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Neutrino Oscillations in the Presence of a Magnetic Field

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Abstract
We calculate oscillation probabilities in the presence of an external magnetic field in a one-generation neutrino framework that includes both Majorana and Dirac mass terms. First, we write down the Euler-Lagrange equations and obtain a system of eight differential equations coupling together eight different neutrino states that can be distinguished by helicity, chirality, and particle/antiparticle-ness. We then solve this system of differential equations in various special cases, exhibiting different types of oscillations. When the magnetic field is in the direction of momentum, there are only four oscillation channels as helicity flip is forbidden. We observe that chirality flips are suppressed by a factor of m^2/E^2, whereas the transitions involving active neutrinos and sterile antineutrinos are not while having a form similar to two-generation flavor oscillations.

Keywords
Neutrino Oscillations, Chiral Oscillations, Spin Oscillations, Neutrino-antineutrino Oscillations, Majorana mass, Dirac mass, CP-violating phase, Sterile neutrinos, Neutrino magnetic moments

Cover Page Footnote
This honor project would not have been possible without the tremendous support and guidance from Professor Tonnis ter Veldhuis. I would also like to express my deepest gratitude towards him for his continued care and invaluable direction during my undergraduate time at Macalester. I would also like to thank my parents, professors, friends, and everyone who have helped me, in some forms or another, over the past four years. Without all these helps, I would not have come this far.

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We calculate oscillation probabilities in the presence of an external magnetic field in a one-generation neutrino framework that includes both Majorana and Dirac mass terms. First, we write down the Euler-Lagrange equations and obtain a system of eight differential equations coupling together eight different neutrino states that can be distinguished by helicity, chirality, and particle/antiparticle-ness. We then solve this system of differential equations in various special cases, exhibiting different types of oscillations. When the magnetic field is in the direction of momentum, there are only four oscillation channels as helicity flip is forbidden. We observe that chirality flips are suppressed by a factor of m^2/E^2, whereas the transitions involving active neutrinos and sterile antineutrinos are not while having a form similar to two-generation flavor oscillations.

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Chapter 1

Introduction

Atmospheric [1] and solar [2] neutrino experiments have indicated that neutrinos change flavors as they propagate, a phenomenon called neutrino flavor oscillation. This provides evidence that neutrinos have non-zero masses, and there is mixing among neutrino mass states and flavour states accompanied by a CP-violating phase. Of particular importance is the CP-violating phase, which can potentially explain the matter/antimatter asymmetry of the universe [3; 4].

By studying neutrino-neutrino oscillations and antineutrino-antineutrino oscillations, current experiments have verified and constrained this CP-violating phase to a great degree [5; 6; 7]. However, there are additional CP-violating phases if neutrinos turn out to be Majorana fermions. Such phases arise only in neutrino-antineutrino oscillations [8], oscillations where neutrinos change into their antiparticles. The probability of such oscillations is small as it is suppressed by the factor of m^2/E^2 [9]. However, it is worth studying, as future experiments might be able to produce neutrino beams with low energy, one example being the Mossbauer electron antineutrino with energy 18.6 keV [10]. Therefore, multiple papers have studied this type of oscillations [9; 11; 12].

The oscillations we have discussed so far are flavor oscillations. There is, however, another type of oscillations called chiral oscillations [14]. In chiral oscillations, we separate the Dirac spinors into their left-handed and right-handed chiral components (also called Weyl spinors) and treat them on their own. This separation makes sense because only left-handed fields are coupled to the W and Z boson [14]. Left-handed Weyl spinors are coupled to right-handed Weyl spinors through the Dirac mass term. As they propagate, left-handed spinors can then change into right-handed spinors or vice versa (see Section 4.6.1). The effect of chiral oscillations is tiny, suppressed by the factor of m^2/E^2. Therefore, we can study chiral oscillations only in low energy conditions (non-relativistic regime). It has been pointed out that cosmic neutrino background ($C\nu B$) provides such conditions and therefore, chiral oscillations have been analyzed in this context [15; 16].

To complicate matters, massive neutrinos have a non-zero magnetic moment [17]. Therefore, in the presence of an external magnetic field, neutrino can flip its spin as well as oscillates between different flavors, which we call a spin-flavor oscillation [18; 19]. It is shown that such oscillations can have important effects on processes within extreme astrophysical environments, for example the creation of neutrinos within supernovae [20].

To sum everything up, there are four things that can change given a Majorana neutrino in the presence of an external magnetic field: its spin, its chirality, its flavour, and its particle-antiparticleness (as characterized by the charged weak interactions). References above have only consider such changes in pairs or alone, because of the complexity of the problem.

References in [21; 22; 23; 24], provide a first attempt at a framework for calculating neutrino–neutrino/antineutrino–antineutrino flavor oscillations, neutrino–antineutrino flavor oscillations, and chiral oscillations in a unified way, by solving the Dirac equation. However, these authors did not consider an external magnetic field where there are spin oscillations as well. Therefore, this paper will extend the same framework to include spin oscillations in the presence of a magnetic field, resulting in flips in chirality and helicity, and change from neutrino to antineutrino all at once. We do not consider flavor change but such addition will follow straightforwardly from our calculations. As mentioned above, to have neutrino-antineutrino oscillations, we require that neutrinos are Majorana particles. To achieve this, in addition to a Dirac mass term, we also add a Majorana mass term that is made up of the right-handed (sterile) neutrinos.

The paper is structured as followed. In Chapter two, I talk about the Lorentz group, its transformation laws, and the spin 1/2 representations. In Chapter three, I write down the chirality operator, the helicity operator, and the charge conjugation matrix. In Chapter four, I write down the Lagrangian and then obtain the Hamiltonian form of the equations of motion that describe the evolution of the eight states a neutrino can take. I then describe what these various states are. Due to the complexity of the problem, I proceed to calculate the oscillation probabilities for some special cases of the problem, each one involving combinations of different type of oscillations.

Chapter 2

Lorentz Invariance

The background knowledge summarized in this chapter is presented here for completeness. It is based on the standard textbooks on Quantum Field theory [25; 26; 27], which readers are encouraged to consult for further clarifications.

2.1 Minkowski Space

A Minkowski space is a 4-dimensionl vector space R^4, representing points in spacetime coordinates, with a inner product given by:

$$\langle (t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \rangle = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2. \tag{2.1}$$

We use units in which the speed of light is $c = 1$ and employ the index notation:

$$x^\mu \equiv \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad \mu = 0, 1, 2, 3. \tag{2.2}$$

Now, we define:

$$x_\mu \equiv (t, -x, -y, -z). \tag{2.3}$$

With this notation, the inner product can be written as:

$$\langle x, y \rangle \equiv xy = x^\mu y_\mu, \tag{2.4}$$

where we have employed the Einstein summation convention, according to which a sum over repeated indices is implicit. For example, the distance between x and the origin in this metric would be given by:

$$x^2 = x^\mu x_\mu = t^2 - x_1^2 - x_2^2 - x_3^2. \tag{2.5}$$

We can raise and lower indices using the Minkowski metric,

$$x^\mu = \eta^{\mu\nu} x_\nu \quad \text{or} \quad x_\mu = \eta_{\mu\nu} x^\nu, \tag{2.6}$$

$$\text{where} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

2.2 Lorentz Group

A Lorentz transformation is a linear transformation Λ acting on x in the Minkowski space such that the distance between x and the origin stays constant after the transformation. We write

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{2.7}$$

where the transformation matrices satify

$$\eta_{\mu\nu} \Lambda^{\mu\rho} \Lambda^{\nu\sigma} = \eta^{\rho\sigma}. \tag{2.8}$$

The set of Minskowski transformations forms the Lorentz group. Note that there are various subgroups within the Lorentz group which can be classified according to

- proper: $det\Lambda = 1$,
- orthochronous: $\Lambda^{0}{}_{0} \geq 1$.

The set of proper and orthochronous transformations, denoted by $SO^{+}(3,1)$, constitute a continuous subgroup of the Lorentz group. Elements outside this subgroup are either improper or non-orthochronous transformations or both. They are mapped to the proper orthochronous transformations by two special discrete spacetime transformations known as parity P and time reversal T. Another discrete transformation of concern is charge conjugation, denoted by C. This transformation interchanges particles with antiparticles and vice versa.

2.3 Lorentz Invariance

A quantity is Lorentz-invariant if it remains constant under a proper-orthochronous Lorentz transformation. By definition of a Lorentz transformation, the norm of a four-vector is constant. In fact, an inner product between any two four-vector is Lorentz invariant:

$$\begin{aligned} x^{\mu} y_{\mu} \to x'^{\mu} y'_{\mu} &= \Lambda^{\mu}{}_{\rho} x^{\rho} \Lambda^{\nu}{}_{\sigma} y^{\sigma} \eta_{\mu\nu} \\ &= x^{\rho} y^{\sigma} \eta_{\rho\sigma} = xy. \end{aligned} \tag{2.9}$$

A scalar that is Lorentz invariant is called a Lorentz scalar, and one way to make a Lorentz scalar is to contract the same upper and lower indices of any objects, which can be four-vectors or tensors, such that there are no free indices. For example, all of the terms below are Lorentz scalars:

$$x^{\mu} x_{\mu}, p^{\mu} p_{\mu}, F^{\mu\nu} F_{\mu\nu},$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor.

The postulate of special relativity states that the laws of physics have the same form in any inertial reference frame. If we take the point of view of an active transformation, this means that if we have an equation that describes a particle or a field, called it ϕ, after applying a continuous Lorentz transformation to that field by rotating it or boosting it, we must still have an equation of the same form. Let D be a differential operator then:

$$D(\phi(x)) = 0 \to D'(\phi'(x')) = 0. \tag{2.10}$$

We say such an equation is Lorentz covariant, which means that the left and right side of the equation transforms in the same way under a Lorentz transformation. Because of the Lagrangian formulation of field theory, an equation is Lorentz covariant if it follows from a Lagrangian density that is a Lorentz scalar. Therefore, our goal is to construct a Lagrangian $\mathcal{L}$ that transforms like a Lorentz scalar.

