bi-triplets here, all possible combinations of right- and left-handed paths give us a total of nine potential terms (λ_8 through λ_{16} below). We find one term for an octet, a triplet, and a conjugate triplet to contract, giving us λ_{17}.

$$\begin{aligned} V(\psi, \phi) =& \lambda_8 \psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\alpha}\phi_k^\gamma\overline{\phi_k^\gamma} + \lambda_9 \psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\gamma}\phi_k^\alpha\overline{\phi_k^\gamma} + \lambda_{10}\psi_{ij}^{\alpha\beta}\psi_{ji}^{\gamma\delta}\phi_k^\sigma\overline{\phi_k^\rho}\epsilon_{\alpha\gamma\rho}\epsilon_{\beta\delta\sigma} \\ &+ \lambda_{11}\psi_{ij}^{\alpha\beta}\psi_{jk}^{\beta\alpha}\phi_i^\gamma\overline{\phi_k^\gamma} + \lambda_{12}\psi_{ij}^{\alpha\beta}\psi_{jk}^{\beta\gamma}\phi_i^\alpha\overline{\phi_k^\gamma} + \lambda_{13}\psi_{ij}^{\alpha\beta}\psi_{jk}^{\gamma\delta}\phi_i^\sigma\overline{\phi_k^\rho}\epsilon_{\alpha\gamma\rho}\epsilon_{\beta\delta\sigma} \\ &+ \lambda_{14}\psi_{ij}^{\alpha\beta}\psi_{kn}^{\beta\alpha}\phi_l^\gamma\overline{\phi_m^\gamma}\epsilon_{ikm}\epsilon_{jnl} + \lambda_{15}\psi_{ij}^{\alpha\beta}\psi_{kn}^{\beta\gamma}\phi_l^\alpha\overline{\phi_m^\gamma}\epsilon_{ikm}\epsilon_{jnl} \\ &+ \lambda_{16}\psi_{ij}^{\alpha\beta}\psi_{kn}^{\gamma\delta}\phi_l^\sigma\overline{\phi_m^\rho}\epsilon_{ikm}\epsilon_{jnl}\epsilon_{\alpha\gamma\rho}\epsilon_{\beta\delta\sigma} + \lambda_{17}\psi_{ij}^{\alpha\beta}\phi_i^\alpha\overline{\phi_j^\beta} \end{aligned} \tag{6.5}$$

Next, $V(\psi, \chi)$ looks quite similar to Equation 6.5, but since χ_L is not involved in the symmetry breaking, we only need three quartic potential terms (the other analogues to the terms in Equation 6.5 involve χ_L). No cubic term is needed here, as it would also require χ_L:

$$V(\psi, \chi) = \lambda_{19}\psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\alpha}\chi^{\gamma}\overline{\chi^{\gamma}} + \lambda_{20}\psi_{ij}^{\alpha\beta}\psi_{ji}^{\beta\gamma}\chi^{\alpha}\overline{\chi^{\gamma}} + \lambda_{21}\psi_{ij}^{\alpha\beta}\psi_{ji}^{\gamma\delta}\chi^{\sigma}\overline{\chi^{\rho}}\epsilon_{\alpha\gamma\rho}\epsilon_{\beta\delta\sigma} \tag{6.6}$$

Finally, $V(\phi, \chi)$ was found by [10]. In index form, we have

$$V(\phi, \chi) = \lambda_{22}\phi_i^{\alpha}\overline{\phi_i^{\alpha}}\chi^{\beta}\overline{\chi^{\beta}} + \lambda_{23}\phi_i^{\alpha}\overline{\phi_i^{\beta}}\chi^{\beta}\overline{\chi^{\alpha}} \tag{6.7}$$

6.5 Bi-Octet Charges and Multiplets

Opening up ψ in terms of W_k^{γ} as in Section 6.1 is useful, but in general the fields which live in Standard Model multiplets are not the fields W_k^{γ}, but rather specific linear combinations of these fields. There are four possible types of linear combinations: four-field combinations (e.g., $W_4^1 - iW_4^2 - iW_5^1 - W_5^2$) corresponding to single entry $\psi_{ij}^{\alpha\beta}$, two-field combinations (e.g., $W_4^6 - iW_5^6$) which appear in exactly two $\psi_{ij}^{\alpha\beta}$ entries, or singlet fields such as W_8^8 and W_8^3 (which appear in six and nine entries of ψ, respectively).

To find the Standard Model multiplets, we use two things: (1) a gauge invariant cubic coupling between ψ and a bi-triplet ϕ and (2) the bi-triplet model discussed in Chapter 5.

Firstly, to be a gauge-invariant tensor, each entry $\psi_{ij}^{\alpha\beta}$ must have a unique set of Standard Model charges (weak hypercharge Y and weak isospin T_3). These charges can be found by pairing ψ with a field we whose charges we already know - such as a bi-triplet (Equation 5.3). The simplest such term is cubic, and (in index form) reads $\psi_{ij}^{\alpha\beta}\phi_i^{\alpha}\overline{\phi_j^{\beta}}$. This is the cubic term that appears in Equation 6.5 accompanied by the coupling constant λ_{17}.

For the Lagrangian to be gauge-invariant, each term must contain fields whose total quantum charges sum to 0 (otherwise a gauge transformation would change the Lagrangian, which would make it no longer a gauge transformation). Since the charges of ϕ_i^α are known, it is simple to extrapolate what the charges of $\psi_{ij}^{\alpha\beta}$ must be by summing over all indices in $\psi_{ij}^{\alpha\beta}\phi_i^\alpha\overline{\phi_j^\beta}$ and looking at each term. For example, the term in the sum containing ψ_{11}^{21} is $E^- E^{c0}\psi_{11}^{21}$. The combined hypercharge and isospin values of E^- and E^{c0} are -2 and 0, respectively, so ψ_{11}^{21} should contain fields with isospin 0 and hypercharge $+2$.

