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**Automatic Leptonic Tensor Generation for Beyond the Standard Model (BSM) Theories**

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Automatic Leptonic Tensor Generation for Beyond the Standard Model (BSM) Theories

Abstract
With the development of the Deep Underground Neutrino Experiment (DUNE) and Tokai-to-Hyper-Kamiokande (T2HK), we are entering the era of high-precision neutrino measurements. The colossal output of data from DUNE, plus the current data from several other neutrino experiments, will require a fast and efficient method of testing our BSM models in event generators. However, current methods for implementing a BSM theory in the event generators are prone to errors and time-consuming. We propose a novel program capable of automatically calculating the leptonic tensor for a given quantum field theory Lagrangian. This program is written in Python and utilizes the Universal FeynRules Output (UFO) format, the Lark package, and the Berends-Giele recursive relations to produce leptonic tensors that can be automatically implemented in several neutrino event generators, including those relevant for DUNE. For this project, we tested our algorithm with three SM processes: $e^- p^+ \rightarrow e^- p^+$, $\nu_e \rightarrow e^- \mu^+$ and $\nu_e p^+ \rightarrow \nu_e p^+$, and $\bar{\nu}_e p^+ \rightarrow e^- p^+$. For each process, we calculated the numerical and analytic $|\mathcal{M}|^2$ and $\sigma$ that we plotted as functions of $\cos\theta$ and $E_{CM}$, respectively. The numerical results for $e^- p^+ \rightarrow e^- p^+$ show good agreement with the analytic results with a cross section numerical to analytic ratio of $\sim 1$ and $\sim 0.9$, respectively. The process $\nu_e \rightarrow e^- \mu^+$ shows deviations from the analytic values with a numerical to analytic ratio of $\sim 1.5$. We believe this deviation stems from inconsistencies in the helicity sum of our program and will investigate this effect further. In the future, we will be correcting these deviations and testing more complex SM processes as well as some BSM theories.

Keywords
neutrino, beyond the standard model, leptonic tensor, event generator

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AUTOMATIC LEPTONIC TENSOR GENERATION
FOR BEYOND THE STANDARD MODEL (BSM) THEORIES

A Thesis
Submitted to the
Department of Physics and Astronomy
of
Macalester College
by
Diego Fabrizio Lopez Gutierrez

In Partial Fulfillment of the
Requirements for the Degree
of
Bachelor of Arts with Honors

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To my family and friends without whom I could not be here today, to my long-term partner, Isabel, whose love and support has been fundamental to my success, and more recently, to our puppy Oliver, who is always so happy to see me and brings balance to my life. I love you all.
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ABSTRACT


With the development of the Deep Underground Neutrino Experiment (DUNE) and Tokai-to-Hyper-Kamiokande (T2HK), we are entering the era of high-precision neutrino measurements. The colossal output of data from DUNE, plus the current data from several other neutrino experiments, will require a fast and efficient method of testing our BSM models in event generators. However, current methods for implementing a BSM theory in the event generators are prone to errors and time-consuming. We propose a novel program capable of automatically calculating the leptonic tensor for a given quantum field theory Lagrangian. This program is written in Python and utilizes the Universal FeynRules Output (UFO) format, the Lark package, and the Berends-Giele recursive relations to produce leptonic tensors that can be automatically implemented in several neutrino event generators, including those relevant for DUNE. For this project, we tested our algorithm with three SM processes: $e^- p^+ \rightarrow e^- p^+$, $\nu_e p^\mu \rightarrow e^- \mu^+$ and $\nu_e p^+ \rightarrow \nu_e p^+$. For each process, we calculated the numerical and analytic $|M|^2$ and $\sigma$ that we plotted as functions of $\cos \theta$ and $E_{CM}$, respectively. The numerical results for $e^- p^+ \rightarrow e^- p^+$ and $\nu_e p^+ \rightarrow \nu_e p^+$ show good agreement with the analytic results with a cross section numerical to analytic ratio of $\sim 1$ and $\sim 0.9$, respectively. The process $\nu_e p^\mu \rightarrow e^- \mu^+$ shows deviations from the analytic values with a numerical to analytic ratio of $\sim 1.5$. We believe this deviation stems from inconsistencies in the helicity sum of our program and will investigate this effect further. For the future, we will be correcting these deviations and testing more complex SM processes as well as some BSM theories.
1. Introduction

The Standard Model (SM) is our most accurate physics theory capable of describing three of the four known fundamental forces of nature along with their corresponding particles. However, the SM is an incomplete theory as it does not include gravity and it fails to explain dark matter, dark energy, and a variety of other phenomena. For example, the SM predicts that only left-handed massless neutrinos exist, contradicting experimental evidence of massive neutrinos via neutrino oscillations as reported by the Super-Kamiokande [1], SNO [2] and KamLAND [3] experiments. Since then, several experiments have found anomalies regarding the behavior of neutrino oscillations at short-baselines, hinting at the existence of a fourth type of neutrino that is sterile to any SM interactions [4–10]. To explain the phenomena of neutrino oscillations, the origin of its mass, the existence of a possible sterile neutrino and other interesting experimental evidence, scientists develop Beyond the Standard Model (BSM) theories. However, many BSM processes are too complex to be evaluated by hand. Instead, we rely on event generators such as Genie [11], NuWro [12], NEUT [13], and GiBUU [14,15] to obtain predictions that we can then compare to experimental data.

Within the next decade, we are entering an era of high-precision neutrino studies. The neutrino community will be enriched with colossal amounts of data coming from the Deep Underground Neutrino Experiment (DUNE) [16] and the Tokai-to-Hyper-Kamiokande (T2HK) [17] collaborations. The unprecedented number of neutrino events coming from these two experiments, plus the data that we already have from experiments such as MicroBooNE [18], will allow for the testing of several BSM theories. However, the current method of manually implementing a BSM theory into an event generator is inadequate. The manual implementation process is prone to errors due to the different code conventions of each event generator, which inevitably lead to human errors, and is time-consuming given that the user has to repeat all the
work for each BSM model. Due to these setbacks and because of the large number of theories to be tested, this current process becomes infeasible. Instead, we propose an algorithm that automates the testing process.

Event generators calculate Feynman diagram amplitudes from a set of input momenta. These amplitudes are closely related to experimental observables such as decay widths and cross sections. For the neutrino interactions we are considering, we can always decompose the amplitudes into two quantities: the leptonic and the hadronic tensor. Any effects of BSM physics that are present in the hadronic tensor would be discovered by experiments such as the Large Hadron Collider (LHC) before being detected at DUNE or T2HK. Consequently, for DUNE and T2HK, it is most useful to focus on analyzing the leptonic tensor instead. We propose a novel program that automates the implementation of BSM theories in event generators by automatically calculating the leptonic tensor of the theory given its Lagrangian. Moreover, our algorithm can be easily interfaced to several neutrino event generators. The program relies on the Universal FeynRules Output (UFO) file [19] as well as the Lark package [20] and the Berends-Giele algorithm [21]. Before we dive into the details of the program, let us review some particle physics and quantum field theory concepts. The following background information is based on Refs. [22–24] as well as this author’s personal notes. For more information, the reader is encouraged to review those textbooks.

1.1 Cross Sections

To study the properties of particles and their interactions, we must rely on experimental observables, quantities that can be measured and that will tell us something about a given event. Among these observables are cross sections and decay widths, which can be measured in particle physics experiments by colliding particles with each other or analyzing their decays. In this section, we will only focus on the cross section.
To study particle collisions, we must understand how particles scatter off each other. Let us briefly review particle scattering theory. Imagine we have a target (i.e. scattering center) and an incident particle with impact parameter $b$. Our incident particle has energy $E$ and, after scattering off the target, will emerge at some scattering angle $\theta$. In general, we can think of particles going through an infinitesimal $d\sigma$ of a cross sectional area and scattering off into an infinitesimal solid angle $d\Omega$ as shown in Fig. 1.1.

![Figure 1.1. Scattering of an incident particle with energy $E$ and impact parameter $b$ on a target. Particle goes through infinitesimal area $d\sigma$ and scatters off into a solid angle $d\Omega$. Figure retrieved from Scattering Theory, Lect. 20 in Advanced Quantum Mechanics (2009), Department of Physics, University of Cambridge.](image)

The infinitesimal cross section $d\sigma$ and the infinitesimal solid angle $d\Omega$ are related by a proportionality factor $\frac{d\sigma}{d\Omega}$ called the differential cross section. The total cross section $\sigma$ will be given by

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega.$$  

While this equation might seem trivial, it is useful because, in general, we can find the differential cross section more easily than we can find the total cross section. For a non-relativistic case, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$
where $f(\theta)$ is called the scattering amplitude. This amplitude tells us the probability that our particle will scatter in a given direction $\theta$.

We can think of a cross section as the effective area that the target particle presents to the incident particle. Each type of scattering event has its own cross section associated with it. For example, we could have the events $e^-e^+ \rightarrow \mu^-\mu^+$ and $e^-e^+ \rightarrow e^-e^+$ and each would have its own cross section $\sigma(e^-e^+ \rightarrow \mu^-\mu^+)$ and $\sigma(e^-e^+ \rightarrow e^-e^+)$. In quantum field theory, the differential cross section is also related to a scattering amplitude $\mathcal{M}$ via

$$\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2.$$  

More generally, we have that for a process with two initial particles $A_1, A_2$ and $n$ final particles $B_1, B_2, \cdots, B_n$, the cross section is given by

$$\sigma(A_1A_2 \rightarrow B_1B_2\cdots B_n) = \frac{S}{2\sqrt{\lambda(E_{CM}^2, m^2_{A_1}, m^2_{A_2})}} \int d\Pi_n |\mathcal{M}|^2; \hspace{1cm} (1.1)$$

where $|\mathcal{M}|^2$ is the square of the amplitude, $\lambda(E_{CM}^2, m^2_{A_1}, m^2_{A_2}) = E_{CM}^4 + m^4_{A_1} + m^4_{A_2} - 2(E_{CM}^2 m^2_{A_1} + m^2_{A_1} m^2_{A_2} + E_{CM}^2 m^2_{A_2})$ is the Källén function with $E_{CM}$ as the center-of-mass energy, $S$ is a symmetry factor and $d\Pi_n$ is the $n$–dimensional phase space given by

$$\int d\Pi_n = \int (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_{A_i} - \sum_{i=1}^{n} p_{B_i} \right) \prod_{j=1}^{n} \frac{1}{2E_{B_j}} \frac{d^3 \vec{p}_{B_j}}{(2\pi)^3}. \hspace{1cm} (1.2)$$

To understand what $\mathcal{M}$ is and how to calculate it, we must rely on a powerful tool of particle physics: Feynman diagrams. But before delving into Feynman diagrams, let us study Lagrangian mechanics as this will be a useful topic when dealing with $\mathcal{M}$.

1.2 Lagrangian Formalism

In an introductory physics class, students usually learn the principles of mechanics using Newton’s Laws of Motion. We call this method of doing physics the Newtonian formalism. However, while this method is true and works for all of the problems encountered in such classes, it is not the only one. For more complex problems, it
becomes easier to use an alternative method: the Lagrangian formalism or Lagrangian mechanics.

In Lagrangian mechanics, we utilize a quantity, the Lagrangian $L$, to calculate the equations of motion of our system. This formalism is based on the principle of least action, which claims that the path that a system (e.g. particle) follows is the one for which the variation of a quantity $S$, called the action, is minimized. The principle of least action is expressed as

$$\delta S = 0.$$ 

This action is related to the Lagrangian of the system via the following relation

$$S = \int_{t_i}^{t_f} L dt.$$ 

For classical processes such as the one seen in introductory physics classes, the Lagrangian can be expressed as

$$L = T - V,$$

where $T$ is the total kinetic energy of the system and $V$ is the potential energy. However, the Lagrangian is more commonly expressed as a function of some generalized coordinates $q$ and $\dot{q}$, so $L = L(q, \dot{q}; t)$. A common example would be to have $q$ correspond to position $x$ and $\dot{q}$ to velocity $v$, although this relation is not always true. Applying the principle of least action $\delta S = 0$ to our equation for $S$ in terms of $L$, we arrive at the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L(q, \dot{q}; t)}{\partial \dot{q}} \right) - \frac{\partial L(q, \dot{q}; t)}{\partial q} = 0,$$ 

where we get one Euler-Lagrange equation for each generalized coordinate $q$. While the Lagrangian formalism described so far is powerful in dealing with a variety of physical processes, it is so far limited only to classical and discrete systems. When studying the quantum field theories of particles, we are dealing with continuous fields permeating all of spacetime. We must therefore derive a Lagrangian formalism that merges quantum mechanics and relativity and that works for continuous fields. For
such quantum fields, we can express the Lagrangian $L$ as the spatial integral of a Lagrangian density $\mathcal{L}$

$$L = \int \mathcal{L} d^3x.$$  

Similarly to the discrete Lagrangian $L$, the Lagrangian density $\mathcal{L}$ is a function of one or more fields $\phi(x)$ and their derivatives $\partial_\mu \phi(x)$, so $\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$. Greek indices (e.g. $\mu$) run from 0 to 3 and represent the time (0) and spatial (1-3) dimensions.

The action $S$ can now be expressed in terms of this Lagrangian density

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x.$$  

Our integral for the action is now over all of spacetime. If we apply the principle of least action again on our equation above, we arrive at the Euler-Lagrange equation of motion for a field

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$  \hspace{1cm} (1.4)

For simplicity, I will refer to the Lagrangian density $\mathcal{L}$ as simply the Lagrangian from now on. The Lagrangian $\mathcal{L}$ of a quantum field theory contains all the information about the particles and its interactions. As an example, let us analyze the photon field.

### 1.2.1 The Maxwell Lagrangian

Maxwell’s equations describe the behavior of electric and magnetic fields in the presence of an electric charge density, $\rho(x)$, and a current density, $\vec{j}(x)$. These equations form the basis for the study of electromagnetic phenomena

$$\nabla \cdot \vec{E} = \rho,$$  \hspace{1cm} (1.5a)

$$\nabla \cdot \vec{B} = 0,$$  \hspace{1cm} (1.5b)

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0,$$  \hspace{1cm} (1.5c)

$$\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \vec{j}.$$  \hspace{1cm} (1.5d)
However, for a relativistic theory, we would prefer to use a quantity that is Lorentz invariant. Therefore, let us define the 4-vector potential

\[ A^\mu(x) = (A^0(x), \vec{A}(x)) = (V(x), \vec{A}(x)), \]

where \( V(x) \) is the electric potential and \( \vec{A}(x) \) is the magnetic vector potential. We can recover our electric and magnetic fields via the following relations

\[ \vec{B} = \vec{\nabla} \times \vec{A}, \]
\[ \vec{E} = -\vec{\nabla} V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}. \]

Using vector calculus identities, it is easy to verify that \( V \) and \( \vec{A} \) automatically satisfy Maxwell’s equations \[1.5b\] and \[1.5c\]. It is customary in quantum field theory to set some fundamental constants of nature to 1 to simplify our formulas. We say that we are working with natural units. For our present case and henceforth, we will set \( c = \hbar = \epsilon_0 = \mu_0 = 1 \). That is, all measurements will be in energy units. To recover SI units, we must only multiply by the corresponding factors of the aforementioned quantities such that our dimensional analysis works out.

Let us now define the electromagnetic field strength tensor

\[ F^{\mu\nu} \equiv \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \] (1.6)

To calculate this quantity, notice that

\[ F^{i0} = -\nabla^i V^j - \partial_i \vec{A} = E^i, \]
\[ F^{ij} = -\nabla^i A^j + \nabla^j A^i = -\epsilon^{ijk} B^k \]

where latin indices run from 1 to 3, and \( \epsilon^{ijk} \) is the 3-dimensional Levi-Civita symbol (also known as the antisymmetric symbol.) We get that the EM field strength tensor is given by

\[ F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}. \]
Notice now the result of differentiating $F^{\mu\nu}$,

$$\partial_\mu F^{\mu0} = \nabla^i E^i = \vec{\nabla} \cdot \vec{E} = \rho,$$

$$\partial_\mu F^{\mu i} = \partial_t (-E^i) + \nabla^j (-\epsilon^{ijk} B^k) = (\vec{\nabla} \times \vec{B})^i - \partial_t E^i = j^i.$$  

We recovered Maxwell equations 1.5a and 1.5d. To tidy up our equations, let us define now the electric 4-current

$$j^\mu(x) = (j^0(x), \vec{j}(x)) = (\rho(x), \vec{j}(x)).$$

We can then express Maxwell equations in terms of this 4-current and the EM field strength tensor

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x). \quad (1.7)$$

Contained within this expression and the definition of $F^{\mu\nu}$ are all four of Maxwell equations in 1.5. Moreover, if we take the derivative of both sides of this equation, we get the electric charge conservation equation from electromagnetism. Note from our explicit formula for $F^{\mu\nu}$ that $F^{\mu\nu} = -F^{\nu\mu}$. This antisymmetry implies that

$$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\mu \partial_\nu F^{\nu\mu} = 0.$$  

Then

$$\partial_\nu \partial_\mu F^{\mu\nu}(x) = \partial_\nu j^\nu(x)$$

$$0 = \partial_\nu j^\nu(x)$$

$$0 = \frac{\partial}{\partial t} \rho(x) + \vec{\nabla} \cdot \vec{j}(x). \quad (1.8)$$

Within a quantum field theoretical framework, $A^\mu(x)$ is the photon field operator. The Lagrangian corresponding to the photon is given by

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu.$$  

Using the Euler-Lagrange equation 1.4 we recover Maxwell equations as expressed in 1.7. Before delving into more advanced quantum field theories, let us briefly review the Dirac equation and its corresponding Lagrangian.
1.2.2 The Dirac Lagrangian

The Dirac equation, proposed in 1928 by Paul Dirac [25], is a relativistic wave equation for free spin-1/2 particles with mass \( m \). The Standard Model particles with spin-1/2 are the electron \((e^-)\), the muon \((\mu^-)\), the tau \((\tau^-)\), their corresponding neutrinos \((\nu_e, \nu_\mu, \nu_\tau)\) and all six of the quarks \((u, d, c, s, t, b)\). We call these particles fermions. Their corresponding antiparticles, the antifermions, are also spin-1/2 particles. The Dirac equation applies to all of these particles. The Dirac equation is

\[
(i\gamma^\mu \partial_\mu - m)\psi(x) = 0,
\]

where \(\psi(x)\) is the Dirac wavefunction or Dirac field operator (“spinor”) when working within quantum field theory, \(m\) is the mass of the particle, \(\partial_\mu = (\partial_t, \vec{\nabla})\) is the 4-derivative, and \(\gamma^\mu\) are the Dirac matrices that satisfy the Dirac anticommutation algebra

\[
\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}.
\]

There is a set of 4 matrices that satisfy these relations for all dimensions \(n \geq 4\). Moreover, the Dirac matrices are not unique and one can go from one basis of the Dirac matrices to another via unitary transformations. In this paper, we will use the Weyl representation of the Dirac matrices given by

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},
\end{align*}
\]

where \(I_{2 \times 2}\) is the 2-dimensional identity matrix and \(\sigma^i\) are the Pauli matrices for \(i = 1, 2, 3\). Then \(\gamma^\mu = (\gamma^0, \vec{\gamma})\). It is worthwhile to point out some properties of the Dirac matrices as well as some useful notation. The product of a 4-vector \(b_\mu\) with the Dirac matrices \(\gamma^\mu\) (i.e. \(b_\mu \gamma^\mu\)) appears frequently in our equations, so we use \(\slashed{b}\) to denote \(b_\mu \gamma^\mu\). This notation is called Feynman slash notation. The Dirac equation in slash notation is then \((i\slashed{\partial} - m)\psi(x) = 0\). The Dirac matrices also have the following properties:

\[
(\gamma^0)^2 = -(\gamma^i)^2 = I_{4 \times 4},
\]

(1.10a)
where $\dagger$ represents the Hermitian conjugate, and $*$ represents the complex conjugate.

