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Mapping the Tesseral Field of Saturn

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Mapping the Tesseral Field of Saturn

Abstract
Saturn's rotation rate is still uncertain, and while it is theorized to exhibit differential rotation much like Jupiter, this claim is somewhat disputed. By analyzing the properties of waves in the ring system of Saturn as measured by the Cassini spacecraft, we aim to provide observational evidence of this phenomenon. The results of wave analysis allow us to characterize the perturbers responsible for the production of these waves, which are believed to be mass anomalies in the interior of Saturn itself. By calculating the masses of these anomalies and attempting to pinpoint their locations inside of the planet, we provide strong support for the presence of internal mass anomalies and for what is known about Saturn's differential rotation.

Keywords
saturn, resonances, planetary science, solar system, celestial mechanics

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Mapping the Tesseral Field of Saturn

Aurora Hiveley

Introduction

The rotation rate of Saturn is poorly constrained, although Saturn’s day is estimated to be about 10.5 Earth hours in duration, which is equivalent to a rotation rate of around $830^\circ$/day. These estimates vary due to discrepancies in observational methods and model construction. However, it has been theorized and recently proven that Saturn, much like Jupiter, exhibits differential rotation. In simpler terms, Saturn has internal layers much like Earth has a core, mantle, and crust. Since Saturn is gaseous, its layers actually rotate at different speeds such that inside the “differential rotation region,” individual particles rotate variably. In this work, we seek to prove the claim that Saturn rotates differentially using an alternative method: using observational evidence extracted from a study of Saturn’s rings.

The Ring System

Saturn’s radius is about 60,630 km, and its ring system extends from about 75,000 to about 140,000 km from the center of the planet. The ring system of Saturn can be thought of as a fluid, made up of individual particles of dust or other matter which are free to move about each other. Depending upon the density and opacity of this fluid, the rings may be subdivided into smaller regions. The C ring presents itself as ideal for investigation since it has the perfect balance of enough matter to present measurable motions but not so much matter that the motions are damped by friction or hidden by a high opacity.

Figure 1: Saturn’s rings, imaged by the Cassini spacecraft. From left to right, we have the D ring, C ring, B ring, Cassini Division, and the A ring.
Saturn has 82 moons, and as each of these moons orbits, they exert a gravitational force on the planet. However, the gravitational force also acts on the individual particles of the ring system, and this effect can create waves in the body of the ring by expanding and compressing the ring material. This is similar to the way that Earth’s moon creates tidal forces on Earth as the freely moving fluids on the Earth’s surface are attracted by gravity towards the side of the planet which is closest to the moon, creating high and low tides.

For Saturn, this phenomena becomes complicated by the presence of not only one, but 80+ moons. In fact, as each moon orbits Saturn at its own orbital speed and period, occasionally multiple moons come into alignment such that if an observer stood on the surface of the planet and looked straight outward, one or more moons would eclipse each other as they align along the observer’s line of sight. In this case, the gravitational force exerted by the moons is amplified in the direction of the observer’s line of sight since the combined effects of each’s individual gravity creates a much stronger holistic force. This alignment of one or more moons is called a resonance, which is characterized by the periods of the individual moons. For example, if Mimas fully orbits Saturn 3 times in the same amount of time that Enceladus orbits 2 times (this is not the case, but if we assume that it is) then they would create a 3 : 2 resonance since this is the integer ratio of their periods.

When these resonances occur, the waves created in the body of the ring have high enough amplitude to be measured by a satellite like Cassini. When the ring’s optical depth is measured, astronomers can determine the position and amplitude of the waves created and use this information to determine the type of resonance creating the wave as well as identify which moons are most likely responsible. However, astronomers have identified waves in the ring which cannot be attributed to any known moons. In fact, the orbital periods responsible for the creation of these resonances and subsequent waves are all very close to 10 hours. This indicates that one or more unknown massive bodies on the order of a moon’s mass, which are rotating at the same approximate speed of Saturn, are producing effects in the ring much like the moons do. The conclusion that this leads us to is that moon-like objects exist inside of Saturn, and we will henceforth refer to those objects as mass anomalies.