Since the fields $\psi_{ij}^{\alpha\beta}$ may not be the Standard Model interaction eigenstates, we need to find the linear combinations that are interaction eigenstates. To determine these, we use the bi-triplet model discussed in Chapter 5, treating the bi-octet not as a Higgs field to be given VEVs, but as a simple scalar field whose mass is determined by the bi-triplet Higgs. Mathematically, we couple the bi-octet ψ to the minimized fields ϕ_1 and ϕ_2 (Equation 5.4) using $V(\psi, \phi)$ (Equation 6.5). In general, we need three separate potentials: $V(\psi, \phi_1)$, $V(\psi, \phi_2)$, and $V(\psi, \phi_1, \phi_2)$ (in the last, all possible combinations of ϕ_1 and ϕ_2 are considered). However, the first two of these turn out to be sufficient to determine the multiplets. This substitution will give the fields mass terms; the fields with the same masses and hypercharges but different isospins must be in the same multiplets.

Quantum mechanical triplets contain fields with charges of $+1$, 0, and -1; doublets take T_3 values of $1/2$ and $-1/2$; and singlets have charges of 0 (except in the case of $U(1)$. Thus the correct multiplets can be found by matching fields with the right charges and the same masses after the bi-triplets break the symmetry to the Standard Model. The new combinations of fields W_k^γ that are in Standard Model multiplets were found and confirmed by substituting the redefined fields into ψ until the precise multiplets were found. The redefined fields and their charges are listed in Appendix A; the Standard Model multiplets found using this process are listed in Appendix B.

CHAPTER 7: Bi-Octet Symmetry Breaking

The symmetry breaking pathway we present here has VEVs in three fields: the bi-octet ψ, the bi-triplet ϕ, and a right-handed triplet χ_R. We give give VEVs to $\langle W_8^3 \rangle = V_\psi$, $\langle \phi_3^3 \rangle = V_n$, and $\langle \chi_3 \rangle = V_\chi$. To replicate the pattern of fermion masses we use three copies of the bi-triplet, although it is sufficient to deal with only one in our analysis [6]. Finally, to break electroweak symmetry, ϕ_1^1 and ϕ_2^2 acquire VEVs at the electroweak scale. Writing the trio of bi-triplets implicitly with the index n, the VEVs considered here are

$$\langle W_8^3 \rangle = V_\psi \quad \langle \chi_R \rangle = \begin{pmatrix} 0 \\ 0 \\ V_\chi \end{pmatrix} \quad \langle \phi_n \rangle = \begin{pmatrix} v_{un} & 0 & 0 \\ 0 & v_{dn} & 0 \\ 0 & 0 & V_n \end{pmatrix} \tag{7.1}$$

This paper is mainly concerned with symmetry breaking to the Standard Model, so we will largely ignore the electroweak-scale VEVs v_{un} and v_{dn}, which are taken to be far smaller than V_ψ, V_n, and V_χ. Since the three bi-triplet V_n VEVs V_1, V_2, and V_3 are all in the same position, only one linear combination of these three VEVs acquires a VEV on the order of V_ψ and V_χ [6], which we will simply write as V_n (the other two combinations are, like v_{un} and v_{dn}, on the electroweak scale).

Since the Standard Model fields all have even T-parity, we can preserve this T-parity symmetry if we give VEVs only to T-odd fields. Since W_8^3 is found along the main diagonal of the bi-octet (where $i = j$ and $\alpha = \beta$), it has an even T-parity according to Equation 6.1. Similarly, since any VEVs in the bi-triplets occur in positions where $i = j$, these fields are also T-even. Finally, since V_χ has index $\alpha = 3$, it also has even T-parity. Since all our VEVs are in T-even fields, the T-parity symmetry is preserved down to $SU(3)_C \times U(1)_{EM}$, allowing us to stabilize any potential dark matter candidates in the future.

It is instructive to examine what each of the three relevant VEVs does to the trinification symmetry on their own. To do so, we must examine the effect of these VEVs on the gauge sector.

7.1 Gauge Sector

Just as in Section 4.2.1, the gauge bosons in trinification gain mass when Higgs fields acquire vacuum expectation values. The gauge boson masses can be found by inserting the VEVs into the covariant derivatives for each field and examining how the Lagrangian changes. For each gauge boson given mass, a Higgs field must become massless; for this reason it is often said that massive gauge bosons "eat" the Higgs. In this section, we will give the precise covariant derivatives for our Higgs fields.

For a single bi-triplet, the covariant derivative is given by [6]

$$\mathcal{L}_{kin} = D^\mu \phi_i^\alpha (D^\mu \phi_i^\alpha)^\dagger$$

where all indices are summed, with the μ representing a Lorentz 4-vector. For multiple bi-triplets, we simply carry out this summation three times. This format can extended quite simply to the bi-octet and triplets to get a total kinetic term Lagrangian:

$$\mathcal{L}_{kin} = D^\mu \phi_i^\alpha (D^\mu \phi_i^\alpha)^\dagger + D^\mu \chi^\alpha (D^\mu \chi^\alpha)^\dagger + D^\mu \psi_{ij}^{\alpha\beta} (D^\mu \psi_{ij}^{\alpha\beta})^\dagger$$

It is sufficient for us to work out D for each type of field and then sum over all indices. The covariant derivative for a single bi-triplet can be found in [6], and is given by

$$D^\mu \phi_i^\alpha = \partial^\mu \phi_i^\alpha - \frac{ig_L}{2}(T \cdot W_L^\mu)_i^k \phi_k^\alpha + \frac{ig_R}{2}(T \cdot W_R^\mu)_k^\alpha \phi_i^k \tag{7.2}$$

where ∂^μ denotes the usual 4-vector derivative; g_L is a coupling constant related to gauge transformations of $SU(3)_L$; g_R is a coupling constant related to $SU(3)_R$; and $T \cdot W_L^\mu$ and $T \cdot W_R^\mu$ are 3×3 matrices containing the gauge bosons of $SU(3)_L$ and $SU(3)_R$, respectively (constructed in the manner of Equation 5.1).