One crucial aspect of the Dirac equation was that it predicted positive and negative energy solutions, corresponding to particles and antiparticles. In reality, both particles and antiparticles have positive energy; we instead refer to the solutions of the Dirac equation as positive and negative frequency solutions. The positive frequency solutions of the Dirac equation are of the form

$$\psi(x) = u(p)e^{-ip \cdot x}, \quad \text{with} \quad p^2 = m^2, \quad p_0 > 0,$$

where $p$ is the 4-momentum of the particle, $m$ is its mass and $u(p)$ is the momentum-space Dirac spinor. Note that $i\partial_\mu$ is the 4-momentum operator and $i\partial_\mu e^{-ip \cdot x} = p_\mu e^{-ip \cdot x}$. We can then get the Dirac equation in momentum space:

$$(i\gamma^\mu \partial_\mu - m)u(p)e^{-ip \cdot x} = 0$$

$$(\gamma^\mu p_\mu - m)u(p)e^{-ip \cdot x} = 0$$

$$(\gamma^0 - m)u(p) = 0.$$  

The spinor $u(p)$ has two linearly-independent solutions $u^s(p)$ where $s = 1, 2$ represents the two possible spins of the particle. The negative frequency solutions of the Dirac equation are of the form

$$\psi(x) = v(p)e^{ip \cdot x}, \quad \text{with} \quad p^2 = m^2, \quad p_0 > 0,$$

where $v(p)$ is another momentum-space Dirac spinor. Similarly, this spinor $v(p)$ also has two linearly-independent solutions $v^s(p)$. The corresponding momentum-space Dirac equation for this spinor is

$$(\gamma^0 + m)v(p) = 0.$$  

When we take the Hermitian conjugate of the Dirac equation, we get

$$\bar{\psi}(i\bar{\gamma}^\mu + m) = 0,$$
where $\bar{\psi} = \psi^\dagger \gamma^0$ is called the adjoint Dirac spinor and where the 4-momentum operator $(i\partial_\mu)$ is acting to the left. Like before, this equation also has positive and negative frequency solutions. The positive frequency solutions are of the form

$$\bar{\psi}(x) = \bar{v}(p)e^{-ip \cdot x}, \quad \text{with} \quad p^2 = m^2, \quad p_0 > 0,$$

where $\bar{v}(p) = v^\dagger(p)\gamma^0$ is the adjoint momentum-space Dirac spinor. This equation also depends on the spin $s = 1, 2$ like before and its corresponding momentum-space adjoint Dirac equation is

$$\bar{v}(p)(\not{p} + m) = 0.$$

Finally, we have the negative frequency solutions of the form

$$\bar{\psi}(x) = \bar{u}(p)e^{ip \cdot x}, \quad \text{with} \quad p^2 = m^2, \quad p_0 > 0,$$

where $\bar{u}(p) = u^\dagger(p)\gamma^0$ is the adjoint momentum-space Dirac spinor with the usual spin dependence. Its corresponding momentum-space adjoint Dirac equation is

$$\bar{u}(p)(\not{p} - m) = 0.$$

When dealing with quantum field theory, the Dirac spinors $\psi$ and $\bar{\psi}$ are promoted to field operators. These field operators create and annihilate particles. The operator $\psi$ annihilates fermions and creates antifermions. The operator $\bar{\psi}$ annihilates antifermions and creates fermions. We are now ready to tackle the Dirac Lagrangian.

The Lagrangian that gives the Dirac equation is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(x)(i\partial - m)\psi(x), \quad (1.11)$$

where $\psi(x)$ is the position-space Dirac spinor and $\bar{\psi}(x)$ is the adjoint position-space Dirac spinor. Let $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. Notice that

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu \psi)$$

$$= (im\bar{\psi})\psi + \bar{\psi}(-im\psi)$$

$$= 0.$$
This equation for $j^\mu$ looks similar to the charge conservation equation of electromagnetism, where $j^\mu$ was the electromagnetic 4-current. Indeed, $j^\mu = \bar{\psi} \gamma^\mu \psi$ is the 4-current of electromagnetism but without a factor of $e$ for the electric charge. We are ready to delve into Quantum Electrodynamics.

1.2.3 The Quantum Electrodynamics Lagrangian

Quantum Electrodynamics (QED) is the quantum field theory that governs the electromagnetic interactions via the exchange of the photon $\gamma$. Before generalizing for all fermions, let us consider the Lagrangian of a universe with only photons and electrons. The QED Lagrangian for this universe would be

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{Dirac}}$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \bar{\psi} (i\partial - m) \psi$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\partial - m) \psi - Q_e e \bar{\psi} \gamma^\mu \psi A_\mu,$$

where $e$ is the elementary charge and $Q_e$ is the electric charge of the electron in units of $e$ (i.e. $Q_e = -1$). The term $-Q_e e \bar{\psi} \gamma^\mu \psi A_\mu$ contains the electron ($\psi$), positron ($\bar{\psi}$) and photon ($A_\mu$) fields and it represents the interaction between our photon and our electron/positron. The corresponding Dirac equation in the presence of a photon field is then

$$(i\partial - Q_e e \sigma - m) \psi(x) = 0.$$}

We can generalize the QED Lagrangian for all fermions/antifermions by taking into account the different charges

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{\text{all fermions}} \bar{\psi}_f (i\partial - Q_f e \sigma - m_f) \psi_f,$$  \hspace{1cm} (1.12)

where $m_f$ is the fermion mass (e.g. $m_e$ for the electron), $Q_f$ is the fermion electric charge (e.g. $-1$ for the electron) and $\psi_f, \bar{\psi}_f$ are the corresponding fermion/antifermion Dirac spinors. This QED Lagrangian is the blueprint of all SM electromagnetic interactions in the universe. Let us now study Feynman diagrams.
1.3 Feynman Diagrams

In Sec. 1.2.3, we saw that the QED Lagrangian for the electron/positron and the photon contains an interaction term $L_{\text{int}} = -Q_e \bar{\psi} \gamma^\mu \psi A^\mu$. This interaction Lagrangian is related to another quantity called the interaction Hamiltonian via the relationship

$$H_{\text{int}} = \int d^4x (-L_{\text{int}}).$$

To find more information about our interacting theory (i.e. QED), we could try to find the eigenvalues and eigenvectors of this interaction Hamiltonian. It turns out, however, that there are no exactly solvable interacting field theories for more than two spacetime dimensions. Our best solution is to find an approximation by expanding our interaction Hamiltonian in a perturbation series.

Let us now go back to our results from Sec. 1.1. We had asserted that the differential cross section of a particle scattering event was proportional to $|\mathcal{M}|^2$. We will now connect $\mathcal{M}$ to $H_{\text{int}}$. Without loss of generality, suppose that we have an event with two initial particles $A$ and $B$ and two final particles $1$ and $2$, so $A + B \rightarrow 1 + 2$. Let our particles have momenta given by $\vec{p}_i$ where $i = A, B, 1, 2$. Our initial and final states in momentum space would be given by

$$\langle \vec{p}_1 \vec{p}_2 | \text{ and } | \vec{p}_A \vec{p}_B \rangle.$$ 

The probability amplitude that this reaction occurs is given by how much these states overlap with one another; that is, it is given by the inner product of these two states

$$\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle.$$ 

This product can also be expressed as follows

$$\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle = \langle \vec{p}_1 \vec{p}_2 | \lim_{t \to \infty} e^{-iH(2t)} | \vec{p}_A \vec{p}_B \rangle,$$
where $H$ is the Hamiltonian of our system (including $H_{\text{int}}$). Expanding the exponential term, we define the transition matrix $T$ as follows

$$1 + iT = \lim_{t \to \infty} e^{-iH(2t)}$$

$$1 + iT = 1 + \lim_{t \to \infty} \left( (-iH(2t)) + \frac{1}{2!}(-iH(2t))^2 + \cdots \right)$$

$$iT = \lim_{t \to \infty} \left( (-iH(2t)) + \frac{1}{2!}(-iH(2t))^2 + \cdots \right).$$

The 1 in the equation above reflects the probability that the two initial particles and the two final particles do not interact at all. We are interested in the $iT$ term, which represents the probability that the particles interact with one another. It can be shown that

$$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{t \to \infty (1 - i\epsilon)} \left( 0 \langle \vec{p}_1 \vec{p}_2 | \exp \left[ -i \int_{-t}^{t} dt' H_{\text{int}} \right] | \vec{p}_A \vec{p}_B \rangle \right)_{\text{connected, amputated}}. \quad (1.13)$$

The reader does not need to worry about the strange new details surrounding this equation. The major takeaway from this equation is the fact that we can express $\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle$ as a power series of the interaction Hamiltonian $H_{\text{int}}$ (recall Taylor expansion of exponential function). Similarly, it can also be shown that

$$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle = (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2) \cdot iM(p_A, p_B \rightarrow p_1, p_2), \quad (1.14)$$

where $M$ is the amplitude discussed in Sec. 1.1 for the current event $A + B \rightarrow 1 + 2$. Comparing Eqs. 1.13 and 1.14, we see that $M$ can also be expressed as a power series of the interaction Hamiltonian $H_{\text{int}}$.

When we expand $M$ in terms of $H_{\text{int}}$, we get terms that are proportional to a coupling constant $\alpha$. For QED, this constant is given by $\alpha = \frac{e^2}{4\pi}$. For simplicity, we will express our expansion in terms of powers of $e$. Thus:

$$M = M_0(e^0) + M_1(e^1) + \cdots$$

where each term $M_j$ would be proportional to $0 \langle \vec{p}_1 \vec{p}_2 | \left[ -i \int_{-t}^{t} dt' H_{\text{int}} \right]^j | \vec{p}_A \vec{p}_B \rangle$. Each term $M_j$ in this perturbative expansion of $M$ corresponds to a spacetime process.
We represent this process via a Feynman diagram. As an example, let us consider the process \( e^+e^- \to \mu^+\mu^- \). A generic representation of this event is given in Fig. 1.2.

\[ e^- \quad p_1 \quad p_3 \quad \mu^+ \]
\[ p_2 \quad e^+ \quad p_4 \quad \mu^- \]

Figure 1.2. Schematic representation of \( e^+e^- \to \mu^+\mu^- \) for particles with 4-momenta \( p_1, p_2, p_3, p_4 \). The blob represents the particle physics that occurs to turn \( e^+e^- \) into \( \mu^+\mu^- \).

In this figure, the blob represents the different possible processes that could be happening between the electron/positron pair and the muon/antimuon pair. We can think of the blob as related to \( \mathcal{M} \). Each term \( \mathcal{M}_j \) in the expansion of \( \mathcal{M} \) corresponds to a possibility of what is happening within the blob.

As we mentioned, each \( \mathcal{M}_j \) is related to \( j \) copies of \( H_{\text{int}} = \int d^3x (-\mathcal{L}_{\text{int}}) \). For QED, we had that

\[ \mathcal{L}_{\text{int}} = \sum_{\text{all fermions}} -Q_f \bar{\psi}_f \gamma^\mu \psi_f A_\mu. \]

For our process \( e^+e^- \to \mu^+\mu^- \), we are only dealing with electrons, muons and their antiparticles. Let us then only focus on the terms (recall \( Q_e = Q_\mu = -1 \))

\[ \mathcal{L}_{\text{int}} = \bar{\psi}_f \gamma^\mu \psi_f A_\mu; \quad f = e, \mu. \]

The first term in \( \mathcal{M} \) is that for which there is no interaction Hamiltonian (and hence no interaction Lagrangian) mediating the process \( e^+e^- \to \mu^+\mu^- \). As we can see from the QED Lagrangian, there is no term that spontaneously transforms an electron/positron pair into a muon/antimuon pair. Such a term would be of the form...
Therefore, the value of $\mathcal{M}_0$ vanishes. Let us now consider $\mathcal{M}_1$, where we have 1 copy of $H_{\text{int}} \propto \mathcal{L}_{\text{int}}$. In this case, we again do not get a term that contains all four of the fermion fields; $\mathcal{M}_1$ also vanishes. However, for $\mathcal{M}_2$, we do have a process that would produce our desired event. Because we have $H^2_{\text{int}}$, we will get a term of the form $e^2 \bar{\psi}_e \gamma^\mu \psi_e A_\mu \bar{\psi}_{\mu} \psi_{\mu}$. Namely, we have an event in which the electron/positron pair produces a photon ($e \bar{\psi}_e \gamma^\mu \psi_e A_\mu$) and this photon then produces a muon/antimuon pair ($e \bar{\psi}_{\mu} \gamma^\mu \psi_{\mu} A_\mu$). The photon created with the electron/positron pair and annihilated with the muon/antimuon pair is undetectable and we say that it is a virtual photon. For this case, the photon will propagate from the creation vertex to the annihilation vertex with 4-momentum $q$ satisfying momentum conservation. That is, $q = p_1 + p_2 = p_3 + p_4$. In general, the virtual photon need not propagate from creation to annihilation vertex; for example, the photon could be created at the electron/positron vertex and propagate until it decays into muons before reaching the annihilation vertex. Our process is depicted in the Feynman diagram shown in Fig. 1.3. Each vertex contributes a factor of $e$. Thus, the diagram shown in Fig. 1.3 corresponds to the term $\mathcal{M}_2(e^2)$ from the perturbation expansion of $\mathcal{M}$.

![Figure 1.3. Tree-level Feynman diagram of $e^+e^- \rightarrow \mu^+\mu^-$ for particles with 4-momenta $p_1, p_2, p_3, p_4$.](https://digitalcommons.macalester.edu/mjpa/vol9/iss1/7)
Similarly, there are other Feynman diagrams involving higher order terms of $\mathcal{M}$. The amplitude $\mathcal{M}$ is a superposition of all these Feynman diagrams.

\[
\mathcal{M} = e^- \gamma \mu^- + e^- e^+ \mu^- + e^- \mu^- e^+ + e^- e^+ \mu^- + \ldots
\]

However, higher order terms also involve higher powers of $e$ and contain loop factors, both of which suppress the diagram’s contribution to the power series. Considering only the first non-zero term is usually a good approximation to the value of $\mathcal{M}$. We say that Feynman diagrams are tree-level diagrams if they correspond to the lowest non-zero term of the perturbation expansion. It is worth pointing out that a process can have more than one Feynman diagram at any order. In the previous equation for $\mathcal{M}$, for example, we have that $\mathcal{M}_4(e^4)$ is made up of five Feynman diagrams. Throughout this paper, we will only focus on tree-level calculations, some of which will involve more than one Feynman diagram.

1.3.1 Feynman Rules

As we mentioned in Sec 1.1, the squared amplitude $|\mathcal{M}|^2$ is proportional to the observables (e.g. cross section) that we are interested in. We must, therefore, be able to calculate $|\mathcal{M}|^2$ (and hence $\mathcal{M}$) if we wish to perform any useful analysis. To calculate the amplitude $\mathcal{M}$ corresponding to a diagram, we utilize the Feynman rules. These rules are obtained from the Lagrangian of our theory. For a thorough
derivation of the Feynman rules for QED and other theories, the reader is encouraged to refer to Ref. [22]. Here are (some of) the rules for QED:

Initial fermion: \[ f \rightarrow p = u^*(p) \]

Initial antifermion: \[ \bar{f} \rightarrow p = \bar{v}^*(p) \]

Final fermion: \[ f \rightarrow p = \bar{u}^*(p) \]

Final antifermion: \[ \bar{f} \rightarrow p = v^*(p) \]

Photon propagator: \[ \mu \sim \gamma \sim \nu \rightarrow q = -i g_{\mu\nu} / q^2 \]

Photon-fermion/antifermion vertex: \[ = i Q_f e \gamma^\mu \]

where the initial and final descriptions refer to external (real) particles and the propagator description refers to internal (virtual) particles. The superscript of the Dirac spinors refers to the spin of the particle. For the vertex, we have let \( e = |e| \) and we will keep this convention from now on. One can notice from the Feynman rules that the QED Lagrangian played a crucial role, most notably in the expressions for the external particles (\( L_{\text{Dirac}} \)) and the QED vertex (\( L_{\text{int}} \)). The propagator, whether for the photon or for fermions, comes from the inverse of the Dirac and Maxwell equations. For the photon, this inverse is ill-defined, which requires the extra addition of the metric tensor to fix the (Feynman) gauge. Using these rules, we can express the tree-level diagram of Fig. 1.3 for the process \( e^+ e^- \rightarrow \mu^+ \mu^- \) in analytical form. For this diagram, we have an initial electron, an initial positron, a final muon, a final
antimuon, a photon propagator and two photon-fermion/antifermion vertices. The amplitude for this diagram is thus given by

\[ M = v^\dagger(p_2) (-ie)\gamma^\mu u^\dagger(p_1) \frac{-ig_{\mu\nu}}{q^2} \bar{v}^\dagger(p_3) (-ie)\gamma^\nu \bar{u}^\dagger(p_4). \] (1.15)

We will stop here for now, but we will calculate \( M \) and \(|M|^2\) in Sec. 1.4.