2.4 Transformation Rules

Now, what are the transformation rules for different objects? Let's consider the scalar field $\phi(x)$ introduced above. We can think of $\phi(x)$ as a value of a some quantity permeated through space. Suppose we rotate or boost all the points in space (as in Eq 2.7), we expect that the transformed field ϕ' at boosted point x will take the same value as the old field ϕ at the old point $\Lambda^{-1}x$. After all, rotating the field does not inherently change the value of the field. Therefore, the transformation rule for the scalar field $\phi(x)$ is given by:

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \tag{2.11}$$

Now, what if we have a vector field $V^\mu(x)$? In this case, not only that the argument will change but the orientation of the field will also change:

$$V^\mu(x) \to \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x). \tag{2.12}$$

If we generalize this to a n-dimensional tensor, $\Phi(x)$, the Lorentz transformation law for such an object is given by:

$$\Phi(x) \to M(\Lambda)\Phi(\Lambda^{-1}x). \tag{2.13}$$

Suppose Λ_1, Λ_2 are two Lorentz transformations. Applying Λ_1 transformation we obtain:

$$\Phi(x) \to M(\Lambda_1)\Phi. \tag{2.14}$$

Now, we apply Λ_2 transformation next:

$$\Phi(x) \to M(\Lambda_2)M(\Lambda_1)\Phi. \tag{2.15}$$

Since $\Lambda_2\Lambda_1$ is still a Lorentz transformation, we can also write:

$$\Phi(x) \to M(\Lambda_2\Lambda_1)\Phi. \tag{2.16}$$

Hence,

$$M(\Lambda_1)M(\Lambda_2) = M(\Lambda_1\Lambda_2). \tag{2.17}$$

This means that $M(\Lambda)$ forms a representation of the Lorentz group. To summarize, to find all the possible transformation rules under a Lorentz transformation, is to find the irreducible representations of the Lorentz group.

2.5 Lorentz Algebra

To find the representations of the Lorentz group[1], we can exponentiate its corresponding representation of the Lie algebra, which is a vector space made up of the corresponding

[1] From here onwards, when we say Lorentz group, we refer to the proper-orthochronous or continuous subgroup $SO^+(3,1)$.

infinitesimal group elements close to the identity. Like representations of the group, representations of the algebra are sets of matrices that preserve the defining characteristics of that algebra, which in this case are the commutation rules. It seems that since there is an infinite number of elements in the algebra, there is an infinite number of rules for them. However, it turns out we can just care about the generators of the algebra, which form a basis of the algebra. Because of the bilinearity of the commutator, the commutation rules just for the generators are sufficient to define the algebra.

The commutation rule for the Lorentz algebra is given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}). \tag{2.18}$$

Note that $J^{\mu\nu}$ is anti-symmetric. We can define the generators of rotation and boosts as:

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{0i}. \tag{2.19}$$

This means we have three rotation generators and three boost generators. Note that this redefinition implies relabeling of our generators, i.e, we relabel J^{12} as J^3.

2.6 Decomposition of the Lorentz Algebra

The Lorentz algebra decomposes into two separate SU(2) algebras. To see this, define

$$\boldsymbol{J}_+ = \frac{1}{2}[\boldsymbol{J} + i\boldsymbol{K}] \quad \wedge \quad \boldsymbol{J}_- = \frac{1}{2}[\boldsymbol{J} - i\boldsymbol{K}]. \tag{2.20}$$

The commutation rules for these newly defined generators are:

$$[J_+^i, J_+^j] = i\epsilon^{ijk}J_+^k, \tag{2.21}$$

$$[J_-^i, J_-^j] = i\epsilon^{ijk}J_-^k, \tag{2.22}$$

$$[J_+^i, J_-^j] = 0. \tag{2.23}$$

Now note that the generators for $SU(2)$, which are the angular momentum operators, also satisfy the same commutation relation as $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$. This means

$$SO(3,1) \cong SU(2) \otimes SU(2). \tag{2.24}$$

Now, why is this remarkable? It turns out that we already have a clear understanding of all the n-dimensional irreducible representation of $SU(2)$, which are labeled by half-integers and integers spin.

2.7 Spin $1/2$ Representation

Consider the spin $1/2$ representation of $SU(2)$. The generators are simply given by the Pauli matrices:

$$\boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}. \tag{2.25}$$

Now, we can make three representations of the Lorentz group out of this spin $1/2$ representation of the $SU(2)$: We label them as

- $(\frac{1}{2}, 0)$: the left-handed spinor representation
- $(0, \frac{1}{2})$: the right-handed spinor representation
- $(\frac{1}{2}, \frac{1}{2})$: the 4-vector representation

The $(\frac{1}{2}, 0)$ representation acts on a two-component objects, which we called left-handed Weyl spinors, denoted by ψ_L. Similarly the $(0, \frac{1}{2})$ representation also acts on two-component objects, which we called a right-handed Weyl spinors, denoted by ψ_R. We can then get the Dirac fermion reducible representation by taking the direct sum of $(\frac{1}{2}, 0)$ representation and $(0, \frac{1}{2})$ representation. This representation therefore acts on a four-component objects, made up of left-handed and right-handed Weyl spinors, which we call Dirac spinors:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \Psi_L + \Psi_R. \tag{2.26}$$

Let,

$$J_+ = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}, \quad J_- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{2.27}$$

Then, we can find J and K, the rotation and boosts generators:

$$J^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad K^i = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}. \tag{2.28}$$

However, if we introduce a set of four 4-dimensional gamma matrices with we can write the generators $J^{\mu\nu}$ in a much more general form: Let,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{2.29}$$

Then the boosts and rotation generators are given by

$$J^{0i} = \frac{i}{4}[\gamma^0, \gamma^i], \quad J^{ij} = \frac{i}{4}[\gamma^i, \gamma^j], \tag{2.30}$$

which in more compact form can be written as

$$J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{2.31}$$

In fact, we can verify $J^{\mu\nu}$ in this form always satisfies the Lorentz algebra if the gamma matrices satisfy the Clifford algebra

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu} \times 1_{n\times n}. \tag{2.32}$$

It turns out that the set of gamma matrices presented above are just one amongst many. However, this particular representation indicates to us very clearly the reducible nature of the Lorentz representation of the Dirac spinor. We call this representation, the chiral representation.

Chapter 3

Chirality, Helicity, and Charge Conjugation

3.1 Chirality

In the previous chapter, we see that the Lorentz group reduces to two copies of the rotation group $SU(2)$. Therefore, we can write a Dirac spinor, a four-component object that transforms under the Lorentz group, as a sum of a left-handed and a right-handed Weyl spinor. Each Weyl spinor, which from now on we will refer to as chiral component, is a four-component object that also transforms separately under the Lorentz group. In this section, we introduce a chiral operator, that teases out these chiral components: We define an additional Gamma matrix that satisfies:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^4. \tag{3.1}$$

In the chiral basis, γ^5 takes the following form:

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Clearly,

- The left-handed chiral component of the Dirac spinor $\Psi_L = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ is an eigenvector of γ^5 with eigenvalue -1.
- Similarly, the right-handed chiral component of the Dirac spinor $\Psi_R = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$ is also an eigenvector of γ^5 with eigenvalue 1.

Therefore, we call the gamma matrix γ^5, the chirality operator. We now refer to Ψ_L and Ψ_R as left-handed and right-handed chiral states, which are eigenvectors of the chirality operator. In fact, we can obtain Ψ_L and Ψ_R from Ψ using a left and right projectors defined as:

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}. \tag{3.3}$$

And with these projectors, we obtain:

$$\Psi_L = P_L\Psi, \quad \Psi_R = P_R\Psi. \tag{3.4}$$

3.2 Helicity

Helicity defines a projection of the spin onto the direction of the momentum. Therefore, we define the helicity operator as:

$$h \equiv \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{3.5}$$

The eigenstates of h with positive eigenvalue $h = \frac{1}{2}$ is called a positive-helicity state, whereas the eigenstates of h with negative eigenvalue $h = -\frac{1}{2}$ is called a negative-helicity state. Note that helicity is not a Lorentz-invariant quantity for massive particles. We can always boost to another frame of reference such that the direction of the momentum is flipped and hence the helicity is also flipped.