The covariant derivative for a single right-handed triplet can be obtained easily from Equation 7.2 by simply ignoring any left-handed indices:

$$D^\mu \chi^\alpha = \partial^\mu \chi^\alpha + \frac{ig_R}{2}(T \cdot W^\mu_R)^\alpha_k \phi^k \tag{7.3}$$

The covariant derivative for ψ is a bit trickier. Firstly, we will need five terms instead of the three seen in the bi-triplet, as we will need to sum over each index of ψ separately. Second, we find that for any sensible breaking pattern, the pattern of negative signs in the terms of $D^\mu \psi^{\alpha\beta}_{ij}$ is slightly different from that found in the bi-triplet, and from the covariant derivative for the otherwise-similar bi-sextet in [6]:

$$\begin{aligned} D^\mu \psi^{\alpha\beta}_{ij} =& \partial^\mu \psi^{\alpha\beta}_{ij} + \frac{ig_L}{2}(T \cdot W^\mu_L)^k_i \psi^{\alpha\beta}_{kj} - \frac{ig_L}{2}(T \cdot W^\mu_L)^j_k \psi^{\alpha\beta}_{ik} \\ &+ \frac{ig_R}{2}(T \cdot W^\mu_R)^\gamma_\alpha \psi^{\gamma\beta}_{ij} - \frac{ig_R}{2}(T \cdot W^\mu_R)^\beta_\gamma \psi^{\alpha\gamma}_{ij} \end{aligned} \tag{7.4}$$

Instead of left-handed terms having negative signs and right-handed terms having positive signs (as for the bi-sextet in [6]), it is necessary to have one term positive and one negative. It seems not to matter precisely *which* term is positive and which negative, but the pattern must be present. If it is not, then no known breaking path from $SU(3)_L \times SU(3)_R$ can be seen in the gauge sector (not to mention electrically charged terms enter into the Lagrangian, which violates $U(1)_{EM}$ symmetry). Since the covariant derivative in Equation 7.4 allows for such paths through certain fields acquiring VEVs, its form appears to be correct. Nevertheless, we have not yet determined why this form is correct using group theory, and efforts to check the breaking path by finding which Higgs fields become massless are incomplete. The reasons for the form taken by this covariant derivative are left to future work.

7.2 Symmetry Breaking and Gauge Boson Masses

Using the covariant derivatives Equation 7.2, Equation 7.3, and Equation 7.4, we can now examine the effects of giving the VEVs in Equation 7.1. The remaining symmetries below are determined by looking at which gauge bosons remain massless.

On its own, V_n breaks trinification to $SU(2)_L \times SU(2)_R \times U(1)$. On the other hand, V_χ breaks the symmetry to $SU(3)_L \times SU(2)_R$. Together, these two VEVs break the symmetry to $SU(2)_L \times SU(2)_R$. Evidently, breaking trinification down to the Standard Model requires another VEV - such as the bi-octet.

V_ψ breaks the trinification symmetry to $SU(2)_L \times U(1)_L \times U(1)_{R1} \times U(1)_{R2}$. As a general rule, Higgs fields in the adjoint representation should not be able to break the rank of a symmetry. Since $SU(3)_L \times SU(3)_R$ is rank 4, it should be the case that the remaining symmetry has four diagonalizable generators. Since $SU(2)$ and $U(1)$ each have one diagonalizable generator, this is precisely what we find. In order to successfully break trinification to the Standard Model, we require two more VEVs (V_n and V_χ) to break the extraneous $U(1)$ symmetries.

Combining V_n, V_ψ, and V_χ gives mass to 12 gauge bosons, leaving us with the residual symmetry $SU(2)_L \times U(1)_R$. Letting this last $U(1)$ (whose associated gauge boson corresponds to W_{3R}) be $U(1)_Y$ gives us the Standard Model symmetry and a viable symmetry breaking pathway. The various symmetry breaking paths are summarized in Table 7.1.

Table 7.1

VEV	Remaining Gauge Symmetry
V_n	$SU(2)_L \times SU(2)_R \times U(1)$
V_χ	$SU(3)_L \times SU(2)_R$
V_ψ	$SU(2)_L \times U(1)_L \times U(1)_{R1} \times U(1)_{R2}$
V_ϕ, V_χ	$SU(2)_L \times SU(2)_R$
V_ϕ, V_χ, V_ψ	$SU(2)_L \times U(1)_R$

The combined VEVs V_n, V_ψ, and V_χ change the Lagrangian $\mathcal{L}$ by

$$\begin{aligned}\Delta\mathcal{L} =& \frac{1}{2}g_L^2V_n^2V_L^{\mu+}V_L^{\mu-} + 6g_L^2V_\psi^2V_L^{\mu+}V_L^{\mu-} + \frac{9}{2}g_R^2V_\chi^2V_R^{\mu+}V_R^{\mu-} + \frac{1}{2}g_R^2V_n^2V_R^{\mu+}V_R^{\mu-} \\ &+ 2g_R^2V_\psi^2V_R^{\mu+}V_R^{\mu-} + \frac{1}{2}g_L^2V_n^2V_L^{\mu 0}V_L^{\mu 0*} + 6g_L^2V_\psi^2V_L^{\mu 0}V_L^{\mu 0*} + \frac{9}{2}g_R^2V_\chi^2V_R^{\mu 0}V_R^* \\ &+ \frac{1}{2}g_R^2V_n^2V_R^{\mu 0}V_R^{\mu 0*} + 2g_R^2V_\psi^2V_R^{\mu 0}V_R^{\mu 0*} + 3g_R^2V_\chi^2(W_{8R}^\mu)^2 + \frac{1}{3}g_L^2V_n^2(W_{8L}^\mu)^2 \\ &+ \frac{2}{3}g_Lg_RV_n^2W_{8L}^\mu W_{8R}^\mu + \frac{1}{3}g_R^2V_n^2(W_{8R}^\mu)^2 + 8g_R^2V_\psi^2W_R^{\mu+}W_R^{\mu-}\end{aligned} \tag{7.5}$$

From Equation 7.5, we can see that on the scale of trinification symmetry breaking, 12 gauge bosons gain mass: $V_L^\pm$, $V_R^\pm$, $V_L^{0,0*}$, $V_R^{0,0*}$, $W_R^\pm$, W_{8R}, and W_{8L}. The remaining four massless gauge bosons are $W_L^\pm$, W_{3L}, and W_{3R}. The first three of these are associated with $SU(2)_L$, while the last is the gauge boson of $U(1)_Y$.