### 1.4 Hadronic and Leptonic Tensor

For neutrino interactions, we can express the squared amplitude as

\[ |M|^2 = L_{\mu\nu} H^{\mu\nu}, \] (1.16)

where \( L_{\mu\nu} \) is called the leptonic tensor and \( H^{\mu\nu} \) is called the hadronic tensor. Eq. (1.16) is valid if we neglect double boson exchange processes; that is, processes where we have two internal virtual bosons. In Ch. 3 we will only focus on events for which we can assume no double boson exchange. This is reasonable given the uncertainties associated with nuclear effects coming from protons and neutrons, which are much larger than the loop corrections from which these double boson exchanges come from.

Moreover, we will also assume point-like nuclear particles for simplicity. To illustrate how \(|M|^2|\) can be split into these tensors, let us finish our calculation of \(|M|^2|\) for the process \( e^+e^- \rightarrow \mu^+\mu^- \).

To get \(|M|^2|\), we must have \( M \) and \( M^\ast \). Let us analyze Eq. (1.15) in more detail. Recall the definition of the Dirac spinors \( u, \bar{u}, v, \bar{v} \) from the momentum-space Dirac equations in Sec. 1.2. The spinors \( u \) and \( v \) are the positive and negative frequency solutions to \((i\slashed{\partial} - m)\psi = 0\). The position-space Dirac spinor \( \psi \) is a 4-dimensional column vector living in Dirac space. In terms of the momentum-space solutions, \( \psi = u(p)e^{-ip\cdot x} \) or \( \psi = v(p)e^{+ip\cdot x} \). Notice that \( \psi \) is related to both \( u \) and \( v \) only by an exponential factor. Therefore, \( u \) and \( v \) must have the same dimensions as \( \psi \): \( u \) and \( v \) are 4-dimensional column vectors living in Dirac space. Let us now investigate the adjoint Dirac equation \( \bar{\psi}(i\slashed{\partial} + m) = 0 \). The spinors \( \bar{u} \) and \( \bar{v} \) are the negative
and positive frequency solutions to the adjoint Dirac equation. By analyzing the dimensions of the adjoint Dirac equation, one can see that for the equation to make sense, the position-space adjoint Dirac spinor $\bar{\psi}$ must be a 4-dimensional row vector living in the adjoint Dirac space. Notice, again, that $\bar{\psi} = \bar{u}e^{-ip\cdot x}$ or $\bar{\psi} = \bar{v}e^{+ip\cdot x}$. Therefore, $\bar{u}$ and $\bar{v}$ are also 4-dimensional row vectors living in adjoint Dirac space.

With this in mind, let us go back to Eq. 1.15. For simplicity, let us rearrange the amplitude to look like this

$$M = ie^2q^2[\bar{v}^s(p_2)\gamma^\mu u^s(p_1)][\bar{u}^{s'}(p_3)\gamma_\mu v^{s'}(p_4)], \quad (1.17)$$

where we have moved the constant factors to the front and we contracted $g_{\mu\nu}\gamma^\nu = \gamma_\mu$. Notice now the dimensions of the quantities in square brackets. We have a 4-dimensional row vector ($\bar{u}$ or $\bar{v}$), a $4 \times 4$ matrix ($\gamma^\mu$ or $\gamma_\mu$) and a 4-dimensional column vector ($u$ or $v$). Therefore, the quantities in brackets are 1-dimensional complex numbers and the amplitude itself is also a 1-dimensional complex number. Because $M$ is just a complex number, $M^* = (M^\dagger)^T = M^\dagger$. Let us thus calculate $M^\dagger$.

The prefactor of $M$ is easy to calculate and it is just $(ie^2q^2)^* = -ie^2q^2$. Let us now focus on one of the square brackets

$$[\bar{v}^s(p_2)\gamma^\mu u^s(p_1)]^\dagger = u^{s\dagger}(p_1)[\bar{v}^{s'}(p_2)\gamma^\mu]^\dagger$$

$$= u^{s\dagger}(p_1)(\gamma^\mu)^\dagger[\bar{v}^{s'}(p_2)]^\dagger$$

$$= u^{s\dagger}(p_1)(\gamma^\mu)^\dagger[u^{s\dagger}(p_2)\gamma^0]^\dagger$$

$$= u^{s\dagger}(p_1)(\gamma^\mu)^\dagger(\gamma^0)^\dagger u^{s'}(p_2)$$

$$= u^{s\dagger}(p_1)\gamma^0\gamma^\mu\gamma^0\gamma^0 u^{s'}(p_2)$$

$$= \bar{u}^s(p_1)\gamma^\mu u^{s'}(p_2),$$

where we have used the Dirac matrices’ properties given in Eq. 1.10a and Eq. 1.10b.

In a similar fashion, we get that

$$[\bar{u}^{s'}(p_3)\gamma_\mu v^{s'}(p_4)]^\dagger = \bar{v}^{s'}(p_4)\gamma_\mu u^{s'}(p_3).$$

Putting our results together, we get the Hermitian conjugate of the amplitude

$$M^\dagger = -ie^2q^2[\bar{u}^s(p_1)\gamma^\mu u^{s'}(p_2)][\bar{v}^{s'}(p_4)\gamma_\mu v^{s'}(p_3)], \quad (1.18)$$
where we have changed the (dummy) index $\mu$ to $\nu$ to avoid confusion with $\mathcal{M}$. Since $\mathcal{M}^\dagger = \mathcal{M}^*$, we can put together Eq. 1.17 and Eq. 1.18 to get the squared amplitude

$$|\mathcal{M}|^2 = \frac{e^4}{q_1^4} [\bar{u}^s(p_2)\gamma^\mu u^s(p_1)] [\bar{v}^s(p_1)\gamma^\nu v^s(p_2)] [\bar{v}^r(p_3)\gamma_\mu v^r(p_4)] [\bar{u}^r(p_4)\gamma_\nu u^r(p_3)],$$

(1.19)

where we have moved around the quantities in square brackets since they are just complex numbers. This squared amplitude $|\mathcal{M}|^2$ is for a given set of momenta and spins. However, in real experiments, we (usually) have an unpolarized beam of incoming particles. This unpolarized beam means that any measurements are an average over the initial state spins $s$ and $s'$. After the process takes place, the detectors measure the aggregated results of the interactions without differentiating between different final spin states. Therefore, any measurements are over a sum of the final state spins $r$ and $r'$. With this in mind, we get that the actual quantity we are looking for is

$$\frac{1}{4} \sum_s \frac{1}{2} \sum_{s'} \sum_r \sum_{r'} |\mathcal{M}|^2 = \frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2,$$

which is given by

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q_1^4} \sum_{s,s',r,r'} [\bar{u}^s(p_2)\gamma^\mu a^s(p_1)] [\bar{u}^s(p_1)\gamma^\nu b^s(p_2)] \times [\bar{v}^r(p_3)\gamma_\mu v^r(p_4)] [\bar{v}^r(p_4)\gamma_\nu u^r(p_3)].$$

We can further simplify this expression for $\frac{1}{4} \sum |\mathcal{M}|^2$. First, we will use the spin-sum relations for Dirac spinors.

The Dirac spinors $u, \bar{u}, v, \bar{v}$ follow the completeness (spin-sum) relations given by

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m,$$

(1.20)

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m.$$

(1.21)

To use Eq. 1.20 and Eq. 1.21, let us first write the expression for $\frac{1}{4} \sum |\mathcal{M}|^2$ explicitly in index notation

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q_1^4} \sum_{s,s',r,r'} [\bar{u}^s_a(p_2)\gamma^\mu a^s_b(p_1)] [\bar{u}^s_c(p_1)\gamma^\nu b^s_d(p_2)] \times [\bar{v}^r_e(p_3)\gamma_\mu e^f p^r_f(p_4)] [\bar{v}^r_g(p_4)\gamma_\nu g^h p^r_h(p_3)].$$
Now, our Dirac spinors and Dirac matrices are expressed in component form; that is, they are just complex numbers that can be moved around. For example, the quantity \( u_{s}^{\mu}(p_{1}) \) represents one component of the spinor \( u^{\mu}(p_{1}) \). To return to our matrix notation in the end, we will just need to rearrange our results in the correct order. Since we are dealing with numbers only, let us rearrange our expression as follows

\[
\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{4} \sum_{s,s',r,r'} u_{s}^{\mu}(p_{1}) \bar{u}_{d}^{\nu}(p_{2}) \gamma_{ab}^{\mu} \gamma_{cd}^{\nu} \times u_{h}^{\nu}(p_{3}) \bar{u}_{e}^{\lambda}(p_{4}) \gamma_{\mu,ef} \gamma_{\nu,gh}.
\]

We can clearly see the expressions for the spin-sum relations from Eqs. 1.20 and 1.21 in our previous equation. Moreover, notice that only the corresponding spinors (\( u \) or \( v \)) are dependent on their spin, so we can apply our spin-sum relations. We get the following

\[
\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{4} (p_{1}^{\mu} + m_{e})_{bc} (p_{2}^{\nu} - m_{e})_{da} \gamma_{ab}^{\mu} \gamma_{cd}^{\nu} \times (p_{3}^{\mu} + m_{\mu})_{he} (p_{4}^{\nu} - m_{\mu})_{fg} \gamma_{\mu,ef} \gamma_{\nu,gh}.
\]

Now that the summation over the spins is gone, we can rearrange our results in the correct order

\[
\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{4} (p_{1}^{\mu} + m_{e})_{bc} (p_{2}^{\nu} - m_{e})_{da} \gamma_{ab}^{\mu} \times (p_{3}^{\mu} + m_{\mu})_{he} (p_{4}^{\nu} - m_{\mu})_{fg} \gamma_{\mu,ef} \gamma_{\nu,gh}.
\]

\[
\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{4} (p_{1}^{\mu} + m_{e}) (p_{2}^{\nu} - m_{e}) \gamma_{\mu,ab}^{\mu} \times (p_{3}^{\mu} + m_{\mu}) (p_{4}^{\nu} - m_{\mu}) \gamma_{\nu,cd}^{\nu}.
\]

\[
\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{4} \text{Tr} \left( (p_{1}^{\mu} + m_{e}) (p_{2}^{\nu} - m_{e}) \gamma_{\mu} \gamma_{\nu} \right) \times \text{Tr} \left( (p_{3}^{\mu} + m_{\mu}) (p_{4}^{\nu} - m_{\mu}) \gamma_{\mu} \gamma_{\nu} \right).
\]
where in the second to last line, we have used the fact that the trace of a matrix \( A \) is \( \text{Tr} [A] = \sum_i A_{ii} = A_{ii} \) in the Einstein summation convention. Our result now only depends on the momenta and masses of our particles. We can further simplify our results by using trace technology.

The traces of Dirac matrices and their products have certain properties. Collectively, these properties are called trace technology and some of them are given below:

\[
\text{Tr} \left[ \text{odd number of } \gamma^\mu \right] = 0, \quad (1.22)
\]

\[
\text{Tr} \left[ \gamma^\mu \gamma^\nu \right] = 4g^{\mu\nu}, \quad (1.23)
\]

\[
\text{Tr} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \right] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \quad (1.24)
\]

We can use these properties to evaluate the traces in the expression of \( \frac{1}{4} \sum |\mathcal{M}|^2 \). However, before delving into the calculation, let us think about the kinematics of our event. Usually, the center-of-mass energies at which particle accelerators operate are of the order of GeV or even TeV. In comparison, the mass of the electron is \( m_e = 0.511 \text{ MeV} \) and the mass of the muon is \( m_\mu = 106 \text{ MeV} \). When working with particle accelerators of such high energies, we can safely neglect the masses of the electron and the muon; that is, we can set \( m_e = m_\mu = 0 \). This assumption will simplify our calculation of the traces. Our new expression for the amplitude is

\[
\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \text{Tr} \left[ \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu \right] \text{Tr} \left[ \gamma^\mu \right] = \frac{e^4}{q^4} p_{1,\alpha} p_{2,\beta} \text{Tr} \left[ \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu \right] p_3^\rho p_4^\sigma \text{Tr} \left[ \gamma^\rho \gamma^\sigma \gamma^\mu \gamma^\nu \right].
\]

We can now use Eq. 1.24 to simplify both traces. Let us focus on the traces individually before plugging them back into our equation:

\[
p_{1,\alpha} p_{2,\beta} \text{Tr} \left[ \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu \right] = p_{1,\alpha} p_{2,\beta} 4(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\nu\beta})
\]

\[
= 4(p_1^\nu p_2^\mu - (p_1 \cdot p_2)g^{\nu\mu} + p_1^\mu p_2^\nu)
\]

\[
p_3^\rho p_4^\sigma \text{Tr} \left[ \gamma^\rho \gamma^\sigma \gamma^\mu \gamma^\nu \right] = p_3^\rho p_4^\sigma 4(g_{\rho\sigma}g_{\mu\nu} - g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\mu\sigma})
\]

\[
= 4(p_3 \cdot p_4)g_{\mu\nu} + p_3 p_4 g_{\mu\nu}
\]
Plugging our results back in, we get

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \frac{1}{4} 4(p_1^\mu p_2^{\nu'} + p_2^\mu p_1^{\nu'} - (p_1 \cdot p_2) g^{\mu\nu}) \cdot 4(p_3,\mu p_4,\nu + p_4,\mu p_3,\nu - (p_3 \cdot p_4) g_{\mu\nu}),$$

where in the first term, we have replaced $g^{\nu\mu}$ by $g_{\mu\nu}$ since the metric tensor is symmetric under this transformation. Before further simplifying our expression, notice that the first term in parenthesis only depends on the momenta of the incoming electron $(p_1)$ and positron $(p_2)$, and the second term in parenthesis only depends on the momenta of the outgoing muon $(p_3)$ and antimuon $(p_4)$. We can therefore define two quantities that contain the information about the electron/positron and the muon/antimuon pairs. These quantities will be the electron leptonic tensor $L_{\mu\nu}^{\mu\nu}$ and the muon leptonic tensor $L_{\mu-\mu\nu}$. In this paper, we do not include the spin-average factor $\frac{1}{4}$ into our definition of the leptonic tensors. We define them as

$$L_{\mu\nu}^{\mu\nu} = \frac{4e^2}{q^2} (p_1^\mu p_2^{\nu'} + p_2^\mu p_1^{\nu'} - (p_1 \cdot p_2) g^{\mu\nu}),$$

$$L_{\mu-\mu\nu} = \frac{4e^2}{q^2} (p_3,\mu p_4,\nu + p_4,\mu p_3,\nu - (p_3 \cdot p_4) g_{\mu\nu}).$$

And our amplitude is given by

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{1}{4} L_{\mu\nu}^{\mu\nu} L_{\mu-\mu\nu}.$$

For now, the definitions of the leptonic tensor are only for educational purposes since they will not affect our calculation of the amplitude. However, it is important for the reader to notice how we could express the amplitude in terms of these two leptonic tensors. Continuing with our calculation, we can now contract both terms in parenthesis to get the following (recall $g^{\mu\nu} g_{\mu\nu} = 4$)

$$(p_1^\mu p_2^{\nu'} + \cdots)(p_3,\mu p_4,\nu + \cdots) = (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4) +$$

$$(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) -$$

$$(p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_2)(p_3 \cdot p_4) \cdot 4$$

$$= 2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3).$$
And so, our amplitude is
\[
\frac{1}{4} \sum_{s,s',r,r'} |M|^2 = \frac{e^4}{q^4} \cdot 4 \cdot 4 \cdot 2 \cdot ((p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3))
\]
\[
= \frac{8e^4}{q^4}((p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)).
\]
Equation 1.27

The expression in Eq. 1.27 is our final result. The initial-spin averaged and final-spin summed square amplitude is given completely in terms of the four momenta of our particles as well as the momenta \( q \) of the virtual photon. The value of \( q \) is calculated with 4-momentum conservation at the vertices of the Feynman diagram in Fig. 1.3.

The expression for the amplitude as given above is general for any reference frame. However, most of the time, we are only interested in the center-of-mass (CM) frame. In this case, there is one more simplification for Eq. 1.27. The general kinematics of 2-to-2 processes, such as the case for \( e^- e^+ \rightarrow \mu^- \mu^+ \), is given in Fig. 1.4. For massless particles in the CM frame, we have that the general form of the 4-momenta of the particles is given by
\[
p_1 = \frac{E_{CM}}{2} (1, 0, 0, 1),
\]
\[
p_2 = \frac{E_{CM}}{2} (1, 0, 0, -1),
\]
\[
p_3 = \frac{E_{CM}}{2} (1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\]
\[
p_4 = \frac{E_{CM}}{2} (1, -\sin \theta \cos \phi, -\sin \theta \sin \phi, - \cos \theta),
\]
Equation 1.28
where $\theta$ and $\phi$ are the polar and azimuthal angles of our interaction in the CM frame, coming from the spherical coordinates system. As can be seen in Fig. 1.4 there is a cylindrical symmetry around the collision ($z$) axis. This symmetry allows us to always work in a frame where $\phi = 0$. Then, the corresponding dot products would be given by

$$
(p_1 \cdot p_3) = \frac{E_{CM}^2}{4} (1 - \cos \theta),
$$

$$
(p_2 \cdot p_4) = \frac{E_{CM}^2}{4} (1 - \cos \theta),
$$

$$
(p_1 \cdot p_4) = \frac{E_{CM}^2}{4} (1 + \cos \theta),
$$

$$
(p_2 \cdot p_3) = \frac{E_{CM}^2}{4} (1 + \cos \theta).
$$

Moreover, notice that the photon momentum $q$ and its square are given by

$$
q = (p_1 + p_2) = (p_3 + p_4) = E_{CM}(1, 0, 0, 0),
$$

$$
q^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2 = E_{CM}^2.
$$

Plugging back these relations into Eq. 1.27 we get

$$
\frac{1}{4} \sum_{s,s',r,r'} |M|^2 = \frac{8e^4}{E_{CM}^4} \left( \frac{E_{CM}^4}{16} (1 - \cos \theta) \cdot (1 - \cos \theta) + \frac{E_{CM}^4}{16} (1 + \cos \theta) \cdot (1 + \cos \theta) \right)
$$

$$
= \frac{8e^4}{E_{CM}^4} \frac{E_{CM}^4}{16} (1 - \cos \theta)^2 + (1 + \cos \theta)^2
$$

$$
= \frac{e^4}{2} (1 + \cos^2 \theta - 2 \cos \theta + 1 + \cos^2 \theta + 2 \cos \theta)
$$

$$
= \frac{e^4}{2} (2 + 2 \cos^2 \theta)
$$

$$
= e^4 (1 + \cos \theta).
$$

The result in Eq. 1.30 is correct for the CM frame. We will perform similar calculations in Ch. 3 for our validation results. Moreover, we will defer the calculation of the cross section $\sigma$ from $\frac{1}{4} \sum |M|^2$ until then.