These mass anomalies must have masses on an order of magnitude similar to Saturn’s moons in order to create perturbations in the ring, however their exact nature is unknown. The most likely explanation is that they are storms or vortices with long lifespans, similar to the Great Red Spot on Jupiter’s surface. These storms are large concentrations of turbulent particles forming a dense mass, so it is possible for such an agglomeration to produce the effects observed in the ring. In fact, ring observations provide a useful window to study the interior of Saturn through since the planet itself is largely opaque and has a low optical depth, making it difficult to image the planet from afar with a satellite like Cassini. Gravitational effects and interactions with nearby objects are thus one of our best resources to extract science from the interior of the planet.

There are currently 6 waves in the ring which have been identified as likely having been produced by mass anomalies. Our task is to characterize the anomalies in an effort to prove that there are masses rotating at different rates inside of the planet. This conclusion would provide concrete evidence of Saturn’s differential rotation based upon observations of the rings since we would prove that there are regions or masses which rotate differentially on the interior of Saturn.

Density Waves and Bending Waves

In order to analyze the ring’s waves measured by Cassini inside the ring, we must understand what these waves are and how they are being produced. The waves in the ring take on one of two types: density waves or bending waves. These waves originate at the position of resonance and have measurable properties which will be of use to us in our analysis.

Density waves are produced by a mass, and they manifest as accordion-like compressions and expansions of ring material in the radial direction. If you imagine laying a piece of fabric out on a table, taping one end down, and pushing the free end towards the taped end, you would see ripples forming in the fabric as the material is compressed, causing peaks and valleys to form in the vertical direction perpendicular to the plane. The same idea is responsible for the formation of density waves as portions of the ring are stretched or compressed due to gravitational interactions. These peaks and valleys are also responsible for the striped appearance of Saturn’s rings in photographs, as the brighter parts are peaks while the darker parts are valleys further from the camera. These waves propagate outward.
radially, and their amplitude is related only to the size of the mass creating them since it is the force of gravity which is “rippling the fabric”, so to speak. Density waves originate at the position of what is called a Lindblad resonance, which is calculated using the laws of orbital mechanics detailed in the data analysis section. Density waves can be thought of as a representation of irregularities in the density of ring particles in the radial direction.

Figure 2: By zooming in on a subdivision of the ring, the accordion-like ripples created by density and bending waves are made visible. The density/thickness of the bands is related to the density wave parameters, while the differences in brightness between the light and dark bands suggests the vertical dispersion characteristic of bending waves. (NASA/JPL)

Bending waves, on the other hand, propagate inwards from the location of a vertical resonance, which is calculated alongside Lindblad resonances in a later section. Bending waves manifest as vertical perturbations in the particles of the ring. Ring particles experience a “forced inclination” caused by a gravitational force directed perpendicularly to the plane of the ring, causing the density of the particles in the vertical direction to vary. This effect may be seen in Figure 2 as the brightness of the peaks is not constant. Certain stripes, or wave peaks, are brighter and thus higher in amplitude than others, so we see that bending waves interact with density waves to create waves with variable wavelengths and amplitudes.

It shouldn’t come as a surprise that if there is a component of the gravitational force which is not perfectly parallel with the ring, then the gravitational perturber (be it a moon or an anomaly inside of the planet) must lay outside of the plane of the ring. In this case, the perturber has an inclination, and the angle $I$ which it makes with the plane of the ring is the angle of inclination of the object. Since bending waves are formed only when there is a vertical component of the gravitational force exerted by a perturber, we conclude that bending waves are created only by perturbers with nonzero inclination, and in fact
the amplitude of the produced bending waves depends upon the magnitude of the angle $I$. Luckily, most perturbers have nonzero inclination, so most will produce bending waves.