Adding the electroweak-scale VEVs v_{un} and v_{dn} makes $\Delta\mathcal{L}$ significantly more complicated, but we can sum up these results using what are called mass matrices. To construct these, we take a basis of fields with the same electric charge and T-parity (say, $x = (W_L^+, W_R^+)$) and write the quadratic couplings in matrix form, so that x^TAx returns us the sum of the terms that we had originally. The off-diagonal terms - such as the coefficient of $W_L^+W_R^-$ - correspond to mixing of the fields; the existence of these terms indicates that the basis we have chosen is not that of the mass eigenstates (i.e., our basis fields do not have definite mass). The advantage of the mass matrix notation is that the eigenvalues of the mass matrix *are* the definite masses, and its eigenvectors are the linear combinations of fields that correspond to definite mass eigenstates. The mass matrices of the gauge bosons after electroweak symmetry breaking are given in Appendix C.

7.3 Minimization Conditions

The final aspect of the bi-adjoint model considered here is the minimization conditions. These conditions must be met for the potential to be minimized, and are found by taking partial derivatives of the vacuum potential V with respect to each of the VEVs V_ψ, V_χ, and V_n. The potential V is the sum of Equation 6.2 through Equation 6.7, or

$$V(\psi,\phi,\chi) = V(\psi) + V(\chi) + V(\phi) + V(\psi,\phi) + V(\psi,\chi) + V(\phi,\chi)$$

To obtain the minimization conditions, we substitute into V the VEVs V_ψ, V_χ, and V_n and set all other fields to zero (that is, we only need the vacuum expectation values of the fields). This gives the vacuum potential

$$\begin{aligned} V =& -4m_\psi^2 - m_\phi^2 V_n^2 - m_\chi^2 V_\chi^2 + 16\lambda_1 V_\psi^4 + 8\lambda_2 V_\psi^4 + 8\lambda_3 V_\psi^4 + 4\lambda_4 V_\psi^4 + \lambda_6 V_n^4 \\ &+ \lambda_7 V_n^4 + 4\lambda_8 V_\psi^2 V_n^2 - 4\lambda_{10} V_\psi^2 V_n^2 + \frac{8\lambda_{11} V_\psi^2 V_n^2}{3} - \frac{8\lambda_{13} V_\psi^2 V_n^2}{3} + \frac{4\lambda_{14} V_\psi^2 V_n^2}{3} \\ &- \frac{4\lambda_{16} V_\psi^2 V_n^2}{3} + \lambda_{18} V_\chi^4 + 4\lambda_{19} V_\chi^2 V_\psi^2 - 4\lambda_{21} V_\chi^2 V_\psi^2 + \lambda_{22} V_\chi^2 V_n^2 + \lambda_{23} V_\chi^2 V_n^2 \end{aligned} \tag{7.6}$$

We can now take the derivatives $\frac{\partial V}{\partial V_\psi}$, $\frac{\partial V}{\partial v_\phi}$, and $\frac{\partial V}{\partial V_n}$ and set them equal to 0:

$$\begin{aligned} \frac{\partial V}{\partial V_\psi} = 0 =& -8m_\psi^2 V_\psi + 64\lambda_1 V_\psi^3 + 32\lambda_2 V_\psi^3 + 32\lambda_3 V_\psi^3 + 16\lambda_4 V_\psi^3 + 8\lambda_8 V_\psi V_n^2 \\ &- 8\lambda_{10} V_\psi V_n^2 + \frac{16\lambda_{11} V_\psi V_n^2}{3} - \frac{16\lambda_{13} V_\psi V_n^2}{3} + \frac{8\lambda_{14} V_\psi V_n^2}{3} \\ &- \frac{8\lambda_{16} V_\psi V_n^2}{3} + 8\lambda_{19} V_\chi^2 V_\psi - 8\lambda_{21} V_\chi^2 V_\psi \\ \frac{\partial V}{\partial V_n} = 0 =& -2m_\phi^2 V_n + 4\lambda_6 V_n^3 + 4\lambda_7 V_n^3 + 8\lambda_8 V_\psi^2 V_n - 8\lambda_{10} V_\psi^2 V_n + \frac{16\lambda_{11} V_\psi^2 V_n}{3} \\ &- \frac{16\lambda_{13} V_\psi^2 V_n}{3} + \frac{8\lambda_{14} V_\psi^2 V_n}{3} - \frac{8\lambda_{16} V_\psi^2 V_n}{3} + 2\lambda_{22} V_\chi^2 V_n + 2\lambda_{23} V_\chi^2 V_n \\ \frac{\partial V}{\partial V_\chi} = 0 =& -2m_\chi^2 V_\chi + 4\lambda_{18} V_\chi^3 + 8\lambda_{19} V_\chi V_\psi^2 - 8\lambda_{21} V_\chi V_\psi^2 + 2\lambda_{22} V_\chi V_n^2 + 2\lambda_{23} V_\chi V_n^2 \end{aligned}$$