Before ending this chapter, let us talk a little more about the leptonic and hadronic tensors. We introduced an analytic version of the leptonic tensor in Eq. 1.25 and...
As we showed, the contraction of these two leptonic tensors gave us the squared amplitude summed over all of the spins (i.e. \( \sum_{s,s',r,r'} |\mathcal{M}|^2 \)). For the cases that we will be analyzing in this work, this is always the case. However, instead of having two leptonic tensors, we will have one leptonic tensor \( L_{\mu\nu} \) and one hadronic tensor \( H_{\mu\nu} \). Conceptually, the only difference between the hadronic and the leptonic tensor is that the leptonic tensor contains the information that pertains to the leptons in the interaction. Similarly, the hadronic tensor contains the information that pertains to the hadrons in the interaction. For example, if we had the following process \( e^- p^+ \rightarrow e^- p^+ \), the leptonic tensor would correspond to the part in the Feynman diagram that contains the incoming and outgoing electron. On the other hand, the hadronic tensor would correspond to the part with the incoming and outgoing proton. Although conceptually similar, the hadronic and leptonic tensors differ in the complexity of their calculations. Whereas the leptonic tensor deals with point-like particles such as electrons and muons, the hadronic tensor deals with hadrons such as protons and neutrons that are made up of quarks and that usually are part of a larger nucleus. This structure means that we must take into account complex nuclear physics and nuclear form factors when dealing with particles of this type. The calculations of hadronic tensors are difficult but can be accomplished with event generators. To avoid the trouble of including calculations of hadronic tensors, our program only calculates the leptonic tensor and leaves the complex nuclear physics of hadronic tensors to the event generators. This separation into leptonic and hadronic tensors is one of the main advantages of our program. We want to separate easily calculable BSM effects from the intricate nuclear effects.
2. Methods

As we mentioned in Ch. 1, we have developed a program that, given a theory’s Lagrangian, automatically produces the leptonic tensor for an event. In this Chapter, we will go over the overall structure and details of our program that allow us to compute leptonic tensors. Let us consider the first step of our program: getting the information of our theory from the Lagrangian.

2.1 Universal FeynRules Output

To perform our amplitude and leptonic tensor calculations, we must be able to access and store the information of our theory. For this, we rely on the Universal FeynRules Output (UFO) format. To obtain the UFO file, we utilize the FeynRules Mathematica package. Given a quantum field theory, FeynRules takes in a text file ModelName.fr that contains the information of the theory such as the particle content, their parameters and the Lagrangian. From this input file, FeynRules will calculate the theory’s Feynman rules (see Sec. 1.3), and will offer the user options to export the theory’s information, including the newly calculated Feynman rules, in different formats. One of these formats is the UFO model file format.

The UFO format is useful because it is “universal.” Other output formats of FeynRules usually involve producing specific text files that must be parsed and interpreted by different tree-level amplitude generators. However, because of their nature, BSM theories are constantly evolving and usually require some extensions to be included in their files. This procedure of editing the files to include these extensions is complicated for a static format such as a text file. Instead, UFO stores its information into Python modules as instances of Python objects defined within UFO. Working within Python classes gives the user more flexibility in the implementation of their BSM
theory. Moreover, unlike other formats, UFO has no prior assumptions on the structures or number of particles that can appear in the model. Because of this lack of assumptions, the UFO file also allows for a larger compatibility with event generators, thus making it “universal.”

Given the Lagrangian of our theory, UFO will export the information into a set of Python files related to the different properties of our theory. Namely, this information is exported to the following six model-dependent files.

- particles.py
- parameters.py
- vertices.py
- couplings.py
- lorentz.py
- coupling_orders.py

Each of these files contains a list of Python instances of their respective objects for the given theory. For example, within the particles.py file, we find a list of all the particles of our theory represented as instances of the Particle class of UFO. An example for the SM electron is given below:

```python
e__minus__ = Particle(pdg_code = 11,
                     name = 'e-',
                     antiname = 'e+',
                     spin = 2,
                     color = 1,
                     mass = Param.ZERO,
                     width = Param.ZERO,
                     texname = 'e-',
                     antitexname = 'e+',
                     charge = -1,
```

GhostNumber = 0,
LeptonNumber = 1,
Y = 0)

The parameters of the particle class are mostly self-descriptive. It is worth pointing out that the spin convention for UFO is $2 \cdot s + 1$, where $s$ is the spin of the particle as given in the theory ($s = 1/2$ for $e^-$). Among the attributes of the electron, we also see that its mass and decay width are given by the object `Param.ZERO`. This object is an instance of the `Parameter` class and can be found within the `parameters.py` file, which contains a list of all the parameters of our theory. The value of `Param.ZERO`, not surprisingly, is 0.

Let us now review an instance of the `Vertex` class. Within the `vertices.py` file, we can find the vertex corresponding to the QED interaction of a photon with an electron/positron ($-ie\gamma^\mu$).

V_77 = Vertex(name = 'V_77',
particles = [ P.e__plus__, P.e__minus__, P.a ],
color = [ '1' ],
lorentz = [ L.FFV1 ],
couplings = {(0,0):C.GC_3})

Let us examine these attributes more carefully. In general, a vertex of $n$ interacting particles can be expressed as the product of a row vector $C$ containing the color tensors, a matrix $G$ containing the couplings, and a column vector $L$ containing the Lorentz structures.

$$V(p_1, \cdots, p_n) = C \times G \times L$$

In this paper, we will not focus on the strong force and its corresponding color structures. Therefore, for all of our calculations, the color tensor will be unity, as is the case for the QED vertex above. That is, we need only care about the couplings and Lorentz structures when calculating vertices. The `couplings` attribute of `V_77` is a dictionary where the keys are tuples and the values are instances of the `Coupling` class.
(a list of all the couplings can be found within the couplings.py file.) The tuples represent coordinates in the coupling matrix \( G \). The \( \text{GC}_3 \) coupling in \( \nu_{77} \) is given by

\[
\text{GC}_3 = \text{Coupling}(\text{name} = \text{\'GC}_3\text{\'}, \\
\text{value} = \text{\'-}(\text{ee}\text{\ast complex}(0,1))\text{\'}, \\
\text{order} = \{\text{\'QED\':}1\} )
\]

where the value \text{'-(ee*complex(0,1))'} is a string that represents \(-ie\). The coupling matrix \( G \) is multiplied to the Lorentz column vector \( L \), so we can let \( G \) be of size \( n \times n \), where \( n \) is the dimension of \( L \). In practice, we would need to account for the size of the color row vector \( C \) when inferring the size of \( G \), but because \( C = 1 \) for all of our calculations, we can just let the size of \( G \) be determined by \( L \) alone. Any unused coordinate of \( G \) will be filled by \( 0 \). Therefore, the coupling matrix for \( \nu_{77} \) is given by

\[
G = \left(\begin{array}{cc}
-ie
\end{array}\right).
\]

The \text{lorentz} attribute of \( \nu_{77} \) is an array where the elements are instances of the \text{Lorentz} class representing the vertex’s Lorentz structures. In the case of the QED vertex, there is only one Lorentz structure, \text{FFV1}. The definition of \text{FFV1} in the UFO file is given by

\[
\text{FFV1} = \text{Lorentz}(\text{name} = \text{\'FFV1\'}, \\
\text{spins} = [\text{2, 2, 3}]), \\
\text{structure} = \text{\'Gamma(3,2,1)\'}
\]

where the \text{structure} attribute gives a string that represents the Lorentz structure. In this case, \text{\'Gamma(3,2,1)\'} represents the Dirac matrices \( \gamma^\mu \). The numbers in parenthesis refer to the Lorentz and spin indices of \( \gamma^\mu \). For this work, the reader does not need to worry about how these numbers play a role; the important takeaway is that the structure that \text{FFV1} represents is \( \gamma^\mu \). Our Lorentz column vector \( L \) is thus given by

\[
L = \left(\begin{array}{c}
\gamma^\mu
\end{array}\right).
\]
From our equation for the vertex of a particle, we see that the vertex $V_{77}$ is given by

$$V(p_e^+, p_e^-, p_\gamma) = C \times G \times L$$

$$= 1 \times \left(-ie\right) \left(\gamma^\mu\right)$$

$$= -ie\gamma^\mu.$$ 

As expected, we recover the QED vertex from the UFO files. For a more complete overview of the different attributes and classes of the UFO files, including a list of common Lorentz and color structures and their definitions, the reader is encouraged to refer to Ref. [19].

One key detail that the reader might have noticed is that the value attribute for the Coupling class and the structure attribute for the Lorentz class are Python strings. In fact, this is true also for the color tensors and parameters of the theory. To retain its universality and compatibility with event generators, UFO stores its mathematical expressions as strings that symbolically represent them. However, for our program to perform any sort of calculations, we must be able to recover these mathematical expressions from the UFO files somehow. To accomplish this, we rely on the Lark package.

2.2 Lark Package

To transform the string outputs of UFO into useful mathematical expressions, we need to rely on a grammar that will act as translator between these two. The Lark package is a parser in Python compatible with most programming and natural languages [20]. Because the Lorentz structures are limited and fixed, we need not use the Lark grammar to compute their values. Instead, we have defined the mathematical expressions in a Python module that maps these to their corresponding objects in the UFO files. This module, for example, properly replaces FFV1 for $\gamma^\mu$. Eventually, we could generalize this procedure of manually implementing the Lorentz structures, but for the present work, this method suffices. To account for the couplings and the
parameters, we developed a Lark grammar for the UFO output strings. This grammar is able to accurately relate a string such as 'MW**2/MZ**2' to $\frac{80.379^2}{91.1876^2}$, where 80.379 and 91.1876 are the masses of the W and Z bosons in GeV, respectively. To access these values of the parameters and the couplings, we load all of the Coupling and Parameter instances of the UFO into a model class that initializes coupling and parameter dictionaries. These dictionaries assign the parameter or coupling name to their respective numerical values using the grammar. Then, a parameter such as

```python
ee = Parameter(name = 'ee',
               nature = 'internal',
               type = 'real',
               value = '2*cmath.sqrt(aEW)*cmath.sqrt(cmath.pi)',
               texname = 'e')
```

gets assigned to 0.3135, which is the value of the elementary charge $e$ in Lorentz-Heaviside units. The 'value' attribute is transformed into this number using the Lark grammar. Now that we have all the information available in mathematical objects, we proceed to calculate the scattering amplitudes.

### 2.3 Berends-Giele Recursive Relations

For the calculation of the amplitudes, we utilize the Berends-Giele recursive relations [21]. Initially proposed to deal with color-ordered multi-gluon amplitude calculations, the Berends-Giele recursive relations are an algorithm that calculates scattering amplitudes by utilizing recursive currents built from the external particles. This recursive nature bolsters the efficiency and speed of the process, as it allows us to reuse currents that appear in more than one diagram. In fact, whereas a general, direct approach to calculate $n$–particle scattering amplitudes scales computationally as $O(n!)$, the Berends-Giele recursive relation algorithm only scales as $O(a^n)$, where $a$ is the highest point vertex. Since its inception, the Berends-Giele algorithm has been extended to deal with general 3–point vertices (i.e. not just gluons), as employed in the
matrix element (amplitude) generator Comix \cite{27}. Furthermore, Ref. \cite{28} generalized
these tree-level calculations with Berends-Giele to $n$–point vertices. The processes
that we are dealing with in this work only require us to calculate amplitudes with
$3$–point vertices, so that is the current extent of our implementation. This limit to
$3$–point vertices also implies that our Berends-Giele calculation scales as $O(3^n)$.

The fundamental objects in the Berends-Giele recursive relations are the $n$–particle
currents $J^\mu_i(\pi)$, where $\pi$ is the set of $n$ particles to which this current corresponds
to, and $i$ is an index labeling the current. The superscript $\mu$ is a multi-index that
represents Lorentz and spin indices, depending on the type of object. If we were
dealing with color structures, we would have the superscript be a multi-index repre-
senting Lorentz, spin and color indices. When a particle $i$ is an external particle, its
current $J^\mu_i$ is given by the particle’s external wavefunctions (e.g. spinors, polarization
vectors, etc.) For $3$–point vertices, two currents $J^\nu_j(\pi_1)$ and $J^\rho_k(\pi_2)$ produce a new
current $J^\mu_i(\pi)$, where $\pi = \pi_1 \oplus \pi_2$. In general, a Berends-Giele $3$–point current can
be expressed as follows

$$
J^\mu_i(\pi) = \sum_{V^i_{j,k}(\pi_1,\pi_2) \in P_2(\pi)} S(\pi_1,\pi_2)P_i(\pi)V^{j,k}_i(\pi_1,\pi_2)J^\nu_j(\pi_1)J^\rho_k(\pi_2),
$$
(2.1)

where $\pi_1, \pi_2$ are set partitions of $\pi$, $P_i(\pi)$ is the propagator corresponding to particle
type $i$, and $V^{j,k}_i$ is the $3$–point vertex connecting these three currents for given par-
titions $\pi_1, \pi_2$. The sums are over all (unordered) set partitions of $\pi$ into $\pi_1, \pi_2$ and
for all the existing $3$–point vertices of the two base currents $J^\nu_j(\pi_1)$ and $J^\rho_k(\pi_2)$ \cite{27}.

By convention, we assume that all particles are outgoing. In that case, if a particle is
incoming, we make it outgoing by flipping the direction of its momentum and chang-
ing it to its antiparticle. For example, an incoming electron $e^-$ with momentum $p$
would become an outgoing positron $e^+$ with momentum $-p$. This convention is also
adopted in the UFO vertices files.
The prefactor $S(\pi_1, \pi_2)$ is a symmetry factor for the corresponding set partition of $\pi$ into $\pi_1 \oplus \pi_2$. This symmetry factor is to account for the antisymmetry of possibly indistinguishable fermionic particles. Its value is given by:

$$S(\pi_1, \pi_2) = (-1)^{S_f(\pi_1, \pi_2)} \quad (2.2)$$

where $S_f(\pi_1, \pi_2)$ is a function that counts the number of (fermion) permutations necessary to achieve a predefined order. By fermion permutations, we mean that, if one of the particles in $\pi_1$ or $\pi_2$ was not a fermion, we would not include it in our calculation of $S_f$. The predefined order for our program is ascending. For example, suppose that we have a positron labeled by the index 2, an electron labeled by the index 4, and an antimuon labeled by the index 1. Let $\pi_1$ be the set corresponding to the current made up of the electron and the positron, then $\pi_1 = [2, 4]$. Let $\pi_2$ correspond to the current made up of just the antimuon, then $\pi_2 = [1]$. The combinations of these currents, if one exists, would involve the set $\pi = \pi_1 \oplus \pi_2 = [2, 4] \oplus [1] = [2, 4, 1]$. In this case, the function $S_f$ would count the number of fermion permutations required to sort this list in ascending order. One can see that, for our present scenario, we require 2 permutations to sort this list; therefore, our symmetry factor is $S(\pi_1, \pi_2) = (-1)^{S_f(\pi_1, \pi_2)} = (-1)^{S_f([2,4,1])} = (-1)^2 = 1$. In general, however, the reader might notice the apparent ambiguity in the selection of $\pi_1$ and $\pi_2$. One could choose $\pi_1$ and $\pi_2$ in such a way that $\pi = \pi_1 \oplus \pi_2$ gives a symmetry factor of, say, $1$. But if instead, we had swapped the definitions of $\pi_1$ and $\pi_2$ (i.e. $\pi_1 \rightarrow \pi_2, \pi_2 \rightarrow \pi_1$), we could get a different value for $S$. Although this is true, this ambiguity plays no role in the larger calculation of the amplitude. As long as we always order the particle sets in the same order for all the currents, the relative sign at the end of our diagram calculations is right. And we only care about this relative sign between terms because the overall sign, whatever it is, will disappear when we square our amplitude.

As an example of Eq. (2.1), let us consider an electron $e^-$ with momentum $p_1$ and a positron $e^+$ with momentum $p_2$ as external particles. For simplicity, let us assume these are already outgoing particles. Let us denote the electron current by $J_{1}^\alpha(e^-)$ and the positron current by $J_{2}^\beta(e^+)$. Then, the electron current would be given by
the outgoing spinor $\bar{u}^\alpha(p_1)$ and the positron current by the outgoing spinor $v^\beta(p_2)$.