It is important to note that Lindblad and vertical resonances, and thus also density and bending waves, exist independently of each other. A single perturber will produce both types of waves assuming that it has mass and a nonzero inclination. The precise positioning of these resonances varies slightly, but they are often quite close together in the ring, and since the waves produced by a perturber propagate from the resonance location, the positions of the waves in the ring may also aid us in our analysis of waves produced by mass anomalies.

Methods

Resonance Calculations

We are tasked with extracting the mass and approximate location of each mass anomaly inside of the planet using the properties of the density and bending waves which have been observed in the ring. In order to do this, we first begin by determining what type of resonance we are looking at according to its integer ratio. We start with an alternative form of Kepler’s third law, which yields a relationship between the gravitational force exerted on a perturber and the perturber’s orbital parameters $a$ and $n$:

$$GM = a^3n^2$$

In this expression, $M$ represents the mass of the parent body (Saturn) and $G$ is Newton’s gravitation constant. Furthermore, $a$ is the length of the semimajor axis of the perturber’s orbit, or the greatest distance between the center of a perturber’s orbit and its location around the parent body. Additionally, $n$ describes the mean motion of the orbiter, or the average orbital speed in radians/second over the course of one full orbit. Since we are concerned with waves produced from resonances, however, we are most primarily concerned with the ratio of two different orbiters’ periods, which can also be related to the mean motion by the fact that all orbiters will traverse $2\pi$ radians in the course of one period. We then have the following relationship for determining the type of resonance produced:

$$\frac{n_1}{n_2} = \text{Resonance Ratio} = \frac{m}{m+1}$$

In this case, $n_1$ and $n_2$ are the mean motions of the two objects in resonance. We make the assumption that the ratio can be written as $\frac{m}{m+1}$ since the strongest resonances (and thus those which produce the waves with the largest and most observable amplitudes) take on this form. So, for example, a 4 : 5 resonance will produce much clearer waves than a 3 : 5 or 4 : 6 resonance, although these other resonances may also exist. Furthermore, it is important to note that there is no difference between a 4 : 5 or a 5 : 4 resonance since we refer only to a ratio, and the order in which we label two perturbers is arbitrary. Generally, the convention is to refer to the larger number first, so we prefer to reference a 5 : 4 ratio as opposed to a 4 : 5 ratio.

Using Kepler’s third law, and given that the two objects are each orbiting Saturn such that $GM$ is equivalent for both expressions, we can rework this resonance expression to find the relationship between the semimajor axes of the perturbers and this resonance ratio:

$$a_1 \frac{n_1^3}{n_2^2} = a_2 \frac{n_1^3}{n_2^2}$$

$$\frac{a_1^3}{a_2^3} = \left(\frac{n_1}{n_2}\right)^2 = \left(\frac{m+1}{m}\right)^2$$

$$a_1 = a_2 \left(\frac{m+1}{m}\right)^{2/3}$$

In this case, it is important to note that we impose $a_1 > a_2$ since $\frac{m}{m+1} < 1$, so $n_1 < n_2$ as Kepler’s law gives us that $a$ and $n$ are inversely proportional. Again, the labelling of our perturbers is arbitrary, but the mathematics now gives us a stricter condition to distinguish the two bodies.
Now that this expression is settled, we can determine the type of resonance from the orbital parameters of the perturbers. As previously mentioned, we assume that the length of Saturn’s day is about 10.5 hours, and if we assume that an anomaly inside of the planet has around the same orbital speed (it is probably not exact since we are assuming some degree of differential rotation), we say that our anomalies have periods somewhere between 10 and 11 hours. This is equivalent to an average orbital speed somewhere between 785 and 865 degrees/day. After converting this to radians/second and using Kepler’s third law, we extract the semimajor axis of the perturber.

But what is our second perturber? Since the resonances are created inside of the body in the ring, we assume that the other orbiting object is an arbitrary particle at the location of the resonance. So, using the above equation, we can calculate the proper \( a \) of a ring particle, which corresponds to the location of the resonance from the semimajor axis of the anomaly (which, in turn, comes from its rotation rate.) It is important to note that since the ring exists outside of the planet, the semimajor axis of the ring is greater than that of the anomaly, so we are solving for \( a_1 \) here as the larger distance.