3.3 Charge Conjugation

In chapter 2, we focus mainly on continuous transformations of the Lorentz group and only briefly mention its discrete transformations. One of the discrete transformations that is of main concern is the charge conjugation transformation. Charge conjugation is a transformation that changes particles into antiparticles, which we also call charge-conjugated fields ψ_c:

$$C : \psi \to \psi^c = i\gamma^2\psi^*. \tag{3.6}$$

The charge conjugation matrix also satisfies:

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu, \quad C^\dagger = C^{-1}, \quad C^T = -C. \tag{3.7}$$

Charge Conjugated field are defined:

$$\psi^c = C(\overline{\psi})^T. \tag{3.8}$$

In this convention, the charge conjuation matrix is therefore given by:

$$C = -i\gamma^0\gamma^2. \tag{3.9}$$

Chapter 4

Neutrino oscillations

4.1 Charged Current Interactions

In the Standard model, three flavors of active neutrinos interact weakly with the W bosons. The interaction is given by:

$$\mathcal{L}_{cc} = \frac{-g}{\sqrt{2}}[\overline{\nu_{\alpha L}}\gamma^{\mu}\ell_{\alpha L}W_{\mu}^{+} + \overline{\ell_{\alpha L}}\gamma^{\mu}\nu_{\alpha L}W_{\mu}^{-}], \tag{4.1}$$

where $\alpha = e, \mu, \tau$. In this paper, we only consider the one-generation neutrino framework, hence the charged current interaction for this toy model is:

$$\mathcal{L}_{cc} = \frac{-g}{\sqrt{2}}[\overline{\nu_{L}}\gamma^{\mu}\ell_{L}W_{\mu}^{+} + \overline{\ell_{L}}\gamma^{\mu}\nu_{L}W_{\mu}^{-}]. \tag{4.2}$$

4.2 The Lagrangian

4.2.1 Four-component notation

Suppose we have a left-handed neutrino field denoted by ψ_L and a right-handed neutrino field denoted by ψ_R. Consider both the Dirac mass term and the Majorana mass term in the Lagrangian,

$$\mathcal{L} \ni -\overline{\psi_L}m_D\psi_R - \overline{\psi_R}m_D^{*}\psi_L - \frac{1}{2}\overline{(\psi_R)^c}M_R\psi_R - \frac{1}{2}\overline{(\psi_R)}M_R^{*}(\psi_R)^c. \tag{4.3}$$

Now we switch on the external magnetic field. The contribution to the Lagrangian from the magnetic moment, μ is:

$$-\mu\overline{\psi_L}F^{\mu\nu}\sigma_{\mu\nu}\psi_R + h.c, \tag{4.4}$$

where $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]$ and $F^{\mu\nu}$ is the electromagnetic field strength tensor. Now setting the electric field to vanish $F^{0i} = 0$, and since the magnetic field $\vec{B}$ can be written as $B_i = \epsilon_{ijk}F^{jk}$, we can write

$$F^{\mu\nu}\sigma_{\mu\nu} = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} B_i = \vec{\Sigma}.\vec{B}, \tag{4.5}$$

where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{4.6}$$

Therefore, the Lagrangian is given by

$$\mathcal{L} \ni -\overline{\psi_L} m_D \psi_R - \frac{1}{2}\overline{(\psi_R)^c} M_R \psi_R - \mu \overline{\psi_L}(\vec{\Sigma}.\vec{B})\psi_R + h.c. \tag{4.7}$$

Note that in this case, we can't do away with fields redefinition to get rid of all complex phases in our parameters, unlike the cases without the contribution from the magnetic moment.

4.2.2 Two-component notation

We can write the neutrino fields in terms of two-component Weyl spinors:

$$\psi_L = \begin{pmatrix} \eta \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \tag{4.8}$$

Hence, the two-component charge conjugated field for the right-handed Weyl spinor is given by

$$(\psi_R)^c = \begin{pmatrix} -i\sigma_2 \xi^* \\ 0 \end{pmatrix}. \tag{4.9}$$

In two-component notation, the Lagrangian is then given by:

$$\mathcal{L} \ni -m_D \eta^\dagger \xi - \frac{1}{2} i M_R \xi^T \sigma_2 \xi - \mu \eta^\dagger (\vec{\sigma}.\vec{B})\xi + h.c. \tag{4.10}$$

Including the kinetic terms, the total Lagrangian of concern is:

$$\mathcal{L} = i\eta^\dagger \tilde{\sigma}^\mu \partial_\mu \eta + i\xi^\dagger \sigma^\mu \partial_\mu \xi - (m_D \eta^\dagger \xi + \frac{1}{2} i M_R \xi^T \sigma_2 \xi + \mu \eta^\dagger (\vec{\sigma}.\vec{B})\xi + h.c). \tag{4.11}$$

4.3 Equations of motion

Using the Euler-Lagrange method [1], we obtain the following equations of motion:

$$i\tilde{\sigma}^\mu \partial_\mu \eta - m_D \xi - \mu(\vec{\sigma}.\vec{B})\xi = 0, \tag{4.14}$$

$$i\sigma^\mu \partial_\mu \xi - m_D^* \eta - \mu^*(\vec{\sigma}.\vec{B})\eta + iM_R^* \sigma_2 \xi^* = 0. \tag{4.15}$$

We look for a solution of the form:

$$\eta(x,t) = \begin{pmatrix} \nu'_{L1}(t)e^{i\vec{p}\vec{x}} + \nu'_{L2}(t)e^{-i\vec{p}\vec{x}} \\ \nu_{L1}(t)e^{i\vec{p}\vec{x}} + \nu_{L2}(t)e^{-i\vec{p}\vec{x}} \end{pmatrix}, \quad \xi(x,t) = \begin{pmatrix} \nu'_{R1}(t)e^{i\vec{p}\vec{x}} + \nu'_{R2}(t)e^{-i\vec{p}\vec{x}} \\ \nu_{R1}(t)e^{i\vec{p}\vec{x}} + \nu_{R2}(t)e^{-i\vec{p}\vec{x}} \end{pmatrix}. \tag{4.16}$$

Without loss of generality, we take the neutrino to be moving in the z-direction, i.e. $\vec{p} = (0,0,p)$. Hence, we obtain:

[1] Note that spinors must be treated as classical anticommuting fields which means its arguments are Grassmann numbers that satisfy certain properties below:

For any grassman numbers α, β, we have

$$(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^* \tag{4.12}$$

$$\frac{d}{d\alpha}(f(\alpha)g(\alpha)) = \frac{df(\alpha)}{d\alpha)}g(\alpha) - f(\alpha)\frac{dg(\alpha)}{d\alpha} \tag{4.13}$$

- Term $e^{i\vec{p}\vec{x}}$:

$$i\frac{d}{dt}\begin{pmatrix}\nu'_{L1}\\ \nu_{L1}\end{pmatrix} = \begin{pmatrix}-p\nu'_{L1}\\ p\nu_{L1}\end{pmatrix} + (m_D + \mu\vec{\sigma}.\vec{B})\begin{pmatrix}\nu'_{R1}\\ \nu_{R1}\end{pmatrix}, \tag{4.17}$$

$$i\frac{d}{dt}\begin{pmatrix}\nu'_{R1}\\ \nu_{R1}\end{pmatrix} = \begin{pmatrix}p\nu'_{R1}\\ -p\nu_{R1}\end{pmatrix} + (m_D^* + \mu^*\vec{\sigma}.\vec{B})\begin{pmatrix}\nu'_{L1}\\ \nu_{L1}\end{pmatrix} + M_R^*\begin{pmatrix}-\nu_{R2}^*\\ \nu_{R2}'^*\end{pmatrix}. \tag{4.18}$$

- Term $e^{-i\vec{p}\vec{x}}$:

$$i\frac{d}{dt}\begin{pmatrix}\nu'_{L2}\\ \nu_{L2}\end{pmatrix} = \begin{pmatrix}p\nu'_{L2}\\ -p\nu_{L2}\end{pmatrix} + (m_D + \mu\vec{\sigma}.\vec{B})\begin{pmatrix}\nu'_{R2}\\ \nu_{R2}\end{pmatrix}, \tag{4.19}$$

$$i\frac{d}{dt}\begin{pmatrix}\nu'_{R2}\\ \nu_{R2}\end{pmatrix} = \begin{pmatrix}-p\nu'_{R2}\\ p\nu_{R2}\end{pmatrix} + (m_D^* + \mu^*\vec{\sigma}.\vec{B})\begin{pmatrix}\nu'_{L2}\\ \nu_{L2}\end{pmatrix} + M_R^*\begin{pmatrix}-\nu_{R1}^*\\ \nu_{R1}'^*\end{pmatrix}. \tag{4.20}$$

Taking the complex conjugate of the equations above, we obtain a system of 16 linear differential equations. However, this system decouples into two systems of eight linear differential equations. The difference between the two systems lies in the difference in the sign of the momentum p. This means that one set of equations explains the evolution of neutrino moving in the positive z direction, whereas the other set of equations explains neutrino moving in the negative z direction. Since momentum is conserved, it makes sense that the two sets of equations decoupled.

Suppose that $\vec{B} = (B_0, \phi, \theta)$. Then the equations of motion for neutrino moving in the positive z direction are given by:

$$i\frac{d}{dt}\begin{pmatrix}\nu'_{L1}\\ \nu_{L1}\\ \nu'_{R1}\\ \nu_{R1}\\ \nu_{L2}'^*\\ \nu_{L2}^*\\ \nu_{R2}'^*\\ \nu_{R2}^*\end{pmatrix} = \begin{pmatrix} -p & 0 & m_D + \mu B_0\cos\theta & +\mu B_0\sin\theta e^{-i\phi} & 0 & 0 & 0 & 0\\ 0 & p & \mu B_0\sin\theta e^{i\phi} & m_D - \mu B_0\cos\theta & 0 & 0 & 0 & 0\\ m_D^* + \mu^* B_0\cos\theta & \mu^* B_0\sin\theta e^{-i\phi} & p & 0 & 0 & 0 & 0 & -M_R^*\\ \mu^* B_0\sin\theta e^{i\phi} & m_D^* - \mu^* B_0\cos\theta & 0 & -p & 0 & 0 & M_R^* & 0\\ 0 & 0 & 0 & 0 & -p & 0 & -m_D^* - \mu^* B_0\cos\theta & -\mu^* B_0\sin\theta e^{i\phi}\\ 0 & 0 & 0 & 0 & 0 & p & -\mu^* B_0\sin\theta e^{-i\phi} & -m_D^* + \mu^* B_0\cos\theta\\ 0 & 0 & 0 & M_R & -m_D - \mu B_0\cos\theta & -\mu B_0\sin\theta e^{i\phi} & p & 0\\ 0 & 0 & -M_R & 0 & -\mu B_0\sin\theta e^{-i\phi} & -m_D + \mu B_0\cos\theta & 0 & -p \end{pmatrix}\begin{pmatrix}\nu'_{L1}\\ \nu_{L1}\\ \nu'_{R1}\\ \nu_{R1}\\ \nu_{L2}'^*\\ \nu_{L2}^*\\ \nu_{R2}'^*\\ \nu_{R2}^*\end{pmatrix}, \tag{4.21}$$

which can be written in the Hamiltonian form:

$$i\frac{d}{dt}\boldsymbol{\nu} = H\boldsymbol{\nu}. \tag{4.22}$$

Due to the complexity of this system of equations, in this paper, we only study the special cases where we can approach it analytically in a meaningful way:

1. $m_D \neq 0, \quad M_R = 0 \quad \vec{B} = (0,0,0)$.
2. $m_D \neq 0, \quad M_R = 0, \quad \vec{B} = (B_x, 0, 0)$.
3. $m_D \neq 0, \quad M_R \neq 0, \quad \vec{B} = (0,0,0)$.
4. $m_D \neq 0, \quad M_R \neq 0, \quad \vec{B} = (0,0,B_z)$.