Let us now calculate the current, denoted by index 3, for a particle set $\pi$ containing the electron and the positron. The only unordered set partition of $\pi$ into two disjoint subsets $\pi_1$ and $\pi_2$ is that where $\pi_1$ corresponds to, say, the electron and $\pi_2$ corresponds to the positron. The symmetry factor for this partition is 1 since $\pi = [1, 2]$. For the vertex summation, we do have more than one term. An electron and a positron may interact via the exchange of a photon or a Z boson. The photon vertex is our familiar QED vertex $-ie\gamma^\mu$. The Z boson vertex is given by the following expression:

$$
e^+ \xrightarrow{Z} e^- = \frac{ie}{2} \left( \frac{\sin(\theta_W)}{\cos(\theta_W)} - \frac{\cos(\theta_W)}{\sin(\theta_W)} \right) \gamma^\mu \left( \frac{\mathbb{I} - \gamma^5}{2} \right)
$$

$$+ ie \left( \frac{\sin(\theta_W)}{\cos(\theta_W)} \right) \gamma^\mu \left( \frac{\mathbb{I} + \gamma^5}{2} \right)$$

where $\cos(\theta_W) = \frac{M_W}{M_Z}$ and $\sin(\theta_W) = \sqrt{1 - \frac{M_W^2}{M_Z^2}}$. $M_W$ and $M_Z$ are the masses of the W and Z bosons. The propagator term for the photon is given by $-i\frac{g_{\mu\nu}q^\nu}{q^2}$, where $g_{\mu\nu}$ is the metric tensor and $q$ is the momentum of the photon. Because the Z boson is a massive vector (i.e. spin-1) particle, its propagator term is slightly different, taking the form:

$$\frac{-ig_{\mu\nu} + i\frac{g_{\mu\nu}q^\nu}{M_Z^2}}{q^2 - M_Z^2 - iM_Z\Gamma_Z}$$

where $\Gamma_Z$ is the decay width of the Z boson, and $q$ is its momentum. By conservation of 4-momentum, $q + p_1 + p_2 = 0$. It is worth mentioning that the propagators contain two Lorentz indices $\mu$ and $\nu$. These indices correspond to the endpoints of the propagator; that is, they are related to the vertices where the virtual particle propagates. For these two propagators, we have included generic Lorentz indices. However, in the real calculation, these two indices would correspond to the currents.
that will be contracted. Putting together all of our terms, we get that the current $J_3^\nu(e^+, e^-)$ is given by:

$$J_3^\nu(e^-, e^+) = (1) \cdot \left( \frac{-ig_{\mu\nu}}{q^2} \right) \cdot \left( -ie\gamma_{\alpha\beta}^\mu \right) \cdot (\pi^\alpha(p_1)) \cdot (v^\beta(p_2))$$

$$+ (1) \cdot \left( \frac{-ig_{\mu\nu} + ig_\mu q_\nu}{q^2 - M_Z^2 - iM_Z\Gamma_Z} \right) \cdot \left( \frac{ie}{2} \left( \frac{\sin(\theta_W)}{\cos(\theta_W)} - \frac{\cos(\theta_W)}{\sin(\theta_W)} \right) \gamma_{\alpha\beta}^\mu \left( \frac{1 - \gamma^5}{2} \right) + ie \left( \frac{\sin(\theta_W)}{\cos(\theta_W)} \right) \gamma_{\alpha\beta}^\mu \left( \frac{1 + \gamma^5}{2} \right) \right)$$

$$\cdot (\bar{\pi}^\alpha(p_1)) \cdot (v^\beta(p_2))$$

As we can see, the current $J_3^\nu(e^-, e^+)$ is expressed by two terms, one corresponding to the $e^+, e^-, \gamma$ vertex and another corresponding to the $e^+, e^-, Z$ vertex. In this case, the current $J_3^\nu$ has Lorentz index $\nu$ because that is the only free (i.e. uncontracted) Lorentz index in our expression. It is worth mentioning that Dirac matrices have three indices: one Lorentz index ($\mu$ in this case) and two spin indices ($\alpha$ and $\beta$ in this case). The spin indices indicate the order of contraction of the Dirac matrices with the Dirac spinors. Now that we know how to calculate the different $n-$particle currents, we can explain their purpose in the calculation of the amplitudes.

The main objective of the Berends-Giele recursive relations is to utilize these $n-$particle currents $J_i^\mu(\pi)$ to calculate scattering amplitudes. We accomplish this by building up Feynman diagrams, and hence scattering amplitudes, from these currents. The current $J_i^\mu(\pi)$ is built up from all the Feynman diagrams that have $\pi$ as its external particles and that include the internal particle or particles that this current represents. In terms of our previous example, we have that $J_3^\nu(e^-, e^+)$ represents the following:

$$J_3^\nu(e^-, e^+) = \begin{array}{c} e^- \\ \gamma \\ e^+ \end{array} + \begin{array}{c} e^- \\ Z \\ e^+ \end{array}$$

Let us now suppose that we have the following scattering event $e^+e^- \rightarrow \mu^+\mu^-$ and we want to calculate its scattering amplitude $\mathcal{M}$. To illustrate how we can obtain
the amplitude, we will calculate all the relevant currents for this process. Let us start with the Feynman diagram for this process:

![Feynman diagram](image)

We must first transform our incoming particles into outgoing particles. That is, our incoming electron with momentum $p_2$ will become an outgoing positron with momentum $-p_2$ and our incoming positron with momentum $p_1$ will become an outgoing electron with momentum $-p_1$.

![Transformed diagram](image)

Now we are ready to calculate our currents and amplitude. For the sake of simplicity, let us ignore the Z boson in these calculations. First, we know that the four
currents corresponding to the outgoing external particles (i.e. $e^+, e^-, \mu^+, \mu^-$) are the Dirac spinors. Let these currents be the following:

$$J_1^\alpha(e^-) = \pi^\alpha(-p_1)$$

$$J_2^\beta(e^+) = \pi^\beta(-p_2)$$

$$J_4^\sigma(\mu^-) = \pi^\sigma(p_4)$$

$$J_8^\lambda(\mu^+) = \pi^\lambda(p_8)$$

From the previous calculations, we know that the currents $J_1^\alpha(e^-)$ and $J_2^\beta(e^+)$ produce the current $J_3^\nu(e^-, e^+)$ as follows

$$J_3^\nu(e^-, e^+) = (1) \cdot \left( \frac{-ig_{\mu\nu}}{q^2} \right) \cdot (-ie\gamma_{\alpha\beta}) \cdot (\bar{u}^\alpha(-p_1)) \cdot (v^\beta(-p_2)).$$

The current $J_3^\nu(e^-, e^+)$ corresponds to the photon. Our other two external currents $J_4^\sigma(\mu^-)$ and $J_8^\lambda(\mu^+)$ correspond to the muon and the antimuon respectively. Therefore, we can combine currents $J_3^\nu(e^-, e^+)$ and $J_4^\sigma(\mu^-)$ because we know that there exists a vertex connecting the photon and the muon. Let us call this current $J_7^\nu(e^-, e^+, \mu^-)$, which will be given by

$$J_7^\nu(e^-, e^+, \mu^-) = S(\pi_1, \pi_2) P_7(e^-, e^+, \mu^-) V_7^{3,4}(\pi_1, \pi_2) J_3^\nu(e^-, e^+) J_4^\sigma(\mu^-)$$

since, ignoring the Z boson, there is only one vertex and one set partition connecting these two currents $J_3^\nu$ and $J_4^\sigma$. Let $\pi_1 = [1, 2]$ from the $J_3^\nu(e^-, e^+)$ current and $\pi_2 = [4]$ from the $J_4^\sigma(\mu^-)$ current. Then $\pi = \pi_1 \oplus \pi_2 = [1, 2, 4]$ and the symmetry factor is $S(\pi) = (-1)^{S_f(\pi_1, \pi_2)} = (-1)^0 = 1$. The vertex between our photon and our muon is $V_7^{3,4} = -ie\gamma^\nu$, where the Lorentz index now matches the second Lorentz index in the photon propagator for the $J_3^\nu$ current. The propagator term is now the propagator corresponding to the antimuon $\mu^+$, since that is the particle corresponding to $J_7(e^-, e^+, \mu^-)$. Fermionic particles have a propagator of the form

$$\frac{i(\not{q} + m_f)}{q^2 - m_f^2},$$
where $q$ is the momentum of the particle and $m_f$ is its mass. Notice that, unlike boson propagators, fermion propagators do not carry a Lorentz index. Using this expression for the antimuon propagator, we arrive at the final expression for $J_{7}(e^{-},e^{+},\mu^{-})$,

$$J_{7}(e^{-},e^{+},\mu^{-}) = \begin{array}{c}
  e^{+} \\
  \gamma \\
  e^{-} \\
  \mu^{-}
\end{array} = (1) \cdot \left( \frac{i(q + m_f)}{q^2 - m_f^2} \right) \cdot (-ie^{\gamma}_{\sigma\rho}) \cdot \left( \frac{1}{1} \cdot \left( -ig_{\mu\nu} \right) \cdot (-ie^{\gamma}_{\alpha\beta}) \cdot (\bar{u}^{\alpha}(-p_1) \cdot (v^{\beta}(-p_2)) \right) \cdot \bar{u}^\sigma(p_4),$$

where we have included a spin index $\rho$ in the vertex term to act as a placeholder. This spin index is the only free index and so we label our current as $J_{7}^{\rho}(e^{-},e^{+},\mu^{-})$. As we can see, this current $J_{7}^{\rho}$ already represents the Feynman diagram corresponding to the scattering event $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}$. To get the scattering amplitude, we must contract the current $J_{7}^{\rho}$ with the current corresponding to the external antimuon, $J_{8}^{\lambda}$. However, this contraction is still off by a factor corresponding to the antimuon propagator, $P_{7}(e^{-},e^{+},\mu^{-})$. Physically, this is because our antimuon is a real external particle, so when we contract currents $J_{7}^{\rho}$ and $J_{8}^{\lambda}$, we do not need a (virtual particle) propagator. To get the correct result for our amplitude, we must divide our contraction by $P_{7}(e^{-},e^{+},\mu^{-})$. Our full scattering amplitude for $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}$ is then given by

$$M(e^{+},e^{-},\mu^{-},\mu^{+}) = J_{8}^{\lambda}(\mu^{+}) \frac{g_{\lambda\rho}}{P_{7}(e^{+},e^{-},\mu^{-})} J_{7}^{\rho}(e^{+},e^{-},\mu^{-})
\begin{array}{c}
  v^{\lambda}(p_8) \cdot g_{\lambda\rho} \cdot (-ie^{\gamma}_{\rho\sigma}) \cdot \left( \frac{-ig_{\mu\nu}}{q^2} \right) \cdot (-ie^{\gamma}_{\alpha\beta}) \cdot \bar{u}^{\alpha}(-p_1) \cdot v^{\beta}(-p_2) \cdot \bar{u}^\sigma(p_4)
\end{array}
\begin{array}{c}
  = \bar{u}(p_4) (-ie^{\gamma}) v(p_8) \frac{-ig_{\mu\nu}}{q^2} \bar{u}(-p_1) (-ie^{\gamma} v(-p_2)
\end{array}.$$
where we have dropped the spin indices in the last line, assuming an implicit summation. From the QED Feynman rules, we can see this result is indeed the amplitude of the $e^+e^- \rightarrow \mu^+\mu^-$ diagram with all particles outgoing. This example illustrates the process behind the Berends-Giele recursive relations. In general, we will have that the $n$–particle scattering amplitude $M$ for a set of particles $\pi$ is given by

$$M(\pi) = J^\alpha_i(n) \frac{g_{\alpha\beta}}{P_{i-1}(\pi - n)} J^\beta_{i-1}(\pi - n),$$

where $J^\alpha_i(n)$ is the current of the $n$th particle, $J^\beta_{i-1}(\pi - n)$ is the current of the set $\pi$ of all particles minus particle $n$, and $P_{i-1}(\pi - n)$ is the propagator term corresponding to $J^\beta_{i-1}(\pi - n)$.

We employ the Berends-Giele recursive relations in the bulk of our calculations. From these examples, we see that the algorithm correctly recovers the amplitude of our diagrams. Moreover, as mentioned at the beginning of this section, one of the advantages of using the Berends-Giele algorithm is that we can reuse the currents for other Feynman diagram calculations within a given process. For example, if we had included the Z boson in our calculations, the current $J^\nu_3(e^-,e^+)$ would have contained the vertex and propagator of the Z boson. Carrying on with the calculations of the current $J^\rho_7(e^-,e^+,\mu^-,\mu^+)$ and the amplitude $M(e^-,e^+\mu^-,\mu^+)$, we would have gotten an amplitude that corresponds to the following

$$M(e^-,e^+\mu^-,\mu^+) = \gamma \begin{array}{c} e^+ \\ \mu^+ \\ e^- \end{array} + Z \begin{array}{c} e^+ \\ \mu^+ \\ \mu^- \end{array}.$$ 

For this calculation, we only would have calculated the currents once, instead of calculating the amplitudes one by one. We now have all the necessary tools to explain our algorithm in the next section.

2.4 Our Program

Let us start with the basics of our program. For this, we assume that the user has uploaded the model into FeynRules, which subsequently produced the corresponding
UFO files in the working directory. We then request a run_card.yml file that contains information about the event to be calculated. An example of a run_card.yml file is given below:

```
Model: SM_NLO
Mode: lmunu
PtCut: 1
EnergyRange: [20, 200, 51]
NEvents: 1000

Particles:
- Particle: [11, in]
- Particle: [13, in]
- Particle: [11, out]
- Particle: [13, out]
```

The first parameter in the file is Model and its input is the name of the directory containing the UFO files. The second parameter is Mode, which indicates the preferred output of the user. If Mode: lmunu as in this case, the program will only calculate the leptonic tensor of the given event. If Mode is anything else, the default output of the program is the amplitude $M$. PtCut refers to a cutoff in the transverse momentum to be included in the calculation of $M$. This cutoff limits the maximum value of $\cos \theta$ and is necessary if we are dealing with massless $t$–channel processes where the amplitude blows up at $\cos \theta = 1$. Following, we have EnergyRange which is a list containing the minimum and maximum center-of-mass (CM) energies (in GeV), as well as the number of evenly spaced samples. NEvents indicates the number of events per $E_{CM}$. Finally, we have the parameter Particles which contains a list of all the particles involved in the event to be calculated. Each Particle argument is given as a list containing the particle’s Particle Data Group code [29], followed by an indicator of whether the particle is incoming or outgoing. If a particle is incoming, our program
will proceed with the proper transformation into an outgoing particle as explained in Sec. 2.3.

Once the run_card is loaded into our program, we proceed to initialize our momenta. Iterating over the range of CM energies, we set the momenta of in particles based on $E_{CM}$. The out particles’ momenta is generated using a simple phase space integrator called Rambo [30]. When the momenta are ready, we iterate over all the helicity states of our particles and produce an instance diagram of our Diagram class with the appropriate particles, momenta, helicities and mode passed onto it. This instance diagram initializes the corresponding external particle currents based on the type of particles passed onto it. The command diagram.generate_currents will calculate the remaining currents following the Berends-Giele recursive relations, where the calculation will include all tree-level Feynman diagrams for the given external particles. That is, our program will generate all the possible combinations of the external particles that produce the desired event (at tree-level), giving a variety of currents. The way our code is written up, we only include a given diagram once, avoiding overcounting and unnecessary calculations. Once the currents are generated, we proceed to calculate either the leptonic tensor or the amplitude depending on the value of Mode in the run_card.yml file.

To calculate the leptonic tensor, let us briefly go back to Sec. 1.4 and Sec. 2.3. In Sec. 1.4, we saw how to calculate the leptonic tensor for the process $e^+ e^- \rightarrow \mu^+ \mu^-$, where, in this case, we had two leptonic tensors. One of the tensors was an electron leptonic tensor $L_{\mu\nu,e^-}$ corresponding to the half of the diagram that contained the incoming electron and positron together with the photon. The other tensor was a muon leptonic tensor $L_{\mu^-}$ corresponding to the other half of the diagram that contained the outgoing muon and antimuon together with the photon. Similarly, to calculate $L_{\mu\nu}$ from our Berends-Giele currents, we must use the currents that correspond to the part of the Feynman diagram that contains the leptons in our event. For the run_card.yml example above, we have the process $e^- \mu^- \rightarrow e^- \mu^-$, whose Feynman diagram is given below.
This event also has an electron leptonic tensor and a muon leptonic tensor. Let us label the in electron with index 1, the in muon with index 2, the out electron with index 4 and the out muon with index 8. Then, the current $J_5$, produced by the currents $J_1$ and $J_4$, would correspond to the upper half of the Feynman diagram for $e^-\mu^-\rightarrow e^-\mu^-$ that contains the incoming and outgoing electron. However, these currents are of shape $(4,1)$ whereas the leptonic tensors are of shape $(4,4)$. The correct expression for the leptonic tensor from these currents is given by

$$L^{\mu\nu} = \sum_{\text{helicity states}} \left( J_i^{\mu} \right) \otimes \left( J_i^{\nu} \right)^\dagger, \quad (2.4)$$

where $J_i$ is the current that corresponds to the leptonic tensor, $\dagger$ is the Hermitian conjugate operator and $\otimes$ is the outer tensor product operator. Currently, our calculation of the leptonic tensor using the currents is dependent on our selection of which current $J_i$ is the appropriate one. For the validation events given in Ch. 3, this manual process works fine. In the future, we will generalize this process. Notice that the indices of $J$ could be covariant (lower) or contravariant (upper). The notation we have taken throughout this section is to just label the currents with an upper index, but the reader should notice the location of the free index within the expression for
$J$ to know the true nature of $J$. Depending on whether the index of $J$ is covariant or contravariant will determine the indices of the leptonic tensor $L$.