For an array of possible rotation rates in the acceptable range, we then calculate the location of the resonance in the ring for a number of possible resonances (integer ratios.) This produces the following plot, which indicates the resonance location along the horizontal axis from a rotation speed on the vertical axis, color coded according to the resonance type. As previously mentioned, the subregion of the ring with the clearest observable data is the C ring, and this region is highlighted in light green on the plot below. From this data, we can see that most of the resonances which fall in the C ring will be 3 : 2 resonances, although there are some 4 : 3 resonances which correspond to objects with higher orbital speeds which do land in the C ring. Based upon the data we have regarding the locations of the pre-identified waves in the ring, we conclude that the resonances we are observing are, in fact, 3 : 2 resonances.

![Figure 3](image-url)

**Figure 3:** From a rotational speed and an integer value for \( m \), the location of the resonance in the ring is calculated and plotted assuming an \( \frac{m+1}{m} \) resonance.

### Amplitude Calculations

Our main goal is to use observed properties of waves in the ring to calculate the physical properties of the mass anomalies. Thus, we want to find the relationship between the properties of the anomaly (chiefly its mass and inclination) which produce ring waves and the properties of those exact waves. Starting very simply, the force at play in the interactions between the anomalies and the ring particles is gravity. We thus invoke Newton’s law of gravitation, and assume that the magnitude of the gravitational force will be proportional to the wave amplitude. Since the gravitational force is responsible for the production of these waves, this isn’t unreasonable.

\[
A \propto F_G = \frac{GMm}{r^2} \approx \frac{GM}{r^2}
\]
Note that we drop $m$ since the mass of a ring particle $m$ will be much, much smaller than the mass of an anomaly $M$ since we expect $M$ to be akin to the mass of a moon in order for waves like the ones we’ve observed to be produced. This gravitational force can be broken into two smaller components: 1.) Keplerian and 2.) resonant terms.

The Keplerian component is the portion of the gravitational force which is dedicated to keeping objects in their (mostly static) orbits. This gravitational force must be equal to the centripetal force acting on a rotating object in order for it to stay on a circular path without flying into free space or spiraling inwards to collide with the parent body. In contrast, the resonant component of the gravitational force is the force that creates these resonances, permitting the ring particles to experience perturbations radially and vertically in order to create density and bending waves. Thus, the resonant term is really the part of the gravitational force which we are most interested in since it is what produces the waves which we observe.

Extracting the individual components from the total of $F_G$ involves deriving the disturbing function \[ \Pi \] (so called because it yields an expression for the resonant term, which disturbs ring particles.) This is a daunting task seeing as we must determine the value of $r$ in order to determine $M$, which is our end goal. However, $r$ is actually the distance between the ring particle and the anomaly, which is constantly changing as the anomaly rotates through the inside of the planet and as the ring particles orbit through the body of the ring. We therefore must define a set of changing angles which will help us define this distance. These angles include $\lambda$, which represents the longitude, or the current position of the orbiter in its orbit with respect to a fixed reference point. This fixed reference point is often the angle $\omega$ of pericenter, which is the point in the orbiter’s orbit where it is closest to the parent body (the planet, in our case). The diagram below showcases how the distance, renamed $\Delta$ for clarity, is constructed from the individual radii of the anomaly and ring particle with respect to the center of the planet.

**Figure 4:** Top down views of the distance vectors over which the gravitational force acts for two variations. **Left:** A moon at a distance $r_s$ acts on a ring particle at a distance $r$ via the vector $\Delta$. **Right:** An interior mass anomaly at a $r_a$ acts on that same ring particle over a reoriented $\Delta$.

However, this only considers two dimensions, which helps us analyze radially acting density wave amplitudes but leaves us high and dry with bending waves. Luckily, most of the derivation is the same for bending waves as for density waves, with the big difference being that a third dimension must be added to our angles from before in order to account for the inclination of an anomaly. An image of this more complex formulation follows.
A mass anomaly at a distance \( r_a \) from the center of the planet acts on a ring particle at a distance \( r \). The anomaly is now inclined at an angle \( i \) which, with the angle \( \phi \) in the equatorial plane, reorients \( \Delta \) to accommodate the 3-D geometry of the bending wave.

