May 2022

Lie Algebras and the Poincare Group

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Abstract

This paper will discuss my research with Professor Tonnis ter Veldhuis on the Poincare Group and other similar algebraic approaches based on the Minkowski Metric. This paper will begin with an introduction discussing group theory and expand on its specific applications in theoretical physics.

Keywords

Cover Page Footnote

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Abstract **– This paper will discuss my research with Professor Tonnis ter Veldhuis on the Poincare Group and other similar algebraic approaches based on the Minkowski Metric. This paper will begin with an introduction discussing group theory and expand on its specific applications in theoretical physics.**

I. Group Theory

Group theory is the analysis of groups, sets of abstract elements along with an operation where the elements must be associative, have an inverse element, and have an identity element e where $\forall x \in A : ex = x$. Basic examples of groups include the integers under addition, where 0 acts as the identity element, any integer has its negative value as an inverse, and where addition is associative. As there is no way to gain an element that is not an integer by adding or subtracting integers, the set of integers is a group. One important property of groups is the concept of a symmetry group, and an invariance. Symmetry describe sets of actions taken on an object that leave specific properties invariant, or unchanged. There are often several properties that can be invariant under the same symmetry group, and proving that a certain property remains invariant under the elements, or actions, of a symmetry group is important in demonstrating properties of what the group is describing. In fact, Noether's Theorem states that differentiable symmetries indicate the presence of a conservation law for a particular system. The final aspect of group theory important in this discussion is the concept of a generator. A generator is an object which can be exponentiated to produce all elements of the group. For instance, all elements in $\mathbb{Z}$ under addition can be found by iterating either 1 or its additive inverse. Generator elements are useful because proving something about the generator allows that proof to be induced onto the whole group.

II. Lie Group

A Lie group is a specific subclass of groups distinguished as also being differentiable manifolds. A manifold is a space with geometry that resembles Euclidian geometry near each point. For instance, a circle can be approximated with a line at every point. For this manifold to be differentiable, there must be a derivative value for each point. A group that meets these definitions is a Lie group, allowing for the use of differential equations when dealing with the group.

III. Minkowski Spacetime

Minkowski Spacetime is a space on which Lie group actions are preformed where the Euclidian spacial coordinates, often indicated as x, y, and z, are combined with the time coordinate to create a manifold. Thus a position is recorded as

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The Minkowski Spacetime has the distinction of keeping a measurement of the spacetime interval, or distance, between two points is the same for every inertial frame of reference. This spacetime interval is defined as

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

or alternatively

$$x^\mu \eta_{\mu\nu} x^\nu = 0$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This form uses the Einstein summation convention, where $A_{\mu,\nu} x_\nu = \sum_\nu . A_{\mu\nu} x_\nu$

Minkowski space is important because it is commonly used as the mathematical structure for special relativity, and because transformations on Minkowski Spacetime are themselves important groups.

IV. Lorentz Group

The Lorentz group can be formally defined as the group of Lorentz transformations on Minkowski Spacetime. The generators for these transformations are the boost transformation,

$$x'^\mu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\beta\gamma c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu$$

where

$$\beta = v/c$$

and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

which describe shifts in velocity, and the rotation transformation

$$x'_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin\theta & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x_{\mu,\nu}$$

These transformations are all of the form $X' = \Lambda X$ and

maintain the relationship $\Lambda^\mu_\nu \Lambda^\nu_\kappa \eta_{\mu,\lambda} = \eta_{\lambda,\kappa}$

V. Poincare Group

The Poincare group is the set of all Minkowski Isometries, which is to say the group of actions that preserve the Minkowski spacetime interval. The Poincare group is an expansion on the Lorentz group incorporating the translation action generator on top of the boost and rotation actions. These translation transformations can be codified in the form

$$x'^\mu = x^\mu + a^\mu$$

for any vector a^μ and represents a shift from one inertial reference frame to another. A common representation of the Poincare group establishes M as the generators of the Lorentz group and P as translations giving the following identities as a complete expression of the entire group.

$$[P_\mu, P_\nu] = 0$$

$$\frac{1}{i}[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu$$

$$\frac{1}{i}[M_{\mu,\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}$$

These generators and identities then can form elements of the form

$$\Lambda^\mu_\nu = e^{\omega_{\mu\nu} M^{\mu\nu}}$$

where ω is the Maurer-Cartan form, a way of representing differentials. It may be noted here that matrix exponentiation is done as

$$e^A = \sum_k \frac{A^k}{k!}$$

as with any other form of e^x. Some of these calculations are made easier with the Baker-Campbell-Hausdorf formulae.

VI. Equations of Motion

With the space well defined, some analysis of a particle situated in the space can finally be done. In this space, particles are represented as scaler fields $\phi(x^m)$ and a Lagrangian density can be constructed as

$$\mathscr{L} = -\frac{1}{2}\partial^m \phi \eta_{mn} \partial^n \phi - V(\phi)$$

which is remarkably similar to the classical Lagrangian $L = T - U$. Then the Euler-Lagrange equation

$$\partial_a \left(\frac{\partial \mathscr{L}}{\partial \partial_a \phi} \right) - \frac{\partial \mathscr{L}}{\partial \phi} = 0$$

provides a partial differential equation which provides equations of motion. For example, a Lagrangian density of the form

$$\mathscr{L} = -\frac{1}{2}\partial^m \phi \eta_{mn} \partial^n \phi - \frac{m\phi^2}{2} - \lambda \phi^4$$

will provide the result

$$-\Box \phi + m^2 \phi + 4\lambda \phi^3 = 0$$

where the D'Alembertain symbol

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

. This result is recognizable as the relativistic wave equation. Similar equations of motion can be derived by constructing different potentials in the Lagrangian density.

VII. Bibliography

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