Let us go through one calculation of $L^{\mu\nu}$ as an example. First, it is worth pointing out that in our program, we assume that the leptonic tensor carries all of the propagator term. This means that the hadronic tensor would have no propagator and the amplitude we get from contracting these two tensors is correct. Let us then calculate the (electron) leptonic tensor for the process $e^- e^+ \rightarrow \mu^- \mu^+$. The current that is associated with $L$ is $J_3^\nu(e^-, e^+)$

$$J_3^\nu(e^-, e^+) = (1) \cdot \left(-i g_{\nu\chi} q^2\right) \cdot (-ie\gamma^\chi_{\alpha\beta}) \cdot (\bar{u}_{\alpha}(-p_1)) \cdot (e^\beta(-p_2))$$

$$= -e g_{\nu\chi} q^2 \bar{u}(p_1)\gamma^\chi u(p_2),$$

where in the last line we have turn our outgoing particles back into incoming ones. We have also reorganized the spinors and the Dirac matrices to match the spin indices. The daggered current $(J_3^\nu)^\dagger$ would be given by

$$(J_3^\nu)^\dagger(e^-, e^+) = -e g_{\nu\xi} q^2 \bar{u}(p_2)\gamma^\xi v(p_1).$$

The leptonic tensor would just be given by Eq. 2.4. Notice that $J_3$, and hence $(J_3)^\dagger$, have their free index lowered. As such, the leptonic tensor would have its indices lowered. The expression for $L_{\mu\nu,e}$ is as follows

$$L_{\mu\nu} = \sum \left(-e g_{\mu\chi} q^2 \bar{u}(p_1)\gamma^\chi u(p_2)\right) \left(-e g_{\nu\xi} q^2 \bar{u}(p_2)\gamma^\xi v(p_1)\right)$$

$$= \sum e^2 q^4 g_{\mu\chi} g_{\nu\xi} \bar{u}(p_1)\gamma^\chi u(p_2)\bar{u}(p_2)\gamma^\xi v(p_1)$$

$$= \sum e^2 q^4 \bar{v}(p_1)\gamma_{\mu} u(p_2)\bar{u}(p_2)\gamma_{\nu} v(p_1).$$

Recalling that the other leptonic tensor, which in this case would act as our hadronic tensor, does not include the propagator term, we recover the correct expression for the amplitude sum without the spin average term (i.e. $\sum |\mathcal{M}|^2$).

As mentioned before, if Mode: 1munu, we will calculate the leptonic tensor as given in Eq. 2.4. Otherwise, our program proceeds to calculate $\mathcal{M}$ as given in Eq. 2.3.
Our results are now ready to be used for cross section calculations and plotting. A schematic of the algorithm is shown in Fig. 2.2.

![Diagram of the algorithm](image)

Figure 2.2. Structure of the algorithm used to calculate $L_{\mu\nu}$ and $M$. 
3. Results and Discussion

Based on the explanations of Ch. 2, we are ready to compute meaningful quantities with our program. To test our program, we will focus on three Standard Model processes outlined below:

1. $e^- p^+ \rightarrow e^- p^+ \text{ with a virtual } \gamma \text{ boson.}$

2. $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ \text{ with a virtual } W \text{ boson.}$

3. $\nu_e p^+ \rightarrow \nu_e p^+ \text{ with a virtual } Z \text{ boson.}$

For each of these processes, we will calculate the analytic and numerical squared amplitudes $M$ and plot their initial-spin averaged, final-spin summed squared amplitude versus $\cos \theta$, the cosine of the polar angle of the outgoing lepton. These amplitude plots will be at six different center-of-mass energies $E_{CM}$ of $20 \text{ GeV, } 60 \text{ GeV, } 100 \text{ GeV, } 140 \text{ GeV, } 180 \text{ GeV and } 200 \text{ GeV}$. To obtain the analytic result, we will employ our knowledge from amplitude calculations used in Sec. 1.4. For the numerical result, we will use our program to calculate the leptonic tensor $L_{\mu \nu}$ and we will contract it with a general hard-coded hadronic tensor $H_{\mu \nu}$. The general form of $H_{\mu \nu}$ as well as the code used to generate it are given below:

$$H_{\mu \nu} = 2 \left( (g_L^2 + g_R^2) \cdot (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1 \cdot p_2) g_{\mu \nu}) + (g_L^2 - g_R^2) \cdot i \cdot \varepsilon^{\mu \nu \alpha \beta} p_{1,\alpha} p_{2,\beta} \right)$$

```python
def HadronicTensor(p1, p2, gl2, gr2):
    symmetric = np.einsum('bi,bj->bij', p1, p2) + np.einsum('bi,bj->bij', p2, p1)
    symmetric -= np.einsum('ij, b -> bij', ls.METRIC_TENSOR, Dot(p1, p2)[:,:0])
    antisymmetric = 1j*np.einsum('ijkl, bk, bl -> bij', ls.EPS, p1, p2)
```
return \(2\times((g_{L}+g_{R})\times\text{symmetric}+(g_{L}-g_{R})\times\text{antisymmetric})\)

where \(g_{L}\) and \(g_{R}\) are the left- and right-handed couplings of the interacting particle, \(p_{1}, p_{2}\) are the momenta and \(\varepsilon^{\mu\nu\alpha\beta}\) is the totally antisymmetric Levi-Civita tensor. For this work, we take the convention that \(\varepsilon^{0123}=+1\). Because of the way that our program is currently structured, we can only account for one type of virtual boson at a time both for the leptonic and for the hadronic tensors. The consequence of this is that for a process like \(e^{-}p^{+}\rightarrow e^{-}p^{+}\), we will only consider the photon \(\gamma\) and will ignore the \(Z\) boson. We will start our discussion of the results with process 1.

3.1 \(e^{-}p^{+}\rightarrow e^{-}p^{+}\)

Let us start with the scattering of an electron \(e^{-}\) with a proton \(p^{+}\). To calculate the squared amplitude \(\vert M\vert^{2}\) for this process, we will rely on some assumptions that will simplify our calculations. In real experiments, the proton is usually bound to a nucleus that contains other protons and neutrons. Moreover, unlike the electron, the proton is not a point-like particle but rather is composed of two up quarks and one down quark. To account for the internal structure of the proton and its bound state within the nucleus, we would need to include nuclear form factors and complex nuclear physics in our calculations. Since the ultimate goal of our project is to leave these difficult calculations to the event generators, we will assume in this chapter that the proton is a point-like, massless particle.

With these assumptions, we can compute the analytic amplitude. First, let us consider the type of Feynman diagram that we would be dealing with. The proton is a spin\(-\frac{1}{2}\) fermion with positive charge \(+1\). Based on these properties, we can assume that there exists a QED vertex coupling a photon \(\gamma\) with a proton/antiproton pair with value \(+ie_{\gamma}^{\mu}\). Since the electron and the proton are different species of particles, we can also assume that there is no QED vertex coupling a photon with these two particles, just like there is no QED vertex coupling an electron with a
particle of a different species, such as the muon. Under these assumptions, our only allowed Feynman diagram would be given by:

\[
\begin{array}{c}
e^-
\end{array}
\begin{array}{c}
p_1
\end{array}
\begin{array}{c}
\gamma
\end{array}
\begin{array}{c}
p_2
\end{array}
\begin{array}{c}
q
\end{array}
\begin{array}{c}
p_3
\end{array}
\begin{array}{c}
p_4
\end{array}
\begin{array}{c}
e^-
\end{array}
\begin{array}{c}
p^+
\end{array}
\begin{array}{c}
p^+
\end{array}
\]

Figure 3.1. Tree-level Feynman diagram of \(e^-p^+ \rightarrow e^-p^+\) for particles with 4-momenta \(p_1, p_2, p_3, p_4\).

Following a similar line of reasoning, we could argue that there exists a Feynman diagram like the one in Fig. 3.1 but with a \(Z\) boson replacing the photon \(\gamma\). However, as mentioned at the beginning of this chapter, we will only focus on one type of virtual boson at a time. For this process, we will only focus on the photon case.

To apply our QED vertex rules to the proton-photon and electron-photon vertices, it is instructional to express all particles as outgoing. The process to do this is outlined in Sec. 2.3. The resulting (equivalent) Feynman diagram is given below:
Figure 3.2. Tree-level Feynman diagram of $e^- p^+ \rightarrow e^- p^+$ for particles with 4-momenta $p_1, p_2, p_3, p_4$. In this case, all particles are considered outgoing, transforming the incoming $e^-$ and $p^+$ into $e^+$ and $p$, respectively, and flipping their momenta.

where the electron $e^-$ with momentum $p_1$ has turned into a positron $e^+$ with momentum $-p_1$ and the proton with momentum $p_2$ has turned into an antiproton $\bar{p}$ with momentum $-p_2$.

From the Feynman diagram in Fig. 3.2 and from the Feynman rules, we can write down an expression for $\mathcal{M}$ as

$$\mathcal{M} = \overline{\nu}(p_3)(-ie\gamma^\mu)v^s(-p_1)\frac{-ig_{\mu\nu}}{q^2}\overline{\nu}'(p_1)(ie\gamma^\nu)v'^s(-p_2),$$

where we have used our knowledge that the proton and antiproton are spin-$1/2$ fermions to write down their spinors. Now that we have an expression for the amplitude, we can go back to having the electron and proton be incoming. We do this by applying the following transformation: $v(-p_1) = u(p_1), v(-p_2) = u(p_2)$. This transformation makes sense when considering the way we turn incoming particles into outgoing ones. After applying these transformations and contracting the metric tensor with the gamma matrices, our amplitude is now expressed as

$$\mathcal{M} = -\frac{ie^2}{q^2} \overline{\nu}(p_3)\gamma^\mu u^s(p_1)\overline{\nu}'(p_4)\gamma_{\mu}u'^s(p_2).$$
As explained in Sec. 1.4, $\mathcal{M}^* = \mathcal{M}^\dagger$ and our result is

$$\mathcal{M}^* = \frac{i e^2}{q^2} \pi^s(p_1) \gamma^\nu u^r(p_3) \pi^{s'}(p_2) \gamma_\nu u^{r'}(p_4),$$

where we have changed our dummy index to $\nu$ to avoid any confusions with $\mathcal{M}$. We can now get an expression for $|\mathcal{M}|^2$. However, as in Sec. 1.4, we still need to average over the initial-state spins and sum over the final-state spins. We get

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \sum_{s,s',r,r'} \pi^s(p_3) \gamma^{\mu} u^r(p_1) \pi^{s'}(p_4) \gamma_\mu u^{r'}(p_2) \times$$

$$\tilde{u}^s(p_1) \gamma^\nu u^r(p_3) \tilde{u}^{s'}(p_2) \gamma_\nu u^{r'}(p_4).$$

Recalling that we have set $m_e = m_p = 0$ and applying our spin-sum completeness relations from Eqs. 1.20 and 1.21 we arrive at the following expression

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{q^4} \sum_{s,s',r,r'} \pi^s(p_3) \gamma^{\mu} u^r(p_1) \pi^{s'}(p_4) \gamma_\mu, c_d u^{r'}(p_2) \times$$

$$\tilde{u}^s(p_1) \gamma^\nu u^r(p_3) \tilde{u}^{s'}(p_2) \gamma_\nu u^{r'}(p_4)$$

$$= \frac{e^4}{q^4} \sum_{s,s',r,r'} \gamma^{\mu} u^r(p_1) \pi^{s'}(p_2) \gamma_\nu, c_d u^{r'}(p_4)$$

$$= \frac{e^4}{q^4} \Tr[\gamma^{\mu} \gamma^\nu] \Tr[\gamma_\mu, c_d \gamma^{s'}] \Tr[\gamma_\nu u^{r'}].$$

Using our trace technology, we arrive at the following results for the traces

$$\Tr[\gamma^{\mu} \gamma^\nu] = p_{3,\alpha} p_{1,\beta} \cdot 4 \cdot (g^{\alpha \mu} g^{\beta \nu} - g^{\alpha \beta} g^{\mu \nu} + g^{\alpha \nu} g^{\mu \beta})$$

$$= 4 \cdot (p_3^{\alpha} p_1^{\beta} - (p_1 \cdot p_3) g^{\mu \nu} + p_1^{\alpha} p_3^{\beta}),$$

$$\Tr[\gamma_\mu, c_d \gamma^{s'}] = p_2^{\alpha} p_2^{\beta} \cdot 4 \cdot (g_{\rho \sigma} g_{\alpha \nu} - g_{\mu \sigma} g_{\rho \nu} + g_{\mu \nu} g_{\rho \sigma})$$

$$= 4 \cdot (p_2^{\alpha} p_2^{\nu} - (p_2 \cdot p_4) g_{\mu \nu} + p_2^{\alpha} p_4^{\nu}).$$

Substituting the results of the traces back into the expression for $\frac{1}{4} \sum |\mathcal{M}|$ and contracting the momenta, we obtain

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{8e^4}{q^4} \left( (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right). \quad (3.1)$$
We can now use the general form of the 4-momenta, as given in Eq. 1.28, to calculate our dot products

\[
(p_1 \cdot p_2) = \frac{E_{CM}^2}{2},
\]

\[
(p_3 \cdot p_4) = \frac{E_{CM}^2}{2},
\]

\[
(p_1 \cdot p_4) = \frac{E_{CM}^2}{4} (1 + \cos \theta),
\]

\[
(p_2 \cdot p_3) = \frac{E_{CM}^2}{4} (1 + \cos \theta).
\]

Let us now focus on the photon momentum. By conservation of 4-momentum, \( q \) is now given by

\[
q = (p_1 - p_3) = \frac{E_{CM}}{2} (0, -\sin \theta \cos \phi, -\sin \theta \sin \phi, 1 - \cos \theta),
\]

\[
q^2 = (p_1 - p_3)^2 = -\frac{E_{CM}^2}{2} (1 - \cos \theta).
\]

Plugging in our values for the dot products from Eq. 3.2 and for the photon momentum from Eq. 3.3 together, we get

\[
\frac{1}{4} \sum_{s,s',r,r'} |M|^2 = 2e^4 \left( \frac{4 + (1 + \cos \theta^2)}{(1 - \cos \theta)^2} \right). \tag{3.4}
\]

This equation is our final result for the analytic amplitude of the process \( e^-p^+ \rightarrow e^-p^+ \). The result depends on the polar angle \( \theta \) and is independent of \( E_{CM} \). As we can see from the denominator of Eq. 3.4, our amplitude blows up as \( \cos \theta \rightarrow 1 \). We will deal with this pole in the numerical calculation of our cross section. We can now compare the analytic value from Eq. 3.4 to the results from our program.
Figure 3.3. Numerical and analytic $\frac{1}{4}|\mathcal{M}|^2$ vs. $\cos(\theta)$ for the process $e^-p^+ \rightarrow e^-p^+$. For each histogram, we used $10^5$ number of events per value of $E_{CM}$ divided into 100 bins. From left to right, top to bottom, the center-of-mass energies are 20, 60, 100, 140, 180 and 200 GeV. The analytic values were computed from Eq. 3.4.

For our program, we calculated the leptonic tensor $L^{\mu\nu}$ associated with the process $e^-p^+ \rightarrow e^-p^+$ and contracted it with its corresponding hadronic tensor as given at the beginning of this chapter. We performed this calculation for a range of center of mass energies from 20 GeV to 200 GeV with 190 evenly spaced samples and for a total of $10^5$ events per value of $E_{CM}$. The results of these calculations were plotted in
6 histograms with 100 bins each, along with a plot of the analytic calculations. These histograms can be found in Fig. 3.3. As we can see, the analytic and computational values agree for the most part for all six of the histograms. Although there is a slight deviation of our numerical values from the analytic ones, we believe these to be coming from the precision of our momentum generator or from the machine. Regardless, this deviation should go away as we improve the accuracy of our program. The analytic and numerical curves are also independent of the center-of-mass energies, as expected from Eq. 3.4. In the plots, we have shown the complete range of $\cos \theta$ values. However, for the calculation of the cross section, we will apply a cut on the values of $\cos \theta$. To prevent the large numbers to interfere with our cross section calculation, we added a cut $\cos \theta < 0.95$. Physically, the events where $\cos \theta = 1$ (and hence $\theta = 0$) are those where the particles do not interact. We are not interested in these types of events and we can safely ignore them. Let us now calculate the total cross section.

In Sec. 1.1, we saw the general form of the cross section in Eq. 1.1. For the cases of 2 incoming and 2 outgoing massless particles, this equation can be further simplified without any knowledge of the amplitude. In the end, our cross section will be proportional to $\int d\Omega |M|^2$. However, the 2-to-2 events we are considering are symmetric around the $z$ axis, which translates as being independent of the azimuthal angle $\phi$. Since $d\Omega = d\phi d\cos \theta$, we can also evaluate the $\phi$ integral from $\int d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos \theta$. Finally, we need to take into account the way our detector works. As mentioned in Sec. 1.4 we need to average over the initial state spins and sum over the final state spins. When we measure the cross section, we take this into account by integrating over the initial-state spin-averaged, final-state spin-summed amplitude. Our expression for the cross section is then given by

$$\sigma = \frac{1}{32\pi E_{CM}^2} \int_{-1}^{1} d\cos \theta \frac{1}{n} \sum |M|^2; \quad (3.5)$$

where $\frac{1}{n}$ is the spin average term. For $e^- p^+ \rightarrow e^- p^+$, $\frac{1}{n}$ is equal to $\frac{1}{4}$. To compare the cross section with our numerical results, we must integrate along the same range of values for $\cos \theta$. Integrating over this new range $[-1, 0.95)$ also prevents our integral from diverging. Plugging in the expression for $\frac{1}{4} \sum |M|^2$ from Eq. 3.4 into Eq. 3.5
we can obtain the analytic result of the cross section. The integral is evaluated using Mathematica.

![Graph showing the cross section $\sigma$ for the process $e^-p^+ \rightarrow e^-p^+$ as a function of center-of-mass energy $E_{CM} = \sqrt{s}$. For each of the 190 evenly spaced values of $E_{CM}$, we calculated $\frac{1}{4}|M|^2$ with $10^5$ events.]

Figure 3.4. Cross section $\sigma$ for the process $e^-p^+ \rightarrow e^-p^+$ as a function of center-of-mass energy $E_{CM} = \sqrt{s}$. For each of the 190 evenly spaced values of $E_{CM}$, we calculated $\frac{1}{4}|M|^2$ with $10^5$ events.