Ultimately, the expressions for the amplitudes of the density waves and bending waves are found to be the following.

\[
A_{DW} = \frac{M}{2\pi\sigma_0^{3/2}R_S^2} \cdot \frac{r - r_T}{r_T^3} \cdot \left((m + 1) + \frac{\alpha}{d} d\right)b_{1/2}^{m+1}(\alpha)
\]

\[
A_{BW} = \frac{M \sin I}{2\pi\sigma_0^{3/2}r_T R_S^2} \cdot \frac{r - r_T}{r_T} \cdot b_{3/2}^{m}(\alpha)
\]

In these expressions, \( M \) is the mass of the anomaly and \( I \) is its inclination. \( A \) is the dimensionless amplitude of the appropriate wave, either density (subscripted DW) or bending (subscripted BW), and \( r \) is the radial location in the ring at which this amplitude is maximal. In contrast, \( r_T \) is the location where the resonance theoretically occurs, computed by the method detailed in the following section. Our resonance type is determined by \( m \), which we set equal to 3 in order to mathematically represent 3 : 2 resonances. The mass and radius of Saturn are \( M_S \) and \( R_S \), respectively, and \( \sigma_0 \) is the surface density of particles in the ring, which we take to be 20 g/cm\(^2\) for the C ring.[4] The variable \( \alpha \) is the quotient of \( r_T \) and \( R_S \), which is then used by the function \( b \) to compute a Laplace coefficient indexed by the type of resonance \( m \).[2]

**Data Analysis**

With the equations from the previous section in hand, we can use the properties of the waves previously identified in the ring to extract physical properties of the mass anomalies. The observed wave parameters include the radial position, rotational speed, peak amplitude, and background amplitude. The results of these observations are summarized in the table below, and depicted as corresponding pairs in the following plot. Note, as mentioned before, that a single anomaly will produce both types of waves assuming that it has nonzero inclination, so the waves are grouped in pairs produced by individual anomalies.

The mass expressions derived in the previous section relied explicitly on the radial position of the wave in the ring (\( r \)), and the peak amplitude takes the place of the maximum amplitude, but beyond that things become a little bit more tricky. In order to derive the location of the resonance in the ring, we return to our initial theory and calculate it from a variation on Kepler’s third law which accounts for additional complexities in the orbit, including the aforementioned angles \( \lambda \) and \( \omega \), as well as corrections for the oblateness (imperfect spherical shape) of Saturn.[2], [7] It is also important to note that unlike before, the methods for calculating the locations of the Lindblad and vertical resonances differ since the geometry of the system differs in the case of whether or not the inclination must be taken into consideration. These values are included in Table 1 alongside the radius of peak amplitude for the sake of comparison. Notably, these two locations are quite close together in the ring, but they are not necessarily identical.
Figure 6: Density and bending waves locations from Cassini data are plotted against the pattern speed of the perturber producing the wave. The wave type is indicated by the datapoint shape, while the color indicates pattern speed (and thus also the perturbing anomaly.)