First, we will clarify what the different neutrino states are and give an overview of the method we use to calculate the amplitudes of transitions between these different states:

4.4 Categorizing different states

We distinguish between eight states:

$$\nu'_L, \nu_L, \nu'_R, \nu_R, \nu'^*_L, \nu^*_L, \nu'^*_R, \nu^*_R. \tag{4.23}$$

We will show that each of these eight states corresponds to a state of definite helicity (positive and negative), definite chirality (left-handed and right-handed), and definite particle/antiparticle-ness simultaneously.

4.4.1 Chiral states

First consider just the field:

$$\psi = \begin{pmatrix} \nu'_L \\ \nu_L \\ \nu'_R \\ \nu_R \end{pmatrix}. \tag{4.24}$$

The left-handed chiral states are given by the eigenvectors of the γ^5 operator with eigenvalues -1. These are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.25}$$

which correspond to ν'_L and ν_L. Similarly, the right-handed chiral states are given by the eigenvectors of the γ^5 operator with eigenvalues 1. These are

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.26}$$

which correspond to ν'_R and ν_R.

4.4.2 Helicity States

The helicity operator for p in the z-direction is given by:

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.27}$$

The positive eigenstates are the eigenvectors of h with eigenvalues $1/2$. These are:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.28}$$

Similarly, the positive eigenstates are the eigenvectors of h with eigenvalues $1/2$. These are:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.29}$$

This means that

- the positive-helicity particle states are represented by ν'_L, ν'_R.
- the negative-helicity particle states are represented by ν_L, ν_R.

4.4.3 Particle/Antiparticle state

Note that the anti-particle field is related to the particle field by:

$$\psi^c \equiv \begin{pmatrix} (\nu'^c)_L \\ (\nu^c)_L \\ (\nu'^c)_R \\ (\nu^c)_R \end{pmatrix} = C\overline{\psi}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \psi^* = \begin{pmatrix} \nu_R^* \\ -\nu_R'^* \\ -\nu_L^* \\ \nu_L'^* \end{pmatrix}. \tag{4.30}$$

This means that

- ν_R^* represents a left-handed, positive-helicity state.
- $\nu_R'^*$ represents a left-handed, negative-helicity state.
- ν_L^* represents a right-handed, positive-helicity state.
- $\nu_L'^*$ represents a right-handed, negative-helicity state.

4.4.4 Summary

To summarize, the eight states that neutrinos can take are represented by:

- ν'_L: left-handed and positive-helicity neutrino.
- ν_L: left-handed and negative-helicity neutrino.
- ν'_R: right-handed and positive-helicity neutrino .
- ν_R: right-handed and negative-helicity neutrino .
- $\nu_L'^*$: right-handed and negative-helicity antineutrino .
- ν_L^*: right-handed and positive-helicity antineutrino .
- $\nu_R'^*$: left-handed and negative-helicity antineutrino.
- ν_R^*: left-handed and positive-helicity antineutrino.

4.5 The oscillation amplitude

The equation

$$i\frac{d}{dt}\boldsymbol{\nu} = H\boldsymbol{\nu}, \tag{4.31}$$

has a solution in the following form:

$$\boldsymbol{\nu}(t) = e^{-iHt}\boldsymbol{\nu}(0). \tag{4.32}$$

Since H is hermitian, there exists a unitary matrix U such that

$$H = U^{\dagger}\Lambda U, \tag{4.33}$$

where Λ is real, diagonal, and made up of the eigenvalues of the Hamiltonian. This means we have,

$$\boldsymbol{\nu}(t) = Ue^{-i\Lambda t}U^{\dagger}\boldsymbol{\nu}(0). \tag{4.34}$$

We can rewrite this as

$$\nu_i(t) = U_{ij}e^{-iE_jt}U^*_{kj}\nu_k(0), \tag{4.35}$$

where i ranges from 1 to 8 and label the eight states described in the previous section. We then impose the initial condition that,

$$\nu_k = 1, \quad \text{for } k = m \quad \text{and} \quad \nu_k = 0, \quad \text{otherwise} \tag{4.36}$$

then the amplitude of oscillation from $\nu_m \to \nu_i$ is given by:

$$A(\nu_m \to \nu_i) = U_{ij}e^{-iE_jt}U^*_{mj}. \tag{4.37}$$

4.5.1 One-particle notation

Let $\boldsymbol{\nu'}(0) = U^{\dagger}\boldsymbol{\nu}(0)$. Then according to Eq 4.35, we have

$$\nu_i(t) = U_{ij}e^{-iE_jt}\nu'_j(0). \tag{4.38}$$

Now, we change to a basis of energy eigenstates where we create an orthonormal basis vectors using $\nu'(0)$. In other words, we have

$$\langle \nu'_i(0)|\nu'_j(0)\rangle = \delta_{ij} \tag{4.39}$$

We can write each neutrino field as a linear combination of energy eigenstates:

$$\langle \nu_i(t)| = U_{ij}e^{-iE_jt}\langle \nu'_j(0)|. \tag{4.40}$$

By taking the complex conjugate and setting $t = 0$, we obtain the initial state:

$$|\nu_m(0)\rangle = U^*_{mn}|\nu'_n(0)\rangle. \tag{4.41}$$

Now the amplitude of oscillation from ν_m to ν_i is given by:

$$A(\nu_m \to \nu_i) = \langle \nu_i|\nu_m\rangle = U_{ij}e^{-iE_jt}U^*_{mn}\langle \nu'_j(0)|\nu'_n(0)\rangle \tag{4.42}$$

$$= U_{ij}e^{-E_jt}U^*_{mj}, \tag{4.43}$$

which is equivalent to using the other notation. We will use this one-particle notation throughout the paper because of its simplicity.

4.6 Special cases of the Hamiltonian

4.6.1 Case with $\vec{B} = (0,0,0), \quad M_R = 0$

Since only m_D is non-zero, we can set it to be real as we can always redefine the fields to absorb its complex phase. After rearranging the fields, the equations of motion are given by

$$i\frac{d}{dt}\begin{pmatrix} \nu'_{L1} \\ \nu'_{R1} \\ \nu_{L1} \\ \nu_{R1} \\ \nu'^*_{L2} \\ \nu'^*_{R2} \\ \nu^*_{L2} \\ \nu^*_{R2} \end{pmatrix} = \begin{pmatrix} -p & m_D & 0 & 0 & 0 & 0 & 0 & 0 \\ m_D & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & m_D & 0 & 0 & 0 & 0 \\ 0 & 0 & m_D & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p & -m_D & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_D & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & -m_D \\ 0 & 0 & 0 & 0 & 0 & 0 & -m_D & -p \end{pmatrix}\begin{pmatrix} \nu'_{L1} \\ \nu'_{R1} \\ \nu_{L1} \\ \nu_{R1} \\ \nu'^*_{L2} \\ \nu'^*_{R2} \\ \nu^*_{L2} \\ \nu^*_{R2} \end{pmatrix}. \tag{4.44}$$

This set of eight equations clearly decouples into four sets of two equations, which only differ in the sign of the momentum p and the Dirac mass m_D. The fields also decouple in an interesting way: they decouple according to their helicities and their particle/antiparticle-nesses. This means that in the case where only the Dirac mass is non-zero, neutrinos only change their chirality. Helicity flips and neutrino-antineutrino oscillations are forbidden. We will only solve the case of positive-helicity neutrinos as the other cases have the exact same results [2]:

$$i\frac{d}{dt}\begin{pmatrix} \nu'_R \\ \nu'_L \end{pmatrix} = \begin{pmatrix} p & m_D \\ m_D & -p \end{pmatrix}\begin{pmatrix} \nu'_R \\ \nu'_L \end{pmatrix}. \tag{4.45}$$

We can diagonalize the Hamiltonian using its eigenvectors and eigenvalues. Let

$$\begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix} = \begin{pmatrix} \lambda_+ & \lambda_- \\ -\lambda_- & \lambda_+ \end{pmatrix}\begin{pmatrix} \nu^- \\ \nu^+ \end{pmatrix}, \tag{4.46}$$

where

$$\lambda_\pm = \sqrt{\frac{E \pm p}{2E}}, \quad E = \sqrt{p^2 + m^2}. \tag{4.47}$$

The equations of motion become:

$$i\frac{d}{dt}\begin{pmatrix} \nu^- \\ \nu^+ \end{pmatrix} = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} \nu^- \\ \nu^+ \end{pmatrix}, \tag{4.48}$$

which means that

$$\nu^\pm(t) = e^{\mp iEt}\nu^\pm(0) \tag{4.49}$$

If we write the neutrino fields in one-particle state notation introduced in Section 4.5.1, we obtain:

$$\begin{cases} |\nu'_R\rangle = \lambda_+|\nu^-\rangle + \lambda_-|\nu^+\rangle, \\ |\nu'_L\rangle = -\lambda_-|\nu^-\rangle + \lambda_+|\nu^+\rangle. \end{cases} \tag{4.50}$$

[2] We have dropped the index 1 in the following equations because it is not important and non-ambiguous.