These equations can easily be solved for m_ψ, m_ϕ, and m_χ:

$$m_\psi^2 = 8\lambda_1 V_\psi^2 + 4\lambda_2 V_\psi^2 + 4\lambda_3 V_\psi^2 + 2\lambda_4 V_\psi^2 + \lambda_8 V_n^2 - \lambda_{10} V_n^2 + \frac{2\lambda_{11} V_n^2}{3} - \frac{2\lambda_{13} V_n^2}{3} + \frac{\lambda_{14} V_n^2}{3} - \frac{\lambda_{16} V_n^2}{3} + V_\chi^2 \lambda_{19} - \lambda_{21} V_\chi^2 \tag{7.7}$$

$$m_\phi^2 = 2\lambda_6 V_n^2 + 2\lambda_7 V_n^2 + 4\lambda_8 V_\psi^2 - 4\lambda_{10} V_\psi^2 + \frac{8\lambda_{11} V_\psi^2}{3} - \frac{8\lambda_{13} V_\psi^2}{3} + \frac{4\lambda_{14} V_\psi^2}{3} - \frac{4\lambda_{16} V_\psi^2}{3} + \lambda_{22} V_\chi^2 + \lambda_{23} V_\chi^2 \tag{7.8}$$

$$m_\chi^2 = 2\lambda_{18} V_\chi^2 + 4\lambda_{19} V_\psi^2 - 4\lambda_{21} V_\psi^2 + \lambda_{22} V_n^2 + \lambda_{23} V_n^2 \tag{7.9}$$

These are the conditions on the coupling constants in the potential necessary for the symmetry breaking to occur. Substituting these conditions into the full potential, with the fields no longer set to 0, allows us to determine the mass matrices for the fields, and find which Higgs fields become massless. This process is not yet complete, and is left to future work.

7.4 Potential Boundedness Conditions

In order for the vacuum potential Equation 7.6 to open upward (i.e., for the vacuum to be a potential minimum), a set of conditions on the coupling constants are needed. Although all the restrictions on the couplings are not easy to determine, some conditions needed for the quartic terms to be bounded below can be given. Following the treatment in [6], it must be the case that the quartic terms given an overall nonnegative contribution to the potential. We can write $x^T = (V_\psi^2, V_\chi^2, V_n^2)$ and format the quartic term coefficients in a matrix A so that $x^T A x$ returns the sum of the quartic potential terms contained in Equation 7.6. In the basis x^T, this coefficient matrix is

$$\begin{pmatrix} 8\lambda_1 + 2\lambda_4 + 4\lambda & 2(\lambda_{19} - \lambda_{21}) & 2(\lambda_8 - \lambda_{10}) + \Lambda \\ 2(\lambda_{19} - \lambda_{21}) & \lambda_{18} & \frac{1}{2}(\lambda_{22} + \lambda_{23}) \\ 2(\lambda_8 - \lambda_{10}) + \Lambda & \frac{1}{2}(\lambda_{22} + \lambda_{23}) & \lambda_6 + \lambda_7 \end{pmatrix}$$

where we have used the convenient substitutions $\lambda = \lambda_2 + \lambda_3$ and $\Lambda = \frac{2}{3}(2\lambda_{11} - 2\lambda_{13} + \lambda_{14} - \lambda_{16})$. In this notation, the potential opens up only if $x^T A x \geq 0$; that is, if the coefficient matrix is copositive [6]. The conditions for a 3×3 matrix to be copositive are given in [12]; for the above matrix, we have the necessary conditions

$$8\lambda_1 + 2\lambda_4 + 4\lambda \geq 0 \tag{7.10}$$

$$\lambda_{18} \geq 0 \tag{7.11}$$

$$\lambda_6 + \lambda_7 \geq 0 \tag{7.12}$$

$$2(\lambda_{19} - \lambda_{21}) \geq -\sqrt{\lambda_{18}(8\lambda_1 + 2\lambda_4 + 4\lambda)} \tag{7.13}$$

$$\frac{1}{2}(\lambda_{22} + \lambda_{23}) \geq -\sqrt{\lambda_{18}(\lambda_6 + \lambda_7)} \tag{7.14}$$

$$2(\lambda_8 - \lambda_{10}) + \Lambda \geq -\sqrt{(8\lambda_1 + 2\lambda_4 + 4\lambda)(\lambda_6 + \lambda_7)} \tag{7.15}$$

Additionally, one of the following must be true:

$$\det A \geq 0 \quad or \tag{7.16}$$

$$2\sqrt{\lambda_6 + \lambda_7}(\lambda_{19} - \lambda_{21}) + \sqrt{8\lambda_1 + 2\lambda_4 + 4\lambda}(\lambda_{22} + \lambda_{23}) + 2\sqrt{\lambda_{18}}(\lambda_8 - \lambda_{10}) \\ + \Lambda\sqrt{\lambda_{18}} + 2\sqrt{(8\lambda_1 + 2\lambda_4 + 4\lambda)\lambda_{18}(\lambda_6 + \lambda_7)} \geq 0 \tag{7.17}$$

CHAPTER 8: Conclusion and Future Work

In this paper, we develop a model of trinification symmetry breaking using a bi-adjoint Higgs field. In the first 5 chapters, we describe the basics of group theory, particle physics, spontaneous symmetry breaking, and trinification.

In Chapter 6, we describe the structure of the bi-octet and find its Standard Model charges and multiplets. We then write the Higgs potential for the bi-octet in tensor form and find its couplings to a bi-triplet and a right-handed triplet.

Our model uses three VEVs to break trinification to the Standard Model: one bi-octet VEV, one bi-triplet VEV, and one VEV in the right-handed triplet. In Chapter 7, with the VEVs in place, we evaluate each Higgs field's covariant derivative to determine which gauge bosons gain mass. This calculation confirms that these three VEVs do indeed break trinification to the Standard Model. Two more VEVs along the diagonal of the bi-triplet then break the Standard Model down to $SU(3)_C \times U(1)_{EM}$, and we calculate the resulting gauge boson masses. We find minimization conditions for the first stage of symmetry breaking (to $SU(3)_C \times SU(2)_L \times U(1)_Y$) and some necessary conditions for the boundedness of the Higgs potential.

Much work has been done to develop this model, but there is more to be done in the future. Two of the most interesting avenues for future work are finding ways that neutrino masses can be generated and determining the phenomenological properties of dark matter candidates. Dark matter can be stabilized in our model due to the preservation of T-parity; the lightest T-odd scalar is the most probable dark matter candidate. Other work that must be done to complete the basic structure of our model includes finding the mass matrices for the Higgs sector and developing the Yukawa sector.