For the numerical cross section, we first applied our cut to the squared amplitudes. We then used our phase space integrator Rambo from Sec. 2.4. Rambo generates the random momenta that we use in our calculation of the amplitudes and it also assigns a weight to each numerical amplitude value. These weights represent the phase space factor $\frac{1}{16\pi}d\Omega$ and we multiply them with our numerical amplitude in the cross section calculation. To get the correct numerical cross section for a given value of $E_{CM}$, we must average over the range of $\cos\theta$ and divide by a flux factor equal to $2E_{CM}^2$. Finally, as mentioned in Sec. 1.2.1, we have been working in natural units where fundamental constants such as $\hbar$ and $c$ are set equal to 1 and all quantities are expressed in terms of energy. However, when we measure the cross section, we do so in squared meters.
Because the cross sections are very small quantities, we rely on the unit barn (b) where 1 b = $1 \times 10^{-28}$ m$^2$. For our examples, we will be expressing $\sigma$ in units of pb where the pico prefix is equivalent to $10^{-12}$. As can be seen from Eq. [3.5], the cross section $\sigma$ is given in units GeV$^{-2}$. To recover pb from our natural units, we must multiply by the conversion factor $(\hbar c)^2 = 1$ GeV$^{-2} = 0.38937966 \times 10^9$ GeV$^2$ pb. Thus, we get the numerical results for the cross section. For $e^-p^+ \rightarrow e^-p^+$, the results can be found in Fig. [3.4]. As we can see in the figure, the analytic and numerical cross sections agree very well throughout the range of energies [20 GeV, 200 GeV]. This agreement can be seen in the subplot at the bottom of the figure. This subplot is the numerical to analytic ratio and it stays constantly near 1 with very small fluctuations. We can also see from the plot the $E_{CM}^{-2}$ dependence of our cross section, as was expected. This concludes our analysis of the electron proton scattering.

**3.2 $\nu_e\bar{\nu}_\mu \rightarrow e^-\mu^+$**

We now proceed to evaluate the process $\nu_e\bar{\nu}_\mu \rightarrow e^-\mu^+$. Like in Sec. [3.1], let us start with the analytic calculation of the amplitude. First, let us consider the Feynman diagram involved in this process. Neutrinos only interact via the weak interaction mediated by the $W^\pm$ and $Z$ bosons. If we turn our incoming particles into outgoing ones, we have $\bar{\nu}_e, \nu_\mu, e^-, \mu^+$. The only interaction that $\bar{\nu}_e$ can have is with the electron $e^-$ via a $W^+$ boson. Similarly, the only interaction that $\nu_\mu$ can have is with the antineutron $\mu^+$ via a $W^-$ boson. Thus, the only tree-level Feynman diagram for this process would be given by:
Figure 3.5. Tree-level Feynman diagram of $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+$ for particles with 4-momenta $p_1, p_2, p_3, p_4$. In this case, all particles are considered outgoing, transforming the incoming $\nu_e$ and $\bar{\nu}_\mu$ into $\bar{\nu}_e$ and $\nu_\mu$, respectively, and flipping their momenta.

We have only labeled the propagator by $W$ because, depending on how one sees it, the propagating particle can be a $W^+$ or a $W^-$. For the electron neutrino vertex, $W$ is an outgoing $W^+$, but this outgoing $W^+$ becomes an incoming $W^+$ in the muon neutrino vertex. Applying our rules for transforming incoming into outgoing particles, this incoming $W^+$ becomes an outgoing $W^-$. Now that we have the Feynman diagram, we proceed to write down the amplitude.

First, we must know the vertex factor coming from the $W$ boson. The coupling of the $W$ boson is given by $\frac{ig_W}{\sqrt{2}}$ where $g_W = \frac{e}{\sin \theta_W}$ and $\theta_W$ is the Weinberg angle given by $\cos \theta_W = \frac{M_W}{M_Z}$. The $W$ boson is special because it only couples to left-handed particles and right-handed antiparticles. For massless particles, this left- and right-handedness refers to the alignment of the spin of a particle with respect to its momentum. If the particle’s spin is parallel to its momentum, it is a right-handed particle; on the contrary, if the spin is antiparallel to the momentum, the particle is left-handed. We
quantify this property of the $W$ boson by including a left projector operator in the vertex of the $W$. Let us define the fifth Dirac matrix $\gamma^5$ as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

with the following properties

$$\begin{align*}
(\gamma^5)^1 &= \gamma^5, \quad (3.6) \\
(\gamma^5)^2 &= 1, \quad (3.7) \\
\{\gamma^5, \gamma^\mu\} &= 0, \quad (3.8)
\end{align*}$$

where 1 refers to the identity matrix. The left projector operator is then defined as

$$P_L = \frac{1 - \gamma^5}{2}. \quad (3.9)$$

Similarly, the right projector operator is defined as

$$P_R = \frac{1 + \gamma^5}{2}. \quad (3.10)$$

Because of the anticommutativity of $\gamma^5$ (see Eq. 3.8), we have the following relationship

$$\begin{align*}
P_R\gamma^\mu &= \frac{1 + \gamma^5}{2}\gamma^\mu \\
\gamma^\mu P_R &= \gamma^\mu \frac{1 - \gamma^5}{2} \\
P_R\gamma^\mu &= \gamma^\mu P_L. \quad (3.11)
\end{align*}$$

When the RHS of this equation acts on a fermion spinor $u$, it only takes the spinor’s left-handed components

$$\gamma^\mu P_L u(p) = u_L(p).$$

Similarly, when the LHS of this equation acts on an antifermion spinor $\bar{v}$, it only takes the spinor’s right-handed components

$$\bar{v}(p) P_R\gamma^\mu = \bar{v}_R(p).$$
This is exactly the behavior that we want for the $W$ boson. Putting all of this together, the $W$ boson vertex is given by

$$\bar{f} \gamma_\mu P_L f = \frac{ie}{\sqrt{2} \sin \theta_W} \gamma_\mu P_L.$$ (3.12)

We are almost ready to write down an expression for $\mathcal{M}$. But, first, we must think about the propagator term for the $W$ boson. Unlike the photon, the $W$ boson is a massive particle with a finite lifetime. To take this into account, we must change the denominator of the propagator to include the mass of the $W$ boson, $M_W$, and its decay width, $\Gamma_W$. The expression for the $W$ boson propagator is then

$$q \gamma_\mu P_L \gamma_\nu P_R = -ig_{\mu\nu} \left( \frac{q^2 - M_W^2}{(q^2 - M_W^2) - iM_W\Gamma_W} \right).$$ (3.13)

We are now ready to write down an expression for the amplitude. Using the Feynman rules for the spinors as given in Sec. 1.3.1, the $W$ boson vertex from Eq. 3.12 and the $W$ boson propagator from Eq. 3.13, the amplitude $\mathcal{M}$ for the process $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+$ is given by

$$\mathcal{M} = \bar{u}^r(p_3) \left( \frac{ie}{\sqrt{2} \sin \theta_W} \gamma_\mu P_L \right) v^s(-p_1) \times \bar{u}^s(-p_2) \left( \frac{ie}{\sqrt{2} \sin \theta_W} \gamma_\nu P_L \right) v^r(p_4)$$

$$\mathcal{M} = \frac{i e^2}{2 \sin^2 \theta_W (q^2 - M_W^2) - i M_W \Gamma_W} \bar{u}^r(p_3) \gamma_\mu P_L u^s(p_1) \bar{u}^s(p_2) \gamma_\nu P_L v^r(p_4),$$

where in the last line we have switched the spinors for $\nu_e$ and $\nu_\mu$ back to the incoming case. Our complex conjugated amplitude is then given by

$$\mathcal{M}^* = \frac{-i e^2}{2 \sin^2 \theta_W (q^2 - M_W^2) + i M_W \Gamma_W} \bar{u}^s(p_1) \gamma_\nu P_L u^r(p_3) \bar{u}^r(p_4) \gamma_\mu P_L v^s(p_2).$$

We can now join our expressions for $\mathcal{M}$ and $\mathcal{M}^*$ to get the squared amplitude $|\mathcal{M}|^2$. To get the initial-state spin-averaged and final-state spin-summed amplitude, we must
think about the spin average term. Neutrinos are special particles to deal with because they only come left-handed if they are neutrinos and right-handed if they are antineutrinos. Since there is only one possibility for neutrinos or antineutrinos, we have a $\frac{1}{1}$ term coming from the incoming $\nu_e$ and a $\frac{1}{1}$ term coming from the incoming $\bar{\nu}_\mu$. The spin averaged and summed amplitude is therefore

$$\sum_{s,s',r,r'} |M|^2 = \frac{e^4}{4 \sin^4 \theta_W} \frac{1}{(q^2 - M_W^2)^2 + (M_W \Gamma_W)^2} \times$$

$$\sum_{s,s',r,r'} \bar{u}^r (p_3) \gamma^\mu P_L u^s (p_1) \bar{v}^{s'} (p_2) \gamma_\mu P_L v^{r'} (p_4) \times$$

$$\bar{u}^s (p_1) \gamma^\nu P_L u^r (p_3) \bar{v}^{r'} (p_4) \gamma_\nu P_L v^{s'} (p_2).$$

Standard Model (anti)neutrinos are massless and we keep our assumption that $m_e = m_\mu = 0$. Like for the electron proton scattering case, we can apply our spin-sum completeness relations to express our amplitude as a product of traces. For the sake of simplicity, let us focus only on the sum term since the prefactor and the propagator are independent of spin. We get the following

$$\sum_{s,s',r,r'} (\cdots) = \sum_{s,s',r,r'} \bar{n}^r (p_3) \gamma^\mu P_L u^s (p_1) \bar{v}^{s'} (p_2) \gamma_\mu P_L v^{r'} (p_4) \times$$

$$\bar{n}^s (p_1) \gamma^\nu P_L h_i^s (p_3) \gamma^s i^r (p_4) \gamma_\nu P_L v^{r'} (p_2)$$

$$= \frac{4i}{4} P_L P_R = P_R P_L = 0,$$

$$P_L P_R = P_R P_L \quad \text{and} \quad P_R P_R = P_R.$$

Before evaluating the traces, let us introduce two important properties of the left and right projection operators

$$P_L P_R = P_R P_L = 0,$$

$$P_L P_L = P_L \quad \text{and} \quad P_R P_R = P_R. \quad (3.14)$$
Now, recalling the property from Eq. 3.11 let us focus on one of the traces (recall that $\gamma = p_\mu \gamma^\mu$)

$$\text{Tr}[\gamma_3 \gamma^\mu P_L \gamma^\nu P_L] = \text{Tr}[\gamma_3 \gamma^\mu \gamma_1 \gamma^\nu P_L]$$
$$= \text{Tr}[\gamma_3 \gamma^\mu \gamma_1 \gamma^\nu P_L]$$
$$= \text{Tr}[\gamma_3 \gamma^\mu \gamma_1 \gamma^\nu P_L].$$

The same argument applies to the other trace. Our spin averaged and summed squared amplitude is therefore

$$\sum_{s,s',r,r'} |M|^2 = \frac{e^4}{4 \sin^4 \theta_W} \left( \frac{1}{(g^2 - M_W^2)^2 + (M_W \Gamma_W)^2} \times \text{Tr}[\gamma_3 \gamma^\mu \gamma_1 \gamma^\nu P_L] \text{Tr}[\gamma_2 \gamma_\mu \gamma_\nu P_L]. \right)$$

Before starting our calculation of the traces, we will add more properties to our trace technology that include our new $\gamma^5$ matrix (see Sec. 1.4)

$$\text{Tr}[\gamma^5] = 0,$$  (3.16)

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = 0,$$  (3.17)

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -4i \varepsilon^{\mu \nu \rho \sigma},$$  (3.18)

where $\varepsilon^{\mu \nu \rho \sigma}$ is the Levi-Civita tensor with convention $\varepsilon^{0123} = +1$. Now we can proceed to calculate the traces. For the first trace, we have

$$\text{Tr}[\gamma_3 \gamma^\mu \gamma_1 \gamma^\nu P_L] = p_{3,\alpha} p_{1,\beta} \text{Tr}[\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu P_L]$$
$$= p_{3,\alpha} p_{1,\beta} \left[ \gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu \left( \frac{1 - \gamma^5}{2} \right) \right]$$
$$= p_{3,\alpha} p_{1,\beta} \frac{1}{2} \left( 4(g^{\alpha \mu} g^{\beta \nu} - g^{\alpha \beta} g^{\mu \nu} + g^{\alpha \nu} g^{\mu \beta} - 4i \varepsilon^{\alpha \mu \beta \nu}) \right)$$
$$= p_{3,\alpha} p_{1,\beta} \cdot 2 \left( g^{\alpha \mu} g^{\beta \nu} - g^{\alpha \beta} g^{\mu \nu} + g^{\alpha \nu} g^{\mu \beta} + i \varepsilon^{\alpha \mu \beta \nu} \right)$$
$$= 2 \left( p_3^\mu p_1^\nu - (p_1 \cdot p_3) g^{\mu \nu} + p_1^\mu p_3^\nu + i \varepsilon^{\alpha \mu \beta \nu} p_{3,\alpha} p_{1,\beta} \right),$$

where in the third line we have used Eq. 1.24 and Eq. 3.18. The second trace follows a similar argument and we end up with

$$\text{Tr}[\gamma_2 \gamma_\mu \gamma_\nu P_L] = 2 (p_{2,\mu} p_{4,\nu} - (p_2 \cdot p_4) g_{\mu \nu} + p_{4,\mu} p_{2,\nu} + i \varepsilon_{\mu \nu \rho \sigma} p_2^\rho p_4^\sigma).$$
Substituting the results of our traces back into the expression for \( \sum |\mathcal{M}|^2 \) and contracting the momenta, we get

\[
\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{4e^4}{\sin^4 \theta_W} \frac{1}{(q^2 - M_W^2)^2 + (M_W \Gamma_W)^2} (p_1 \cdot p_4)(p_2 \cdot p_3).
\] (3.19)

Since the kinematic set up for \( \nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ \) is the same as \( e^- p^+ \rightarrow e^- p^+ \), we can use our results from Eq. 3.2 and Eq. 3.3 to express our amplitude as a function of \( E_{CM} \) and \( \cos \theta \):

\[
\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{4 \sin^4 \theta_W} \frac{E_{CM}^4 (1 + \cos \theta)^2}{\left( \frac{E_{CM}^2}{2} (1 - \cos \theta) + M_W^2 \right)^2 + M_W^2 \Gamma_W^2}.
\] (3.20)

This equation is our final result for the analytic amplitude of the process \( \nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ \).

Unlike Eq. 3.4, this result depends nontrivially on both the polar angle \( \theta \) and the center-of-mass energy \( E_{CM} \), as well as on the mass and decay width of the \( W \) boson.

Unlike in the case for \( e^- p^+ \rightarrow e^- p^+ \), the denominator in Eq. 3.20 does not blow up because \( M_W \neq 0 \) and \( \Gamma_W \neq 0 \) prevent it from ever going to 0. Let us compare this analytic result from Eq. 3.20 to the results of our program.

Like in Sec. 3.1, we calculated the leptonic tensor \( L^{\mu \nu} \) associated with the process \( \nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ \) and contracted it with its corresponding hadronic tensor as given at the beginning of the chapter, dividing by the appropriate spin average term. Our center-of-mass energies ranged from 20 GeV to 200 GeV with 190 evenly spaced samples and for a total of \( 10^5 \) events per value of \( E_{CM} \). The results of these calculations were plotted in 6 histograms with 100 bins each along with a plot of the analytic calculations; the different values of \( E_{CM} \) selected were given at the beginning of this chapter. The histograms can be found in Fig. 3.6. Unlike the electron proton scattering case, our numerical and analytic results do not match as nicely. The deviation starts at the endpoints of our range (i.e. \( \cos \theta = \pm 1 \)) and diminishes as we approach the middle of the plot (i.e. \( \cos \theta = 0 \)). We are not yet certain as to the cause for this disagreement; however, our current believe is that it might be coming from the way we sum over the helicities. Our program might be accidentally including helicities that are not allowed, maybe due to the way we have coded our spinors or
Figure 3.6. Numerical and analytic $\sum |M|^2$ vs. $\cos(\theta)$ for the process $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+$. For each histogram, we used $10^5$ number of events per value of $E_{CM}$ divided into 100 bins. From left to right, top to bottom, the center-of-mass energies are 20, 60, 100, 140, 180 and 200 GeV. The analytic values were computed from Eq. 3.20.

due to the uncertainty of the machine. We will explore this effect further. Let us now obtain the cross section.

As we explained in Sec. 3.1, the cross section for a 2-to-2 event like $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+$ is given by Eq. 3.5 where our spin average term is $\frac{1}{4}$ as explained earlier. Notice that,
in this case, the amplitude does not blow up as $\cos \theta \to 1$, so we do not need a cut on the cosine. To compare the cross section with our numerical results, we integrate over the range $[-1, 1]$ for $\cos \theta$. Plugging in our expression for $\sum |\mathcal{M}|^2$ from Eq. 3.20 into Eq. 3.5, we get the analytic result for our cross section; this calculation is done using Mathematica.

Like before, we use our phase space integrator Rambo to generate momenta for our event and to assign weights to each numerical amplitude value. The numerical cross section is calculated by averaging over the range of $\cos \theta$ and dividing by the flux factor $2E_{CM}^2$. To get the correct units of picobarn (pb), we multiply by the conversion factor $(\hbar c)^2 = 1 \text{ GeV}^{-2} = 0.38937966 \times 10^9 \text{ GeV}^2 \text{ pb}$. Our results for the numerical cross section can be found in Fig. 3.7. As we can see in the figure, the analytic and numerical cross sections are overall deviated, with a numerical to analytic ratio

Figure 3.7. Cross section $\sigma$ for the process $\nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+$ as a function of the center-of-mass energy $E_{CM} = \sqrt{s}$. For each of the 190 evenly spaced values of $E_{CM}$, we calculated $\sum |\mathcal{M}|^2$ with $10^5$ events.
of approximately 1.5 at low energies and 1.2 at higher energies. This difference is not surprising given that the analytic and numerical amplitudes do not completely agree throughout the range of \( \cos \theta \). The important aspect of the cross section is that the numerical result has more or less has the same shape as the analytic one, which is a good sign and shows us that the problem is coming from the amplitude. We expect this difference to vanish as we fix the amplitude calculation. This concludes our analysis of the \( \nu_e \bar{\nu}_\mu \rightarrow e^- \mu^+ \) process.