Taking the complex conjugate, we obtain:

$$\begin{cases} \langle \nu'_R| = \lambda_+ e^{iEt}\langle \nu^-| + \lambda_- e^{-iEt}\langle \nu^+|, \\ \langle \nu'_L| = -\lambda_- e^{iEt}\langle \nu^-| + \lambda_+ e^{-iEt}\langle \nu^+|, \end{cases} \tag{4.51}$$

where

$$\langle \nu^-|\nu^-\rangle = 1, \quad \langle \nu^+|\nu^+\rangle = 1, \quad \langle \nu^-|\nu^+\rangle = 0. \tag{4.52}$$

Now, we can calculate the amplitudes of oscillation:

$$A(\nu'_L \to \nu'_L) = \langle \nu'_L|\nu'_L\rangle = \cos(Et) - \frac{ip}{E}\sin(Et), \tag{4.53}$$

$$A(\nu'_L \to \nu'_R) = \langle \nu'_R|\nu'_L\rangle = -i\frac{m_D}{E}\sin Et. \tag{4.54}$$

Hence, the oscillation probabilities are given by,

$$P(\nu'_L \to \nu'_L) = \cos^2(Et) + \frac{p^2}{E^2}\sin^2(Et) = 1 - \frac{m_D^2}{E^2}\sin^2(Et), \tag{4.55}$$

$$P(\nu'_L \to \nu'_R) = \frac{m_D^2}{E^2}\sin^2(Et), \tag{4.56}$$

which is consistent with results in [15]. We can also verify that

$$P(\nu'_L \to \nu'_L) + P(\nu'_L \to \nu'_R) = 1. \tag{4.57}$$

This means that in this case, only chirality is flipped, and is suppressed by the factor m^2/E^2, which is why chiral oscillations are usually ignored in the literature. When $m_D = 0$, there is no oscillation probability, since the Dirac mass m_D is the only parameter in this case that couples the left-handed fields to the right-handed ones.

4.6.2 Case with $\vec{B} = (B_x, 0, 0)$ and $M_R = 0$

We can redefine the fields to get rid of complex phases in m_D and μ, since $M_R = 0$. This means that we can take m_D and μ to be real. The equations of motion are therefore given by:

$$i\frac{d}{dt}\begin{pmatrix} \nu'_{L1} \\ \nu_{L1} \\ \nu'_{R1} \\ \nu_{R1} \\ \nu'^*_{L2} \\ \nu^*_{L2} \\ \nu'^*_{R2} \\ \nu^*_{R2} \end{pmatrix} = \begin{pmatrix} -p & 0 & m_D & \mu B_x & 0 & 0 & 0 & 0 \\ 0 & p & \mu B_x & m_D & 0 & 0 & 0 & 0 \\ m_D & \mu B_x & p & 0 & 0 & 0 & 0 & 0 \\ \mu B_x & m_D & 0 & -p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p & 0 & -m_D & -\mu B_x \\ 0 & 0 & 0 & 0 & 0 & p & -\mu B_x & -m_D \\ 0 & 0 & 0 & 0 & -m_D & -\mu B_x & p & 0 \\ 0 & 0 & 0 & 0 & -\mu B_x & -m_D & 0 & -p \end{pmatrix}\begin{pmatrix} \nu'_{L1} \\ \nu_{L1} \\ \nu'_{R1} \\ \nu_{R1} \\ \nu'^*_{L2} \\ \nu^*_{L2} \\ \nu'^*_{R2} \\ \nu^*_{R2} \end{pmatrix}. \tag{4.58}$$

The set of eight differential equations decouples into two sets of four differential equations such that oscillations between neutrinos and anti-neutrinos are forbidden. Now focusing on just the oscillations of neutrino, the equations of motion reduce to[3]:

$$i\frac{d}{dt}\begin{pmatrix} \nu'_L \\ \nu_L \\ \nu'_R \\ \nu_R \end{pmatrix} = \begin{pmatrix} -p & 0 & m_D & \mu B_x \\ 0 & p & \mu B_x & m_D \\ m_D & \mu B_x & p & 0 \\ \mu B_x & m_D & 0 & -p. \end{pmatrix}\begin{pmatrix} \nu'_L \\ \nu_L \\ \nu'_R \\ \nu_R \end{pmatrix}. \tag{4.59}$$

[3]We have dropped the index 1 in the following equations because it is not important and non-ambiguous.

Now, we block diagonalize the above matrix by redefining the fields in the following way:

$$\begin{pmatrix} \nu_L' \\ \nu_L \\ \nu_R' \\ \nu_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \nu_{a1} \\ \nu_{a2} \\ \nu_{a3} \\ \nu_{a4} \end{pmatrix}. \tag{4.60}$$

The equations of motion become:

$$i\frac{d}{dt} \begin{pmatrix} \nu_{a1} \\ \nu_{a2} \\ \nu_{a3} \\ \nu_{a4} \end{pmatrix} = \begin{pmatrix} \mu B_x - p & m_D & 0 & 0 \\ m_D & \mu B_x + p & 0 & 0 \\ 0 & 0 & -\mu B_x - p & -m_D \\ 0 & 0 & -m_D & -\mu B_x + p \end{pmatrix} \begin{pmatrix} \nu_{a1} \\ \nu_{a2} \\ \nu_{a3} \\ \nu_{a4} \end{pmatrix}. \tag{4.61}$$

The last step to obtain a diagonal Hamitonian is given by redefining:

$$\begin{pmatrix} \nu_{a1} \\ \nu_{a2} \\ \nu_{a3} \\ \nu_{a4} \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \nu_1^- \\ \nu_2^- \\ \nu_1^+ \\ \nu_2^+ \end{pmatrix}, \tag{4.62}$$

which results in:

$$i\frac{d}{dt} \begin{pmatrix} \nu_1^- \\ \nu_2^- \\ \nu_1^+ \\ \nu_2^+ \end{pmatrix} = \begin{pmatrix} -E_1 & 0 & 0 & 0 \\ 0 & -E_2 & 0 & 0 \\ 0 & 0 & E_1 & 0 \\ 0 & 0 & 0 & E_2, \end{pmatrix} \begin{pmatrix} \nu_1^- \\ \nu_2^- \\ \nu_1^+ \\ \nu_2^+ \end{pmatrix}, \tag{4.63}$$

where

$$E_1 = -\mu B_x - \sqrt{m_D^2 + p^2}, \quad E_2 = -\mu B_x + \sqrt{m_D^2 + p^2}. \tag{4.64}$$

This means that

$$\nu_i^{\pm}(t) = e^{\mp i E_i t} \nu_i^{\pm}(0), \quad i = 1, 2. \tag{4.65}$$

Tracing back the steps to the original fields, we obtain:

$$\begin{pmatrix} \nu_L' \\ \nu_L \\ \nu_R' \\ \nu_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 & 0 \\ 0 & 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & 0 & \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \begin{pmatrix} \nu_1^- \\ \nu_2^- \\ \nu_1^+ \\ \nu_2^+ \end{pmatrix} \tag{4.66}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_1 & \cos\alpha_1 & \sin\alpha_2 & \cos\alpha_2 \\ \sin\alpha_1 & \cos\alpha_1 & -\sin\alpha_2 & -\cos\alpha_2 \\ \cos\alpha_1 & -\sin\alpha_1 & -\cos\alpha_2 & \sin\alpha_2 \end{pmatrix}, \tag{4.67}$$

where

$$\tan\alpha_1 = \frac{E+p}{m_D}, \quad \tan\alpha_2 = \frac{E-p}{m_D}, \quad E = \sqrt{p^2 + m_D^2}. \tag{4.68}$$

We can also deduce that:

$$\cos^2\alpha_1 + \cos^2\alpha_2 = \sin^2\alpha_1 + \sin^2\alpha_2 = 1, \tag{4.69}$$

$$\cos^2\alpha_1 - \cos^2\alpha_2 = \frac{-p}{E}, \tag{4.70}$$

$$\sin^2\alpha_1 - \sin^2\alpha_2 = \frac{p}{E}, \tag{4.71}$$

$$\cos\alpha_1 \sin\alpha_1 = \cos\alpha_2 \sin\alpha_2 = \frac{m_D}{2E}. \tag{4.72}$$

Using these relationships, we can obtain the oscillation amplitudes:

$$A(\nu'_L \to \nu'_L) = \frac{1}{2}\Big[\cos(E_1 t) + \cos(E_2 t) - \frac{ip}{E}[\sin(E_1 t) - \sin(E_2 t)]\Big], \tag{4.73}$$

$$A(\nu'_L \to \nu_L) = \frac{m_D^2}{2E^2}[\cos(E_1 t) - \cos(E_2 t)], \tag{4.74}$$

$$A(\nu'_L \to \nu'_R) = \frac{m_D^2}{2E^2}[\sin(E_1 t) - \sin(E_2 t)], \tag{4.75}$$

$$A(\nu'_L \to \nu'_L) = \frac{1}{2}\Big[i(\sin(E_1 t) + \sin(E_2 t)) - \frac{p}{E}[\cos(E_1 t) - \cos(E_2 t)]\Big]. \tag{4.76}$$

Finally, the oscillation probabilities are given by:

$$P(\nu'_L \to \nu'_L) = \cos^2(\mu B_x t)[1 - \frac{m_D^2}{E^2}\sin^2(Et)], \tag{4.77}$$

$$P(\nu'_L \to \nu_L) = \sin^2(\mu B_x t)\frac{m_D^2}{E^2}\sin^2(Et), \tag{4.78}$$

$$P(\nu'_L \to \nu'_R) = \cos^2(\mu B_x t)\frac{m_D^2}{E^2}\sin^2(Et), \tag{4.79}$$

$$P(\nu'_L \to \nu_R) = \sin^2(\mu B_x t)[1 - \frac{m_D^2}{E^2}\sin^2(Et)]. \tag{4.80}$$

Again, we can easily show that

$$P(\nu_L \to \nu_L) + P(\nu_L \to \nu'_L) + P(\nu_L \to \nu'_R) + P(\nu_L \to \nu_R) = 1. \tag{4.81}$$

In this case, the oscillations between neutrinos and antineutrinos are forbidden because we set $M_R = 0$, hence effectively imposing that the neutrinos are Dirac fermions. Note that neutrino-antineutrino oscillations happen only when neutrinos are Majorana fermion.