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Wave Pattern Speed (deg/day)</th>
<th>Radius (km)</th>
<th>Resonance Location (km)</th>
<th>Peak A</th>
<th>Background A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>804.5</td>
<td>86812</td>
<td>86814</td>
<td>0.235</td>
<td>0.086</td>
</tr>
<tr>
<td>Density</td>
<td>804.9</td>
<td>86061</td>
<td>86063</td>
<td>0.032</td>
<td>0.022</td>
</tr>
<tr>
<td>Density</td>
<td>812.6</td>
<td>86248</td>
<td>86246</td>
<td>0.087</td>
<td>0.057</td>
</tr>
<tr>
<td>Density</td>
<td>812.7</td>
<td>85510</td>
<td>85511</td>
<td>0.164</td>
<td>0.030</td>
</tr>
<tr>
<td>Density</td>
<td>813.5</td>
<td>86182</td>
<td>86163</td>
<td>0.115</td>
<td>0.065</td>
</tr>
<tr>
<td>Density</td>
<td>813.6</td>
<td>85445</td>
<td>85448</td>
<td>0.108</td>
<td>0.025</td>
</tr>
<tr>
<td>Density</td>
<td>832.9</td>
<td>84863</td>
<td>84864</td>
<td>0.279</td>
<td>0.119</td>
</tr>
<tr>
<td>Density</td>
<td>832.5</td>
<td>84148</td>
<td>84149</td>
<td>0.036</td>
<td>0.019</td>
</tr>
<tr>
<td>Density</td>
<td>833.4</td>
<td>84828</td>
<td>84830</td>
<td>0.436</td>
<td>0.147</td>
</tr>
<tr>
<td>Density</td>
<td>833.7</td>
<td>84067</td>
<td>84068</td>
<td>0.031</td>
<td>0.019</td>
</tr>
<tr>
<td>Density</td>
<td>842.0</td>
<td>84262</td>
<td>84262</td>
<td>0.126</td>
<td>0.052</td>
</tr>
<tr>
<td>Density</td>
<td>842.0</td>
<td>83510</td>
<td>83515</td>
<td>0.108</td>
<td>0.020</td>
</tr>
</tbody>
</table>

Table 1: Density and bending wave parameters from Cassini observations. The wave pattern speed is equivalent to the rotation rate of the anomaly, while the radius is the location where the amplitude of the wave is at its maximum. We also include the calculated resonance location for the sake of comparison.

As discussed in the previous section, the mass will be calculated from the density wave’s properties while the inclination will be calculated from the bending wave’s. In order to calculate the approximate height of the anomaly off of the equatorial plane, we make an assumption about the inner region of the planet in which all of the anomalies live. Iess, et al. (2019) asserts that the region of Saturn which experiences irregular rotation is the outermost region of the planet bounded approximately by \( \frac{11}{12} R_S \), with an uncertainty of \( \frac{1}{12} R_S \). Since we believe that the anomalies reside in the region of differential rotation as a result of their irregular rotation rates, we can make the approximation that the anomalies are located in the middle of this region at \( \approx \frac{11}{12} R_S \). From this radius and the calculated inclination, we use simple trigonometry to extract the approximate height of each anomaly. It is, however, important to note that we have no way of determining whether the inclination angle is positive or negative; we can only determine its magnitude. Thus, it is unclear whether the anomalies reside in the upper or lower hemispheres of the planet, so while we present
our results with the implication that all are in the upper hemisphere, we do not know this for sure.

The uncertainty in the height brings up an important question about the uncertainties in our masses and inclinations. In our observed data, the only quantity which we measured uncertainty for was the amplitude, and it was expressed in the form of the background amplitude. While the peak amplitude was explicitly given, the background amplitude implies that this peak could be shifted up or down by as much as the size of the background, making the background amplitude equal to $\delta a$. We then propagate this error from the density waves to determine $\delta m$, and propagate $\delta m$ with $\delta a$ of bending waves to determine $\delta I$. This $\delta I$ is also propagated with $\delta R = \frac{1}{17} R_S$ to determine the uncertainty in the height, $\delta h$.

**Results & Conclusions**

The results of our wave analysis and error propagation are summarized in the table below.