APPENDIX A: Bi-Octet Fields and Charges

Here we list the redefined bi-octet fields that live in Standard Model multiplets, along with their hypercharge and weak isospin values. The electric charges of the fields are shown in superscript in the name of the field.

Table A.1

New Notation	Field Combination	Y	T_3
W_3^3	W_3^3	0	0
W_3^8	W_3^8	0	0
W_8^3	W_8^3	0	0
W_8^8	W_8^8	0	0
$2Y_{45}^0$	$W_4^4 - iW_4^5 - iW_5^4 - W_5^5$	-1	$+\frac{1}{2}$
$2Y_{45}^{0*}$	$W_4^4 + iW_4^5 + iW_5^4 - W_5^5$	$+1$	$-\frac{1}{2}$
$2Y_{45}^{++}$	$W_4^4 - iW_4^5 + iW_5^4 + W_5^5$	$+3$	$+\frac{1}{2}$
$2Y_{45}^{--}$	$W_4^4 + iW_4^5 - iW_5^4 + W_5^5$	-3	$-\frac{1}{2}$
$2Z_{45}^{-}$	$W_6^4 + iW_6^5 + iW_7^4 - W_7^5$	-1	$-\frac{1}{2}$
$2Z_{45}^{+}$	$W_6^4 - iW_6^5 - iW_7^4 - W_7^5$	$+1$	$+\frac{1}{2}$
$2Z_{54}^{+}$	$W_6^4 + iW_6^5 - iW_7^4 + W_7^5$	$+3$	$-\frac{1}{2}$
$2Z_{54}^{-}$	$W_6^4 - iW_6^5 + iW_7^4 + W_7^5$	-3	$+\frac{1}{2}$
$2X_{45}^{++}$	$W_1^4 + iW_1^5 - iW_2^4 + W_2^5$	$+2$	$+1$
$2X_{45}^{--}$	$W_1^4 - iW_1^5 + iW_2^4 + W_2^5$	-2	-1
$2X_{45}^{0}$	$W_1^4 - iW_1^5 - iW_2^4 - W_2^5$	-2	$+1$
$2X_{45}^{0*}$	$W_1^4 + iW_1^5 + iW_2^4 - W_2^5$	$+2$	-1
$2Y_{12}^{0}$	$W_4^1 + iW_4^2 + iW_5^1 - W_5^2$	$+1$	$-\frac{1}{2}$
$2Y_{12}^{0*}$	$W_4^1 - iW_4^2 - iW_5^1 - W_5^2$	-1	$+\frac{1}{2}$
$2Y_{12}^{--}$	$W_4^1 - iW_4^2 + iW_5^1 + W_5^2$	-3	$-\frac{1}{2}$
$2Y_{12}^{++}$	$W_4^1 + iW_4^2 - iW_5^1 + W_5^2$	$+3$	$+\frac{1}{2}$

Table A.1 (*contd.*)

New Notation	Field Combination	Y	T_3
$2X_{12}^{++}$	$W_1^1 + iW_1^2 - iW_2^1 + W_2^2$	$+2$	$+1$
$2X_{12}^{--}$	$W_1^1 - iW_1^2 + iW_2^1 + W_2^2$	-2	-1
$2X_{12}^{0}$	$W_1^1 - iW_1^2 - iW_2^1 - W_2^2$	-2	$+1$
$2X_{12}^{0*}$	$W_1^1 + iW_1^2 + iW_2^1 - W_2^2$	$+2$	-1
$2Z_{12}^{+}$	$W_6^1 + iW_6^2 + iW_7^1 - W_7^2$	$+1$	$+\frac{1}{2}$
$2Z_{12}^{-}$	$W_6^1 - iW_6^2 - iW_7^1 - W_7^2$	-1	$-\frac{1}{2}$
$2Z_{21}^{+}$	$W_6^1 + iW_6^2 - iW_7^1 + W_7^2$	$+3$	$-\frac{1}{2}$
$2Z_{21}^{-}$	$W_6^1 - iW_6^2 + iW_7^1 + W_7^2$	-3	$+\frac{1}{2}$
$\sqrt{2}Z_{67}^{0}$	$W_6^6 + iW_7^6$	-1	$+\frac{1}{2}$
$\sqrt{2}Z_{67}^{0*}$	$W_6^6 - iW_7^6$	$+1$	$-\frac{1}{2}$
$\sqrt{2}Z_{76}^{0}$	$W_7^7 - iW_6^7$	-1	$+\frac{1}{2}$
$\sqrt{2}Z_{76}^{0*}$	$W_7^7 + iW_6^7$	$+1$	$-\frac{1}{2}$
$\sqrt{2}Y_{67}^{-}$	$W_4^6 + iW_5^6$	-1	$-\frac{1}{2}$
$\sqrt{2}Y_{67}^{+}$	$W_4^6 - iW_5^6$	$+1$	$+\frac{1}{2}$
$\sqrt{2}Y_{76}^{-}$	$W_5^7 - iW_4^7$	-1	$-\frac{1}{2}$
$\sqrt{2}Y_{76}^{+}$	$W_5^7 + iW_4^7$	$+1$	$+\frac{1}{2}$
$\sqrt{2}X_{67}^{-}$	$W_1^6 + iW_2^6$	0	-1
$\sqrt{2}X_{67}^{+}$	$W_1^6 - iW_2^6$	0	$+1$
$\sqrt{2}X_{76}^{-}$	$W_2^7 - iW_1^7$	0	-1
$\sqrt{2}X_{76}^{+}$	$W_2^7 + iW_1^7$	0	$+1$
$\sqrt{2}V_{L3}^{-}$	$W_4^3 + iW_5^3$	-1	$-\frac{1}{2}$
$\sqrt{2}V_{L3}^{+}$	$W_4^3 - iW_5^3$	$+1$	$+\frac{1}{2}$
$\sqrt{2}V_{L8}^{-}$	$W_4^8 + iW_5^8$	-1	$-\frac{1}{2}$
$\sqrt{2}V_{L8}^{+}$	$W_4^8 - iW_5^8$	$+1$	$+\frac{1}{2}$
$\sqrt{2}W_{L3}^{-}$	$W_1^3 + iW_2^3$	0	-1
$\sqrt{2}W_{L3}^{+}$	$W_1^3 - iW_2^3$	0	$+1$
$\sqrt{2}W_{L8}^{-}$	$W_1^8 + iW_2^8$	0	-1
$\sqrt{2}W_{L8}^{+}$	$W_1^8 - iW_2^8$	0	$+1$