However, there are both chiral oscillations and helicity oscillations. Helicity is not conserved, because there is an external magnetic field in the x direction, which breaks the conservation of the spin angular momentum in the z direction. Interestingly, if there is only a chirality flip or only helicity flip, then the probability is suppressed by a factor of m^2/E^2. However, if there is both a flip in helicity and chirality, the probability is not suppressed, meaning that it can have a large value. To see this clearly, setting the term m_D^2/E^2 to zero, we observe that

$$P(\nu'_L \to \nu'_L) = \cos^2(\mu B_x t), \tag{4.82}$$

$$P(\nu'_L \to \nu_R) = \sin^2(\mu B_x t). \tag{4.83}$$

It is interesting that when we set $m_D = 0$, we still obtain a chirality flip. This is reasonable as when $m_D \to 0$, chirality and helicity coincide. In other words, in the case of $m_D = 0$, when helicity flip, chirality must also flip.

4.6.3 Case with $\vec{B} = (0, 0, 0)$ and $M_R \neq 0$

We can let m_D and M_R be real, as we can get rid of their phases through the redefinition of the neutrino fields. After rearranging the fields, the equations of motion are therefore given by:

$$i\frac{d}{dt}\begin{pmatrix}\nu_{L2}^{*}\\ \nu_{R1}'\\ \nu_{L1}'\\ \nu_{R2}^{*}\\ \nu_{L2}'^{*}\\ \nu_{R1}\\ \nu_{L1}\\ \nu_{R2}'^{*}\end{pmatrix}=\begin{pmatrix}p & 0 & 0 & -m_D & 0 & 0 & 0 & 0\\ 0 & p & m_D & -M_R & 0 & 0 & 0 & 0\\ 0 & m_D & -p & 0 & 0 & 0 & 0 & 0\\ -m_D & -M_R & 0 & -p & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -p & 0 & 0 & -m_D\\ 0 & 0 & 0 & 0 & 0 & -p & m_D & M_R\\ 0 & 0 & 0 & 0 & 0 & m_D & p & 0\\ 0 & 0 & 0 & 0 & -m_D & M_R & 0 & p\end{pmatrix}\begin{pmatrix}\nu_{L2}^{*}\\ \nu_{R1}'\\ \nu_{L1}'\\ \nu_{R2}^{*}\\ \nu_{L2}'^{*}\\ \nu_{R1}\\ \nu_{L1}\\ \nu_{R2}'^{*}\end{pmatrix}. \tag{4.84}$$

In this case, the set of eight equations decouples into two similar sets of four equations. Now, helicity flips are forbidden, but chiral and neutrino-antineutrino oscillations are allowed. We will consider just the positive-helicity states, which evolve with respect to the following equations [4]:

$$i\frac{d}{dt}\begin{pmatrix}\nu_{L}^{*}\\ \nu_{R}'\\ \nu_{L}'\\ \nu_{R}^{*}\end{pmatrix}=\begin{pmatrix}p & 0 & 0 & -m_D\\ 0 & p & m_D & -M_R\\ 0 & m_D & -p & 0\\ -m_D & -M_R & 0 & -p\end{pmatrix}\begin{pmatrix}\nu_{L}^{*}\\ \nu_{R}'\\ \nu_{L}'\\ \nu_{R}^{*}.\end{pmatrix}. \tag{4.85}$$

Let $X=\begin{pmatrix}0 & -m_D\\ m_D & -M_R\end{pmatrix}$. Denotes m_1, m_2 as the eigenvalues of X. They are given by

$$m_{1,2}=\frac{-M_R\pm\sqrt{M_R^2+4m_D^2}}{2}.$$

Let

$$\begin{pmatrix}\nu_{L}^{*}\\ \nu_{R}'\\ \nu_{L}'\\ \nu_{R}^{*}\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0 & 0\\ \sin\alpha & \cos\alpha & 0 & 0\\ 0 & 0 & \cos\alpha & \sin\alpha\\ 0 & 0 & -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix}, \tag{4.86}$$

where

$$\tan\alpha=-\frac{\sqrt{m_D^2+M_R^2/4}+M_R/2}{m_D}.$$

Then, we obtain:

$$i\frac{d}{dt}\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix}=\begin{pmatrix}p & 0 & m_1 & 0\\ 0 & p & 0 & m_2\\ m_1 & 0 & -p & 0\\ 0 & m_2 & 0 & -p\end{pmatrix}\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix}. \tag{4.87}$$

Now, we rearrange the fields by letting:

$$\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix}. \tag{4.88}$$

[4] We have dropped the indices 1 and 2 in these equations because they are not important and non-ambiguous.

We then obtain,

$$i\frac{d}{dt}\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix} = \begin{pmatrix} p & m_1 & 0 & 0 \\ m_1 & -p & 0 & 0 \\ 0 & 0 & p & m_2 \\ 0 & 0 & m_2 & -p \end{pmatrix}\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix}. \tag{4.89}$$

The final step to diagonalize the Hamiltonian is by redefining the fields one last time. Let,

$$\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_{-1} & 0 & 0 \\ -\lambda_{-1} & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_{-2} \\ 0 & 0 & -\lambda_{-2} & \lambda_2 \end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}, \tag{4.90}$$

where

$$\lambda_{\pm i} = \sqrt{\frac{E_i \pm p}{2E_i}}, \quad E_i = \sqrt{p^2 + m_i^2}. \tag{4.91}$$

Finally, we obtain:

$$i\frac{d}{dt}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix} = \begin{pmatrix} -E_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & -E_2 & 0 \\ 0 & 0 & 0 & E_2 \end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}, \tag{4.92}$$

which means

$$\nu_i^{\pm}(t) = e^{\mp iE_i t}\nu_i^{\pm}(0). \tag{4.93}$$

Now tracing back to the original fields we obtain:

$$\begin{pmatrix}\nu_L^*\\ \nu_R'\\ \nu_L'\\ \nu_R^*\end{pmatrix} = \begin{pmatrix} \lambda_1\cos\alpha & \lambda_{-1}\cos\alpha & -\lambda_2\sin\alpha & -\lambda_{-2}\sin\alpha \\ \lambda_1\sin\alpha & \lambda_{-1}\sin\alpha & \lambda_2\cos\alpha & \lambda_{-2}\cos\alpha \\ -\lambda_{-1}\cos\alpha & \lambda_1\cos\alpha & -\lambda_{-2}\sin\alpha & \lambda_2\sin\alpha \\ \lambda_{-1}\sin\alpha & -\lambda_1\sin\alpha & -\lambda_{-2}\cos\alpha & \lambda_2\cos\alpha \end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}, \tag{4.94}$$

Using the same method as above, we calculate the amplitudes of oscillation:

$$A(\nu_L' \to \nu_L') = \cos^2\alpha\Big[\cos(E_1t) - \frac{ip}{E_1}\sin(E_1t)\Big] + \sin^2\alpha\Big[\cos(E_2t) - \frac{ip}{E_2}\sin(E_2t)\Big], \tag{4.95}$$

$$A(\nu_L' \to \nu_L^*) = -\frac{im_1}{E_1}\cos^2\alpha\sin E_1t + \frac{im_2}{E_2}\sin^2\alpha\sin E_2t, \tag{4.96}$$

$$A(\nu_L' \to \nu_R') = i\cos\alpha\sin\alpha\Big[\frac{m_1}{E_1}\sin(E_1t) + \frac{m_2}{E_2}\sin(E_2t)\Big], \tag{4.97}$$

$$A(\nu_L' \to \nu_R^*) = \cos\alpha\sin\alpha\Big[\cos(E_1t) - \frac{ip}{E_1}\sin(E_1t) - \cos(E_2t) + \frac{ip}{E_2}\sin(E_2t)\Big]. \tag{4.98}$$

The probability of oscillation is given by, which agrees with results in [15]:

$$
\begin{aligned}
P(\nu'_L \to \nu'_L) &= \cos^4\alpha\Big[1 - \frac{m_1^2}{E_1^2}\sin^2(E_1 t)\Big] + \sin^4\alpha\Big[1 - \frac{m_2^2}{E_2^2}\sin^2(E_2 t)\Big] \\
&\quad + 2\cos^2\alpha\sin^2\alpha\Big[\cos(E_1 t)\cos(E_2 t) + \frac{p^2}{E_1 E_2}\sin(E_1 t)\sin(E_2 t), \qquad (4.99) \\
P(\nu'_L \to \nu^*_L) &= \frac{m_1^2}{E_1^2}\cos^4\alpha\sin^2(E_1 t) + \frac{m_2^2}{E_2^2}\sin^4\alpha\sin^2(E_2 t) \\
&\quad - \frac{2m_1 m_2}{E_1 E_2}\cos^2\alpha\sin^2\alpha\sin(E_1 t)\sin(E_2 t), \qquad (4.100) \\
P(\nu'_L \to \nu'_R) &= \cos^2\alpha\sin^2\alpha\Big[\frac{m_1^2}{E_1^2}\sin^2(E_1 t) + \frac{m_2^2}{E_2^2}\sin^2(E_2 t) \\
&\quad + \frac{2m_1 m_2}{E_1 E_2}\sin(E_1 t)\sin(E_2 t)\Big], \qquad (4.101) \\
P(\nu'_L \to \nu^*_R) &= \cos^2\alpha\sin^2\alpha\Big[2 - \frac{m_1^2}{E_1^2}\sin^2(E_1 t) - \frac{m_2^2}{E_2^2}\sin^2(E_2 t) \\
&\quad - 2\cos(E_1 t)\cos(E_2 t) - 2\frac{p^2}{E_1 E_2}\sin(E_1 t)\sin(E_2 t)\Big]. \qquad (4.102)
\end{aligned}
$$

Note that in the limit that $M_R = 0$,

$$\alpha = -\pi/4, \quad m_1 = m_2 = m_D. \qquad (4.103)$$

This is just the case of non-zero Dirac mass; therefore $P(\nu'_L \to \nu'_L)$ and $P(\nu'_L \to \nu'_R)$ consistently reduce to results in section 4.6.1, and $P(\nu'_L \to \nu^*_L) = P(\nu'_L \to \nu^*_R) = 0$ as expected.