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>$A$</th>
<th>Mass (kg)</th>
<th>Inclination ($^\circ$)</th>
<th>Height (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>$0.235 \pm 0.086$</td>
<td>$(6.47 \pm 2.37) \times 10^{15}$</td>
<td>$3.19 \pm 0.14$</td>
<td>$3079 \pm 2415$</td>
</tr>
<tr>
<td>Bending</td>
<td>$0.032 \pm 0.022$</td>
<td>$(3.05 \pm 2.00) \times 10^{15}$</td>
<td>$60.06 \pm 58.63$</td>
<td>$47920 \pm 32890$</td>
</tr>
<tr>
<td>Density</td>
<td>$0.087 \pm 0.057$</td>
<td>$(4.00 \pm 2.26) \times 10^{15}$</td>
<td>$10.77 \pm 1.244$</td>
<td>$10340 \pm 6383$</td>
</tr>
<tr>
<td>Bending</td>
<td>$0.164 \pm 0.030$</td>
<td>$(2.06 \pm 0.88) \times 10^{16}$</td>
<td>$1.25 \pm 0.018$</td>
<td>$1207 \pm 826$</td>
</tr>
<tr>
<td>Density</td>
<td>$0.115 \pm 0.065$</td>
<td>$(9.65 \pm 3.25) \times 10^{15}$</td>
<td>$2.06 \pm 0.052$</td>
<td>$1985 \pm 1400$</td>
</tr>
<tr>
<td>Bending</td>
<td>$0.108 \pm 0.025$</td>
<td>$(2.37 \pm 0.98) \times 10^{17}$</td>
<td>$(8.2 \pm 0.0053) \times 10^{-2}$</td>
<td>$79.18 \pm 36.54$</td>
</tr>
</tbody>
</table>

Table 2: For each wave pair, the calculated anomaly parameters are shown with their uncertainties. This includes the mass, inclination, and height, alongside the amplitudes and amplitude uncertainties for each wave type.

Interesting to note is that the orders of magnitudes of the anomalies are consistent with our expectations of a reasonably sized mass anomaly. The largest moons of Saturn have masses on the order of $10^{23} - 10^{25}$ kg, however a mass this large inside of the planet likely wouldn’t be detected by some other method before now (for example, the Great Red Spot on Jupiter is an instance of a very large and visible mass anomaly which was detectable with primitive telescopes.) Thus, the calculated masses in the range of $10^{15} - 10^{17}$ kg are realistic for the astronomical context but still large enough to produce the waves which have been observed.

Additionally, most of the calculated inclinations are quite small, sitting at $10^\circ$ or less. The reason for this is unclear. It could be the result of a small sample size, or it could be because of some physical phenomena which we do not yet understand. It seems unrealistic that so many anomalies would be clustered very close to the equator, but perhaps there is some reason why they are more numerous or why we observe the bending waves they produce more readily. This would be a prudent point of further observation.

In conclusion, the density and bending wave data from Saturn’s rings has been used to reverse engineer the parameters of the perturbers responsible for their creation, which we know to be mass anomalies inside of Saturn rather than moons on the exterior. Since each of these wave pairs has a corresponding pattern speed $\Omega$, which is also the orbital speed of the anomaly inside of the planet, we have proven that there are mass anomalies, with the physical characteristics determined in this paper, which rotate at variable rates on the interior of Saturn. Through this method, we provide proof of Saturn’s differential rotation from observational evidence.
Future Work

As previously mentioned, the examination of why we have observed so many minimally inclined anomalies would be a useful point of further study. Additionally, to more precisely pinpoint the locations of the anomalies within the planet, we would also need to calculate the phase of each anomaly from some reference point/angle. This would be similar to determining the longitude of a location on Earth, as thus far we have only determined height/latitude.

The reader may also recall that the resonances observed in the data set analyzed in this paper were 3 : 2 resonances located in the C ring. The sample size question broached before suggests that there may be other waves of lower amplitude which were missed in previous data analysis. However, there may also be 4 : 3 resonances which reside in the C ring, and they may have been overlooked before since they are located close to the outer limit of the C ring. And of course, there are other resonances in other subregions of the ring which we have not observed due to observational difficulties encountered in regions with more or less opacity/density than the ideal C ring. These resonances may have been produced by other mass anomalies which have not yet been detected and which may be able to shed further light on the internal structure of the planet.

These questions are of interest to us since not only can it help prove Saturn’s differential rotation with observational evidence, but the answers may help us better understand planetary formation physics. Eventually, astronomers may extend this knowledge to better understand the initial formation of our solar system and universe, as well as the possibility of life outside of Earth.

References


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