Table A.1 (*contd.*)

New Notation	Field Combination	Y	T_3
$\sqrt{2}V^+_{R3}$	$W^4_3 + iW^5_3$	$+2$	0
$\sqrt{2}V^-_{R3}$	$W^4_3 - iW^5_3$	-2	0
$\sqrt{2}V^+_{R8}$	$W^4_8 + iW^5_8$	$+2$	0
$\sqrt{2}V^-_{R8}$	$W^4_8 - iW^5_8$	-2	0
$\sqrt{2}U^0_{R3}$	$W^6_3 + iW^7_3$	0	0
$\sqrt{2}U^{0*}_{R3}$	$W^6_3 - iW^7_3$	0	0
$\sqrt{2}U^0_{R8}$	$W^6_8 + iW^7_8$	0	0
$\sqrt{2}U^{0*}_{R8}$	$W^6_8 - iW^7_8$	0	0
$\sqrt{2}W^+_{R3}$	$W^1_3 + iW^2_3$	$+2$	0
$\sqrt{2}W^-_{R3}$	$W^1_3 - iW^2_3$	-2	0
$\sqrt{2}W^+_{R8}$	$W^1_8 + iW^2_8$	$+2$	0
$\sqrt{2}W^-_{R8}$	$W^1_8 - iW^2_8$	-2	0
$\sqrt{2}U^0_{L3}$	$W^3_6 + iW^3_7$	-1	$+\frac{1}{2}$
$\sqrt{2}U^{0*}_{L3}$	$W^3_6 - iW^3_7$	$+1$	$-\frac{1}{2}$
$\sqrt{2}U^0_{L8}$	$W^8_6 + iW^8_7$	-1	$+\frac{1}{2}$
$\sqrt{2}U^{0*}_{L8}$	$W^8_6 - iW^8_7$	$+1$	$-\frac{1}{2}$

APPENDIX B: Bi-Octet Multiplets

Here we list the bi-octet's multiplets under the Standard Model symmetry $SU(2)_L \times U(1)_Y$, organized by hypercharge. Doublets and triplets are organized so that isospin decreases downward (the highest-isospin field is labeled at the top). Singlets are written without parentheses.

- $Y = +3$

$$\begin{pmatrix} Y_{45}^{++} \\ Z_{54}^{+} \end{pmatrix} \quad \begin{pmatrix} Y_{12}^{++} \\ Z_{21}^{+} \end{pmatrix}$$

- $Y = +2$

$$\begin{pmatrix} X_{45}^{++} \\ V_{R3}^{+} \\ X_{45}^{0*} \end{pmatrix} \quad \begin{pmatrix} X_{12}^{++} \\ W_{R3}^{+} \\ X_{12}^{0*} \end{pmatrix}$$

$$V_{R8}^{+} \quad W_{R8}^{+}$$

- $Y = +1$

$$\begin{pmatrix} Z_{45}^{+} \\ Y_{45}^{0*} \end{pmatrix} \quad \begin{pmatrix} Y_{67}^{+} \\ Z_{67}^{0*} \end{pmatrix} \quad \begin{pmatrix} Y_{76}^{+} \\ Z_{76}^{0*} \end{pmatrix} \quad \begin{pmatrix} V_{L3}^{+} \\ U_{L3}^{0*} \end{pmatrix} \quad \begin{pmatrix} V_{L8}^{+} \\ U_{L8}^{0*} \end{pmatrix} \quad \begin{pmatrix} Z_{12}^{+} \\ Y_{12}^{0} \end{pmatrix}$$

- $Y = 0$

$$\begin{pmatrix} X_{67}^{+} \\ U_{R3}^{0} \\ X_{76}^{-} \end{pmatrix} \quad \begin{pmatrix} W_{L3}^{+} \\ W_{3}^{3} \\ W_{L3}^{-} \end{pmatrix} \quad \begin{pmatrix} W_{L8}^{+} \\ W_{3}^{8} \\ W_{L8}^{-} \end{pmatrix}$$

$$W_8^3 \quad W_8^8 \quad U_{R8}^0$$

These multiplets fully determine the structure of the bi-octet fields once the symmetry has broken to the Standard Model. The remaining 28 fields are simply the

Hermitian conjugates of the above multiplets. For convenience, these remaining multiplets are listed below.

- $Y = -3$

$$\begin{pmatrix} Z_{54}^{-} \\ Y_{45}^{--} \end{pmatrix} \begin{pmatrix} Z_{21}^{-} \\ Y_{12}^{--} \end{pmatrix}$$

- $Y = -2$

$$\begin{pmatrix} X_{45}^{0} \\ V_{R3}^{-} \\ X_{45}^{--} \end{pmatrix} \begin{pmatrix} X_{12}^{0} \\ W_{R3}^{-} \\ X_{12}^{--} \end{pmatrix}$$

$$V_{R8}^{-} \quad W_{R8}^{-}$$

- $Y = -1$

$$\begin{pmatrix} Y_{45}^{0} \\ Z_{45}^{-} \end{pmatrix} \begin{pmatrix} Z_{67}^{0} \\ Y_{67}^{-} \end{pmatrix} \begin{pmatrix} Z_{76}^{0} \\ Y_{76}^{-} \end{pmatrix} \begin{pmatrix} U_{L3}^{0} \\ V_{L3}^{-} \end{pmatrix} \begin{pmatrix} U_{L8}^{0} \\ V_{L8}^{-} \end{pmatrix} \begin{pmatrix} Y_{12}^{0*} \\ Z_{12}^{-} \end{pmatrix}$$

- $Y = 0$

$$\begin{pmatrix} X_{76}^{+} \\ U_{R3}^{0*} \\ X_{67}^{-} \end{pmatrix}$$

$$U_{R8}^{0*}$$

APPENDIX C: Gauge Boson Mass Matrices

Using the VEVs in Equation 7.1, we calculate the mass-squared matrices of the gauge bosons (including breaking of electroweak symmetry). Although the entries of these matrices are technically squared masses, we will simply refer to them as mass matrices.