### 3.3 \( \nu_e p^+ \rightarrow \nu_e p^+ \)

Our last process for this chapter is the electron neutrino proton scattering \( \nu_e p^+ \rightarrow \nu_e p^+ \). Like in Sec. 3.1, we will assume that the proton is a massless, point-like particle and we will start this section with the analytic calculation of the amplitude. Like we mentioned in Sec. 3.2, the neutrino only interacts via the \( W^\pm \) and \( Z \) bosons. If we turn our incoming particles into outgoing ones, we have \( p_e, \bar{p}, \nu_e, p^+ \). The only possibility for the neutrinos to interact is with each other via the \( Z \) boson. Similarly, the \( Z \) boson can interact with the proton/antiproton pair. Thus, the only tree-level Feynman diagram for this process would be given by:
Figure 3.8. Tree-level Feynman diagram of $\nu_e p^+ \to \nu_e p^+$ for particles with 4-momenta $p_1, p_2, p_3, p_4$. In this case, all particles are considered outgoing, transforming the incoming $\nu_e$ and $p^+$ into $\bar{\nu}_e$ and $\bar{p}$, respectively, and flipping their momenta.

With this Feynman diagram, we can proceed to write down the amplitude. First, we must know the vertex factor coming from the $Z$ boson. The coupling of the $Z$ boson is given by $i g_Z$ where $g_Z = \frac{e}{\sin^2 \theta_W \cos \theta_W}$. Unlike the $W$ boson but similar to the photon, the $Z$ boson couples with both left- and right-handed particles. However, unlike the photon, the coupling of the $Z$ to left- and right-handed fermions is different and it is given by

$$g_L = i g_Z (I^f_3 - \sin^2 \theta_W Q_f),$$
$$g_R = i g_Z (-\sin^2 \theta_W Q_f),$$

where $I^f_3$ is the isospin of the fermion, $Q_f$ is its electric charge, and $g_Z$ is the $Z$ boson coupling. We can calculate now the couplings specific to the neutrinos and the protons. Neutrinos have no electric charge and their isospin is $+\frac{1}{2}$. Given that their electric charge is 0, the right-handed coupling of neutrinos to the $Z$ boson is 0, as expected since we only allow left-handed neutrinos. The left coupling for neutrinos is given by $g_{L, \nu} = \frac{ie}{2 \sin \theta_W \cos \theta_W}$. Protons, on the other hand, do have an electric charge of +1 and their isospin is also $+\frac{1}{2}$. So, their left coupling to the $Z$ is given by
The left coupling is given by

\[ g_{L,p} = \frac{ie}{\sin \theta_W \cos \theta_W} \left( \frac{1}{2} - \sin^2 \theta_W \right) = ie \left( \frac{\cos \theta_W}{2 \sin \theta_W} - \frac{\sin \theta_W}{2 \cos \theta_W} \right). \]

The right coupling is given by

\[ g_{R,p} = -ie \frac{\sin \theta_W}{\cos \theta_W}. \]

To differentiate between left- and right-handed couplings, we must include a projector with each term. Putting these results together, we arrive at the vertices for our diagram

\[
\begin{align*}
\bar{\nu}_e & \quad \nu_e = \frac{ie}{2 \sin \theta_W \cos \theta_W} \gamma^\mu P_L, \\
\bar{p} & \quad p^+ = \left( ie \left( \frac{\cos \theta_W}{2 \sin \theta_W} - \frac{\sin \theta_W}{2 \cos \theta_W} \right) \gamma^\mu P_L \right) + \left( -ie \frac{\sin \theta_W}{\cos \theta_W} \gamma^\mu P_R \right).
\end{align*}
\]

Lastly, similarly to the W boson, the Z boson will have a propagator that takes into account its mass and finite lifetime. The expression for the Z boson propagator is

\[
\mu \quad \nu = \frac{-ig_{\mu \nu}}{(q^2 - M_Z^2) - iM_Z \Gamma_Z}.
\]

We can now put our results from Eq. 3.21 Eq. 3.22 Eq. 3.23 together with the Feynman rules from Sec. 1.3.1 to write down an expression for our amplitude

\[
\mathcal{M} = \bar{\mu}(p_3) g_{L,\nu} \gamma^\mu P_L u^s(p_1) \times \frac{-ig_{\mu \nu}}{(q^2 - M_Z^2) - iM_Z \Gamma_Z} \times \bar{\nu}(p_4) (g_{L,p} \gamma^\nu P_L + g_{R,p} \gamma^\nu P_R) u^s(p_2) = \frac{-i}{(q^2 - M_Z^2) - iM_Z \Gamma_Z} \bar{\mu}(p_3) g_{L,\nu} \gamma^\mu P_L u^s(p_1) \bar{\nu}(p_4) (g_{L,p} \gamma^\mu P_L + g_{R,p} \gamma^\mu P_R) u^s(p_2),
\]

where I have directly written down the expression with the incoming spinors. Our complex conjugated amplitude is then given by

\[
\mathcal{M}^* = \frac{+i}{(q^2 - M_Z^2) + iM_Z \Gamma_Z} \bar{\mu}^s(p_1) g_{L,\nu}^* \gamma^\mu P_L \bar{u}^r(p_3) \bar{\nu}^s(p_4) (g_{L,p}^* \gamma^\mu P_L + g_{R,p}^* \gamma^\mu P_R) u^r(p_2).
\]

Joining our expressions for \( \mathcal{M} \) and \( \mathcal{M}^* \), we can get the squared amplitude \( |\mathcal{M}|^2 \).

As mentioned in Sec. 3.2, neutrinos only contribute a \( \frac{1}{3} \) term to the spin average.
However, now we have a neutrino and a proton in the initial state. The proton is not limited like the neutrino and contributes a factor of $\frac{1}{2}$ to the spin average term. The spin averaged and summed amplitude is therefore

$$\frac{1}{2} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{1}{2} \frac{1}{(q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \times$$

$$\sum_{s,s',r,r'} \left( (g_{L,\nu} g_{L,\nu})^2 \bar{u}^r(p_3) \gamma^\mu P_L u^s(p_1) \bar{u}^{r'}(p_4) \gamma_\mu P_L u^{s'}(p_2) \times \bar{u}^s(p_1) \gamma^\nu P_L u^r(p_3) \bar{u}^{s'}(p_2) \gamma_\nu P_L u^{r'}(p_4) \right) +$$

$$\left( (g_{L,\nu} g_{R,\nu} g_{L,\nu}^*) \bar{u}^r(p_3) \gamma^\mu P_L u^s(p_1) \bar{u}^{r'}(p_4) \gamma_\mu P_R u^{s'}(p_2) \right) \gamma^\nu P_L u^r(p_3) \bar{u}^{s'}(p_2) \gamma_\nu P_L u^{r'}(p_4) +$$

$$\left( (g_{L,\nu} g_{R,\nu} g_{L,\nu}^*) \bar{u}^r(p_3) \gamma^\mu P_L u^s(p_1) \bar{u}^{r'}(p_4) \gamma_\mu P_L u^{s'}(p_2) \right) \gamma^\nu P_R u^r(p_3) \bar{u}^{s'}(p_2) \gamma_\nu P_R u^{r'}(p_4) +$$

$$\left( (g_{L,\nu} g_{R,\nu})^2 \bar{u}^r(p_3) \gamma^\mu P_L u^s(p_1) \bar{u}^{r'}(p_4) \gamma_\mu P_R u^{s'}(p_2) \right) \gamma^\nu P_L u^r(p_3) \bar{u}^{s'}(p_2) \gamma_\nu P_R u^{r'}(p_4) \right).$$

Although this equation is long and complicated, we will find out that some of its terms are zero. For the sake of simplicity, we will skip the step of expressing these terms in index notation. Instead, we will just write down the expression in terms of traces. The curious reader is encouraged to check that these traces are indeed correct.

$$\frac{1}{2} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{1}{2} \frac{1}{(q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \times$$

$$\left( (g_{L,\nu} g_{L,\nu})^2 \right) \text{Tr} [\gamma^\mu P_L \bar{u}^r] \text{Tr} [\gamma^\nu P_L \bar{u}^{r'}] \text{Tr} [\gamma^\nu P_L \bar{u}^{s'}] \text{Tr} [\gamma_\mu P_R \bar{u}^r] +$$

$$\left( g_{L,\nu} g_{R,\nu} g_{L,\nu}^* \right) \text{Tr} [\gamma^\mu P_L \bar{u}^r] \text{Tr} [\gamma^\nu P_L \bar{u}^{r'}] \text{Tr} [\gamma^\nu P_R \bar{u}^r] \gamma_\nu P_L \bar{u}^s \gamma_\nu P_L \bar{u}^{s'} +$$

$$\left( g_{L,\nu} g_{R,\nu} g_{L,\nu}^* \right) \text{Tr} [\gamma^\mu P_L \bar{u}^r] \text{Tr} [\gamma^\nu P_L \bar{u}^{r'}] \text{Tr} [\gamma_\mu P_R \bar{u}^r] \gamma_\nu P_L \bar{u}^s \gamma_\nu P_L \bar{u}^{s'} +$$

$$\left( g_{L,\nu} g_{R,\nu} \right) \text{Tr} [\gamma^\mu P_L \bar{u}^r] \text{Tr} [\gamma^\nu P_L \bar{u}^{r'}] \text{Tr} [\gamma^\nu P_R \bar{u}^r] \gamma_\nu P_L \bar{u}^s \gamma_\nu P_R \bar{u}^{s'} \right).$$
Before attempting to solve these traces, recall Eq. 3.14 and notice the form from the second trace in the second and third term

\[
Tr \left[ p_{4}^{\gamma} \gamma_{\mu} P_{R} p_{2}^{\gamma_{\nu}} P_{L} \right] = Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{L} \right] = Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{R} P_{L} \right] = Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} 0 \right] = 0,
\]

\[
Tr \left[ p_{4}^{\gamma} \gamma_{\mu} P_{L} p_{2}^{\gamma_{\nu}} P_{R} \right] = Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{R} P_{L} \right] = Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} 0 \right] = 0.
\]

Thus, these two terms vanish. We can further simplify the remaining traces by moving the projection operators to one side just like we did in Sec. 3.2 and we are left with

\[
\frac{1}{2} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{1}{2} \frac{1}{(q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \times
\]

\[
\left((g_{L,\nu} g_{L,p})^2 Tr \left[ p_{3}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{L} \right] Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{L} \right] + (g_{L,\nu} g_{R,p})^2 Tr \left[ p_{3}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{L} \right] Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{R} \right]\right).
\]

Let us evaluate each trace individually. Notice that all four of these traces have the same form but with different momenta and projection operators (and two of them are the same). Moreover, we evaluated some of these traces in Sec. 3.2. Referring back to those results, our traces can be expressed as

\[
Tr \left[ p_{3}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{L} \right] = 2 \left( p_{3}^{\mu} p_{1}^{\nu} - (p_{1} \cdot p_{3}) g^{\mu\nu} + p_{1}^{\mu} p_{3}^{\nu} + i \varepsilon^{\alpha\mu\beta\nu} p_{3,\alpha} p_{1,\beta} \right),
\]

\[
Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{L} \right] = 2 \left( p_{4,\mu} p_{2,\nu} - (p_{2} \cdot p_{4}) g_{\mu\nu} + p_{2,\mu} p_{4,\nu} + i \varepsilon_{\mu\sigma\nu\sigma} p_{2}^{\sigma} p_{4}^{\sigma} \right),
\]

\[
Tr \left[ p_{4}^{\gamma} \gamma_{\mu} p_{2}^{\gamma_{\nu}} P_{R} \right] = 2 \left( p_{4,\mu} p_{2,\nu} - (p_{2} \cdot p_{4}) g_{\mu\nu} + p_{2,\mu} p_{4,\nu} - i \varepsilon_{\mu\sigma\nu\sigma} p_{2}^{\sigma} p_{4}^{\sigma} \right).
\]
Notice that the only difference between the traces with $P_L$ and the traces with $P_R$ is a relative minus sign. Substituting the results of our traces back into the expression for $\frac{1}{2} \sum |\mathcal{M}|^2$ and contracting the momenta, we get

$$\frac{1}{2} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{8g^2_{L,\nu}}{(q^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2} \left( g^2_{L,p}(p_1 \cdot p_2)(p_3 \cdot p_4) + g^2_{R,p}(p_1 \cdot p_4)(p_2 \cdot p_3) \right).$$

(3.24)

We can use Eq. 1.29 to express our amplitude in terms of $\cos \theta$ and $E_{CM}$. After substituting the dot products and using our expressions for $g_{L,\nu}, g_{L,p}$ and $g_{R,p}$, we arrive at the following result

$$\frac{1}{2} \sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{8 \cos^4 \theta_W \sin^4 \theta_W} \frac{1}{E_{CM}^2 (1 - \cos \theta) + M_Z^2} = \frac{1}{2} \times$$

$$E_{CM}^4 (1 - 4 \sin^2 \theta_W \cos^2 \theta_W + \sin^4 \theta_W (1 + \cos \theta)^2).$$

(3.25)

This equation is our final result for the analytic amplitude of the process $\nu_e p^+ \rightarrow \nu_e p^+$. The result depends on both the polar angle $\theta$ and the center-of-mass energy $E_{CM}$, as well as the mass and decay width of the $Z$ boson. We can now compare the analytic value from Eq. 3.25 to the results of our program.

Like in the previous sections, we calculate the leptonic tensor $L^{\mu \nu}$ associated with our process $\nu_e p^+ \rightarrow \nu_e p^+$ and contract it with its corresponding hadronic tensor, dividing by 2 to account for the spin average term. The range for our center-of-mass energies goes from 20 GeV to 200 GeV with 190 evenly spaced samples and for a total of $10^5$ events per value of $E_{CM}$. We plotted our results in 6 histograms with 100 bins each along with plots of the analytic calculations. The six values of $E_{CM}$ are given at the beginning of this chapter and can also be found within the each plot.

The histograms can be found in Fig. 3.9. As we can see, our analytic and numerical results do not match at any point of the range of $\cos \theta$. However, the numerical plot follows the same behavior as the analytic and it is off by about a factor of 2. Since our plots would agree well if we multiplied our numerical result by 2, this makes us believe that the only problem might be a misplaced 2 somewhere in our code. At the time of this writing, we had not yet found the misplaced factor of 2, but once we
Figure 3.9. Numerical and analytic $\frac{1}{2} \sum |\mathcal{M}|^2$ vs. $\cos(\theta)$ for the process $\nu_e p^+ \rightarrow \nu_e p^+$. For each histogram, we used $10^5$ number of events per value of $E_{CM}$ divided into 100 bins. From left to right, top to bottom, the center-of-mass energies are 20, 60, 100, 140, 180 and 200 GeV. The analytic values were computed from Eq. 3.25.

...do, this discrepancy should vanish. The important part about these histogram plots is that both the numerical and the analytic exhibit the same behavior. We shift our focus now to the cross section.
As usual, our cross section is given by Eq. 3.5 where our spin average term is $\frac{1}{2}$ as explained earlier. Like in the case for $\nu_e \bar{\nu}_\mu \to e^- \mu^+$, our amplitude does not blow up as $\cos \theta \to 1$, so we do not need a cut on the cosine. To obtain the analytic cross section, we follow our usual procedure of integrating over the range $[-1, 1]$ for $\cos \theta$ using *Mathematica* and multiplying by the conversion factor from GeV$^{-2}$ to pb. For the numerical cross section, we follow the same method outlined in Sec. 3.1 and Sec. 3.2. We average over our range of $\cos \theta$, divide by our flux factor $2E_{CM}^2$ and multiply by the conversion factor. Our results for the numerical cross section can be found in Fig. 3.10. Despite the difference in the amplitude plots, the analytic and numerical cross sections agree very well as can be seen in the figure. The numerical to analytic ratio starts at about 0.9 at low energies and grows closer to 1 at higher energies. This agreement is a little surprising given the difference in the previous plot;
we will also investigate this further. Again, however, we see the correct behavior for
the numerical and analytic cross sections, meaning that any disagreement should not
be fundamental. This concludes our analysis of the $\nu_e p^+ \to \nu_e p^+$ process.
4. Conclusion and Future Steps

We live in an exciting time for particle physics, particularly for neutrino physics. Within the coming decades, two experiments, the Deep Underground Neutrino Experiment (DUNE) and Tokai-to-Hyper-Kamiokande (T2HK), will provide us with unprecedented data of neutrino events that will hopefully hold the key to new physical phenomena unexplained by the Standard Model (SM). With this colossal amount of data available, physicists will require a fast and efficient method of testing the myriad of Beyond the Standard Model (BSM) theories. For this project, we proposed an algorithm capable of achieving this by automatically calculating the leptonic tensor of given processes. For the development of our algorithm, we utilized the **Universal FeynRules Output** (UFO) \(^{[19]}\) file format to obtain the relevant information of the theory, and relied on the Lark package \(^{[20]}\) and the Berends-Giele recursive relations \(^{[21]}\) for proper parsing and computation of the leptonic tensors. To test our program, we computed the numerical squared amplitudes and cross sections of three SM processes: \(e^-p^+ \rightarrow e^-p^+, \ \nu_e p_\mu \rightarrow e^-\mu^+\) and \(\nu_e p^+ \rightarrow \nu_e p^+\). While we got excellent agreement for the \(e^-p^+ \rightarrow e^-p^+\) case, our amplitudes and cross sections for \(\nu_e p_\mu \rightarrow e^-\mu^+\) and \(\nu_e p^+ \rightarrow \nu_e p^+\) still had some issues that need to be fixed. However, as explained in Ch. \(^{[3]}\), we should be able to correct these disagreements in our code and recover the correct results for both processes. For the future, we hope to extend our algorithm to deal with more complicated SM events such as the neutrino trident process \(\nu_\mu N \rightarrow \nu_\mu N \mu^+\mu^-\) as well as with BSM theories. We will also seek to extend our treatment of the proton as a particle with an internal structure to get results more appropriate to the real behavior of these particles. Despite the shortcoming, our preliminary results are promising for the continuation of this project.
REFERENCES
REFERENCES