Since $M_R \neq 0$, this means that there are neutrino-antineutrino oscillations as expected. Comparing to section 4.6.2, there is no longer any existence of helicity flip because we switch off the magnetic field. Moreover, the two oscillation channels that involve chirality flips are suppressed. The unconventional oscillation probability involving a sterile antineutrino is not suppressed. If we let $m^2/E^2 \to 0$, we observe that

$$P(\nu'_L \to \nu^*_R) = 2\cos^2\alpha\sin^2\alpha\Big[1 - \cos(E_1 t)\cos(E_2 t) - \frac{p^2}{E_1 E_2}\sin(E_1 t)\sin(E_2 t)\Big]. \qquad (4.104)$$

4.6.4 Case with $\vec{B} = (0, 0, B_z)$ and $M_R \neq 0$

We now turn to the case where the magnetic field is parallel to the direction of motion. After rearranging the fields, the equations of motion are given by:

$$
i\frac{d}{dt}\begin{pmatrix} \nu^*_{L2} \\ \nu'_{R1} \\ \nu'_{L1} \\ \nu^*_{R2} \\ \nu'^*_{L2} \\ \nu_{R1} \\ \nu_{L1} \\ \nu'^*_{R2} \end{pmatrix} = \begin{pmatrix}
p & 0 & 0 & -m_D^* + \mu^* B_z & 0 & 0 & 0 & 0 \\
0 & p & m_D^* + \mu^* B_z & -M_R^* & 0 & 0 & 0 & 0 \\
0 & m_D + \mu B_z & -p & 0 & 0 & 0 & 0 & 0 \\
-m_D + \mu B_z & -M_R & 0 & -p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -p & 0 & 0 & -m_D^* - \mu^* B_z \\
0 & 0 & 0 & 0 & 0 & -p & m_D^* - \mu^* B_z & M_R^* \\
0 & 0 & 0 & 0 & 0 & m_D - \mu B_z & p & 0 \\
0 & 0 & 0 & 0 & -m_D - \mu B_z & M_R & 0 & p
\end{pmatrix} \begin{pmatrix} \nu'_{L1} \\ \nu'_{R1} \\ \nu_{L1} \\ \nu_{R1} \\ \nu'^*_{L2} \\ \nu'^*_{R2} \\ \nu^*_{L2} \\ \nu^*_{R2} \end{pmatrix}. \qquad (4.105)
$$

Similar to the case with $\vec{B} = (0,0,0)$, the set of eight equations still decouples into two similar sets of four equations. Again, helicity is conserved, whereas particle/antiparticle-ness and chirality are not. We will consider just the positive-helicity states, which evolve with respect to the following equations [5]:

$$i\frac{d}{dt}\begin{pmatrix}\nu_L^*\\ \nu_R'\\ \nu_L'\\ \nu_R^*\end{pmatrix} = \begin{pmatrix} p & 0 & 0 & -m_D^*+\mu^* B_z \\ 0 & p & m_D^*+\mu^* B_z & -M_R^* \\ 0 & m_D+\mu B_z & -p & 0 \\ -m_D+\mu B_z & -M_R & 0 & -p \end{pmatrix}\begin{pmatrix}\nu_L^*\\ \nu_R'\\ \nu_L'\\ \nu_R^*\end{pmatrix}. \tag{4.106}$$

In two-component vectors X and Y, we can write:

$$i\frac{d}{dt}\begin{pmatrix}X\\ Y\end{pmatrix} = \begin{pmatrix} p\mathbb{1} & M \\ M^\dagger & -p\mathbb{1}\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}, \tag{4.107}$$

where $M = \begin{pmatrix} 0 & -m_D^*+\mu^* B_z \\ m_D^*+\mu^* B_z & -M_R^*\end{pmatrix}$ is almost a general complex 2×2 matrix (we can treat it as a general matrix but we later impose the condition that $M_{11} = 0$). According to the singular value decomposition theorem, there exists two unitary matrices U and V such that

$$U^\dagger M V = D, \quad \text{where } D \text{ is real, positive, and diagonal.} \tag{4.108}$$

Let

$$\begin{pmatrix}X\\ Y\end{pmatrix} = \begin{pmatrix}U & 0\\ 0 & V\end{pmatrix}\begin{pmatrix}X'\\ Y'\end{pmatrix} \quad \& \quad D = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix} \tag{4.109}$$

Then we have,

$$i\frac{d}{dt}\begin{pmatrix}X'\\ Y'\end{pmatrix} = \begin{pmatrix}U^\dagger & 0\\ 0 & V^\dagger\end{pmatrix}\begin{pmatrix} p\mathbb{1} & M \\ M^\dagger & -p\mathbb{1}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & V\end{pmatrix}\begin{pmatrix}X'\\ Y'\end{pmatrix} \tag{4.110}$$

$$= \begin{pmatrix} p\mathbb{1} & D \\ D^\dagger & -p\mathbb{1}\end{pmatrix}\begin{pmatrix}X'\\ Y'\end{pmatrix}. \tag{4.111}$$

In four-component form, we have:

$$i\frac{d}{dt}\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix} = \begin{pmatrix} p & 0 & m_1 & 0 \\ 0 & p & 0 & m_2 \\ m_1 & 0 & -p & 0 \\ 0 & m_2 & 0 & -p\end{pmatrix}\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix}. \tag{4.112}$$

Now, we rearrange the fields by letting:

$$\begin{pmatrix}\nu_{a1}\\ \nu_{a2}\\ \nu_{a3}\\ \nu_{a4}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix}. \tag{4.113}$$

[5] We have dropped the indices 1 and 2 in the following equations because they are not important and non-ambiguous.

We then obtain,

$$i\frac{d}{dt}\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix}=\begin{pmatrix}p & m_1 & 0 & 0\\ m_1 & -p & 0 & 0\\ 0 & 0 & p & m_2\\ 0 & 0 & m_2 & -p\end{pmatrix}\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix}. \tag{4.114}$$

Now, we reach the last step in our attempt to diagonalize. Our last linear transformation is made up of eigenvectors:

$$\begin{pmatrix}\nu_{b1}\\ \nu_{b2}\\ \nu_{b3}\\ \nu_{b4}\end{pmatrix}=\begin{pmatrix}\lambda_1 & \lambda_{-1} & 0 & 0\\ -\lambda_{-1} & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & \lambda_{-2}\\ 0 & 0 & -\lambda_{-2} & \lambda_2\end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}, \quad \lambda_{\pm i}=\sqrt{\frac{E_i\pm p}{2E_i}}. \tag{4.115}$$

With this transformation, we finally obtain:

$$i\frac{d}{dt}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}=\begin{pmatrix}-E_1 & 0 & 0 & 0\\ 0 & E_1 & 0 & 0\\ 0 & 0 & -E_2 & 0\\ 0 & 0 & 0 & E_2\end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}, \tag{4.116}$$

which means

$$\nu_i^{\pm}(t)=e^{\mp iE_it}\nu_i^{\pm}(0). \tag{4.117}$$

Now tracing back our steps to the original (X,Y), we obtain:

$$\begin{aligned}\begin{pmatrix}\nu_L^*\\ \nu_R'\\ \nu_L'\\ \nu_R^*\end{pmatrix}&=\begin{pmatrix}U_{11} & U_{12} & 0 & 0\\ U_{21} & U_{22} & 0 & 0\\ 0 & 0 & V_{11} & V_{12}\\ 0 & 0 & V_{21} & V_{22}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\lambda_1 & \lambda_{-1} & 0 & 0\\ -\lambda_{-1} & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & \lambda_{-2}\\ 0 & 0 & -\lambda_{-2} & \lambda_2\end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix},\\ &=\begin{pmatrix}\lambda_1U_{11} & \lambda_{-1}U_{11} & \lambda_2U_{12} & \lambda_{-2}U_{12}\\ \lambda_1U_{21} & \lambda_{-1}U_{21} & \lambda_2U_{22} & \lambda_{-2}U_{22}\\ -\lambda_{-1}V_{11} & \lambda_1V_{11} & -\lambda_{-2}V_{12} & \lambda_2V_{12}\\ -\lambda_{-1}V_{21} & \lambda_1V_{21} & -\lambda_{-2}V_{22} & \lambda_2V_{22}\end{pmatrix}\begin{pmatrix}\nu_1^-\\ \nu_1^+\\ \nu_2^-\\ \nu_2^+\end{pmatrix}.\end{aligned} \tag{4.118}$$