Carrying out the summations in Equation 7.2, Equation 7.3, and Equation 7.4 gives us four mass matrices. Two matrices are uncharged and have opposite T-parity (so the fields do not mix); the other two are charged and have opposite T-parity. Lorentz indices for the fields are suppressed.

The charged T-even gauge bosons W_L^+, W_L^-, W_R^+, and W_R^- form a 2×2 mass matrix. Using the basis (W_L^+, W_R^-), we obtain

$$M_{W^\pm}^2 = \begin{pmatrix} \frac{g_L^2 v_{dn}^2}{2} + \frac{g_L^2 v_{un}^2}{2} & g_L g_R v_{dn} v_{un} \\ g_L g_R v_{dn} v_{un} & 8 g_R^2 v_\psi^2 + \frac{g_R^2 v_{dn}^2}{2} + \frac{g_R^2 v_{un}^2}{2} \end{pmatrix} \tag{C.1}$$

Clearly, v_ψ is responsible for giving mass to the $W_R^\pm$ fields, breaking $SU(3)_R$ at energies far above the electroweak scale.

The four T-odd charged gauge bosons V_L^+, V_L^-, W_R^+, and W_R^- form another 2×2 mass matrix; in the basis (V_L^+, V_R^-) this gives

$$M_{V^\pm}^2 = \begin{pmatrix} 6 g_L^2 v_\psi^2 + \frac{g_L^2 v_n^2}{2} + \frac{g_L^2 v_{un}^2}{2} & g_L g_R v_n v_{un} \\ g_L g_R v_n v_{un} & 2 g_R^2 v_\psi^2 + \frac{9 g_R^2 v_\chi^2}{2} + \frac{g_R^2 v_n^2}{2} + \frac{g_R^2 v_{un}^2}{2} \end{pmatrix} \tag{C.2}$$

Next are the four T-odd uncharged gauge bosons: V_L^0, V_L^{0*}, V_R^0, and V_R^{0*}. We write their mass matrix in the basis (V_L^0, V_R^0), giving us

$$M_{V^{0,0*}}^2 = \begin{pmatrix} 6 g_L^2 v_\psi^2 + \frac{g_L^2 v_{dn}^2}{2} + \frac{g_L^2 v_n^2}{2} & g_L g_R v_n v_{dn} \\ g_L g_R v_n v_{dn} & 2 g_R^2 v_\psi^2 + \frac{9 g_R^2 v_\chi^2}{2} + \frac{g_R^2 v_{dn}^2}{2} + \frac{g_R^2 v_n^2}{2} \end{pmatrix} \tag{C.3}$$

The four T-even neutral gauge bosons form the final matrix, this time 4×4 due to mixing between all the fields. Writing this matrix in the basis (W_L^8, W_R^8, W_R^3, and W_L^3) gives us the mass-squared matrix

$$\begin{pmatrix} \frac{g_L^2 v_{dn}^2}{12}+\frac{g_L^2 v_n^2}{3}+\frac{g_L^2 v_{un}^2}{12} & \frac{g_L g_R v_{dn}^2}{12}+\frac{g_L g_R v_n^2}{3}+\frac{g_L g_R v_{un}^2}{12} & -\frac{\sqrt{3} g_L g_R v_{dn}^2}{12}+\frac{\sqrt{3} g_L g_R v_{un}^2}{12} & \frac{g_L^2 v_{dn}^2}{4}+\frac{g_L^2 v_{un}^2}{4} \\ \frac{g_L g_R v_{dn}^2}{12}+\frac{g_L g_R v_n^2}{3}+\frac{g_L g_R v_{un}^2}{12} & 3g_R^2 v_\chi^2+\frac{g_R^2 v_{dn}^2}{12}+\frac{g_R^2 v_n^2}{3}+\frac{g_R^2 v_{un}^2}{12} & -\frac{\sqrt{3} g_R^2 v_{dn}^2}{12}+\frac{\sqrt{3} g_R^2 v_{un}^2}{12} & \frac{g_L^2 v_{dn}^2}{4}+\frac{g_L^2 v_{un}^2}{4} \\ -\frac{\sqrt{3} g_L g_R v_{dn}^2}{12}+\frac{\sqrt{3} g_L g_R v_{un}^2}{12} & -\frac{\sqrt{3} g_R^2 v_{dn}^2}{12}+\frac{\sqrt{3} g_R^2 v_{un}^2}{12} & \frac{g_R^2 v_{dn}^2}{4}+\frac{g_R^2 v_{un}^2}{4} & \frac{g_L^2 v_{dn}^2}{4}+\frac{g_L^2 v_{un}^2}{4} \\ -\frac{\sqrt{3} g_L^2 v_{dn}^2}{12}+\frac{\sqrt{3} g_L^2 v_{un}^2}{12} & -\frac{\sqrt{3} g_L g_R v_{dn}^2}{12}+\frac{\sqrt{3} g_L g_R v_{un}^2}{12} & \frac{g_L g_R v_{dn}^2}{4}+\frac{g_L g_R v_{un}^2}{4} & \frac{g_L^2 v_{dn}^2}{4}+\frac{g_L^2 v_{un}^2}{4} \end{pmatrix} \tag{C.4}$$

Plugging in benchmark values for the couplings and VEVs demonstrates that this matrix has 1 massless field (corresponding to the photon) and 3 massive fields, as desired. Solving for the precise eigenvalues and eigenvectors to get the physical masses and field combinations is left to future work.

APPENDIX C: Bibliography

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