Using the above method, the oscillation amplitudes are found to be:

$$\langle\nu_L',\nu_L'\rangle=\sum_{k=1}^{2}|V_{1k}|^2\big[\cos(E_kt)-\frac{ip}{E_k}\sin(E_kt)\big], \tag{4.119}$$

$$\langle\nu_R',\nu_L'\rangle=\sum_{k=1}^{2}-i\frac{m_k}{E_k}V_{1k}U_{2k}^*\sin(E_kt), \tag{4.120}$$

$$\langle\nu_L'^*,\nu_L'\rangle=\sum_{k=1}^{2}i\frac{m_k}{E_k}U_{1k}^*V_{1k}\sin(E_kt), \tag{4.121}$$

$$\langle\nu_R^*,\nu_L'\rangle=\sum_{k=1}^{2}V_{1k}V_{2k}^*\big[\cos(E_kt)-\frac{ip}{E_k}\sin E_kt\big]. \tag{4.122}$$

We can parametrize U and V in the following way:

$$V=\begin{pmatrix}e^{i\rho_1}\cos\theta_1 & e^{i(\rho_1+\phi_1)}\sin\theta_1\\ -e^{i\kappa_1}\sin\theta_1 & e^{i(\kappa_1+\phi_1)}\cos\theta_1\end{pmatrix}, \quad U=\begin{pmatrix}e^{i\rho_2}\cos\theta_2 & e^{i(\rho_2+\phi_2)}\sin\theta_2\\ -e^{i\kappa_2}\sin\theta_2 & e^{i(\kappa_2+\phi_2)}\cos\theta_2\end{pmatrix}, \tag{4.123}$$

where as a result of the condition that $M_{11} = 0$, we also have:

$$\phi_1 = \phi_2, \quad m_1 c_1 c_2 + m_2 s_1 s_2 = 0. \tag{4.124}$$

Then the oscillation probabilities are given by:

$$\begin{aligned}
P(\nu'_L \to \nu'_L) &= 1 - 4s_1^2 c_1^2 \sin^2 \frac{(E_2 - E_1)t}{2} - 2s_1^2 c_1^2 (1 - \frac{p^2}{E_1 E_2}) \sin(E_1 t) \sin(E_2 t) \\
&\quad - \Big[c_1^4 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + s_1^4 \frac{m_2^2}{E_2^2} \sin^2(E_2 t)\Big], \qquad (4.125) \\
P(\nu'_L \to \nu_R^*) &= 4s_1^2 c_1^2 \sin^2 \frac{(E_2 - E_1)t}{2} + 2s_1^2 c_1^2 (1 - \frac{p^2}{E_1 E_2}) \sin(E_1 t) \sin(E_2 t) \\
&\quad - s_1^2 c_1^2 \Big[\frac{m_1^2}{E_1^2} \sin^2(E_1 t) + \frac{m_2^2}{E_2^2} \sin^2(E_2 t)\Big], \qquad (4.126) \\
P(\nu'_L \to \nu_L^*) &= c_1^2 c_2^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + s_1^2 s_2^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) \\
&\quad + 2c_1 c_2 s_1 s_2 \cos(\phi_1 - \phi_2) \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t), \qquad (4.127) \\
P(\nu'_L \to \nu_R) &= c_1^2 s_2^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + s_1^2 c_2^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) \\
&\quad - 2c_1 c_2 s_1 s_2 \cos(\phi_1 - \phi_2) \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t). \qquad (4.128)
\end{aligned}$$

Note that $\phi_1 = \phi_2$ in this case as noted in Eq. 4.124; therefore, the physical phase drops out. (See section 4.8 for why we have included it in our results even though it is zero.)

4.7 Discussion

Having explored all the four special cases, we observe three types of oscillations, which in total involve eight neutrino states. The three types are:

- chirality flip: due to m_D and magnetic field in the direction perpendicular to the direction of propagation
- neutrino-antineutrino oscillation: due to M_R.
- helicity flip: due to the magnetic field in the direction perpendicular to the neutrino's motion.

In the case where the external magnetic field is only in the z-direction, which is parallel to the propagation direction of the neutrino, we only have four oscillation channels, meaning that helicity flip is forbidden:

$$P(\nu'_L \to \nu_L) = P(\nu'_L \to \nu_R) = P(\nu'_L \to \nu_R'^*) = P(\nu'_L \to \nu_L'^*) = 0. \tag{4.129}$$

When the magnetic field $\vec{B}$ is parallel to the propagation direction, we expect that the spin angular momentum in the direction of motion is still conserved. Hence, helicity is conserved in this case. Therefore, helicity flip is only induced by the perpendicular component of the magnetic field, since with its non-zero torque, the spin angular momentum in the propagation direction is no longer conserved. This is confirmed by the special case we explore in section 4.6.2.

Again, any process that involves chirality flip without helicity flip is supressed by a factor of m^2/E^2. In the case of a magnetic field in the z-direction, setting m^2/E^2, we observe:

$$P(\nu_L' \to \nu_L') = 1 - 4s_1^2c_1^2 \sin^2 \frac{(E_2 - E_1)t}{2} - 2s_1^2c_1^2(1 - \frac{p^2}{E_1E_2}) \sin(E_1t) \sin(E_2t), \tag{4.130}$$

$$P(\nu_L' \to \nu_R^*) = 4s_1^2c_1^2 \sin^2 \frac{(E_2 - E_1)t}{2} + 2s_1^2c_1^2(1 - \frac{p^2}{E_1E_2}) \sin(E_1t) \sin(E_2t), \tag{4.131}$$

$$P(\nu_L' \to \nu_L^*) = 0, \tag{4.132}$$

$$P(\nu_L' \to \nu_R) = 0. \tag{4.133}$$

The oscillation into sterile antineutrino resembles that of two generation neutrinos with different flavors, as noted by [24]. Note that this oscillation probability is substantial as it is not suppressed by the factor m^2/E^2.

4.8 Remarks on CP-violating phases

We conjecture that there must be a physical phase in the oscillation probabilties in the most general case even for a one-generation neutrino, as indicated by our spurion analysis. The Lagrangian of concern is given by

$$\mathcal{L} \ni -\overline{\psi_L} m_D \psi_R - \frac{1}{2}\overline{(\psi_R)^c} M_R \psi_R - \mu \overline{\psi_L}(\vec{\Sigma}.\vec{B})\psi_R + h.c. \tag{4.134}$$

Consider the $U(1) \times U(1)$ transformations:

$$\psi_L \to \psi_L e^{i\alpha}, \quad \psi_R \to \psi_R e^{i\beta}. \tag{4.135}$$

Then, for $\mathcal{L}$ to be invariant, the parameters m_D, m_R, and μ, considered as spurion fields, must transform as the following:

$$m_D \to m_D e^{i(\alpha-\beta)}, \tag{4.136}$$

$$M_R \to M_R e^{-2i\beta}, \tag{4.137}$$

$$\mu \to \mu e^{i(\alpha-\beta)}. \tag{4.138}$$

Any physically observable quantity must be made out of invariant objects under the transformation rules of the spurion fields. These invariants are:

$$\mu m_D^*, \quad m_D m_D^*, \quad M_R M_R^*, \quad \mu\mu^*. \tag{4.139}$$

The physical phase that arises in the oscillation probabilities of the most general case must therefore come from the phase of the term μm_D^*.

In fact, the oscillation probabilities (Eq 4.125 $\to$ 4.128) also apply to the case where we have the Majorana mass term for the left-handed field, M_L. If we include such term, we find that the same calculation follows except that M_{11} that we set to zero turns out to be M_L. Without the constraint that $M_{11} = 0$, we have an additional phase $\delta' \equiv \phi_1 - \phi_2$, that is non-zero. Note that this phase is also physical as verified by the spurion analysis. The Lagrangian with the additional term added is given by:

$$\mathcal{L} \ni -\overline{\psi_L} m_D \psi_R - \frac{1}{2}\overline{(\psi_R)^c} M_R \psi_R - \frac{1}{2}\overline{(\psi_L)^c} M_L \psi_L - \mu \overline{\psi_L}(\vec{\Sigma}.\vec{B})\psi_R + h.c. \tag{4.140}$$

In addition, M_L transforms:

$$M_L \to M_L e^{-2i\alpha}. \tag{4.141}$$

The invariant objects are:

$$\mu m_D^*, M_L M_L^*, \quad m_D m_D^*, \quad M_R M_R^*, \quad \mu\mu^*, \quad M_L m_D \mu M_R^*. \tag{4.142}$$

The two physical phases that arise in the oscillation probabilities must therefore come from the phases of the terms μm_D^* and $M_L m_D \mu M_R^*$.

4.9 Further Directions

Even though our calculations are complete, there are a few points that need to be addressed or further explored:

- Due to time constraints, we were not able to finish calculating the oscillation probabilities for a magnetic field in an arbitrary direction. We hope to continue working on this.
- In our calculation, we show that there are eight states that a neutrino can take, whereas the usual literature says that neutrinos can only have four degrees of freedom. Our way of reconciling with this is that in the standard literature, helicity is always conserved; therefore, there are only four states. However, when we break helicity conservation, then we would expect neutrinos to have eight states.
- We mention that there must be a complex phase from the mixing matrix that is physical. We wonder if that phase is CP-violating.
- We believe that our method can be extended straightforwardly to include flavor oscillations for two or three generation neutrinos but the oscillation channels would then increase to 16 and 24, respectively.
- We wonder how to connect our results to ongoing or future neutrino experiments. Most pressing point of concern is to produce neutrinos that are in definite states of both helicity and chirality.

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