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Particle Dynamics from the Method of Nonlinear Realizations and Maxwell Group

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Abstract

Professor Tonnis ter Veldhuis provides Macalester students with research opportunities in theoretical physics. In the Summer of 2020, a team of three students were introduced to the method of nonlinear realization of symmetries by studying Prof. Veldhuis's prior research regarding an application of the method to membrane dynamics (1). Subsequent developments of this prior work involved research into torsion and riemann curvature tensors, as well as metric compatibility, highlighting the connection between symmetry and space-time structure. The foundation of this work was the D=4 Poincare algebra in the D=3 Lorentz group covariant form. After this introduction, each team member developed their own individual project. For my own original research, I began by constructing the D=4 Maxwell algebra, and obtained an invariant action corresponding to the dynamics of a charged particle in an external electromagnetic field through application of the same method of nonlinear realizations that I was introduced to earlier in the program. Further developments of my research involved a super-symmetrization of the Maxwell algebra, as well as an attempt to retrieve the action corresponding to spinning charged particles.

Keywords

Coset method, method of nonlinear realizations, particle mechanics, particle dynamics, Maxwell algebra, Maxwell Group

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1 Introduction

The Poincare group is of great significance to physical theories: a relavistic quantum field theory requires a Poincare-Invariant action [4], for example. And although the Poincare group has been intensely studied [1][3][4][6], there is also an extension of it, the Maxwell group, which is rather recent [7][8] and not as well studied. As such, developing research into the Maxwell group is of great interest. Thus, the objective of this research was to construct an action for a particle invariant under the Maxwell group. This was achieved by using the method of nonlinear realizations, which requires knowledge about symmetries and their spontaneous breaking, continuous groups and their associated algebras, the role of covariant objects in constructing invariant quantities, and finally about how to apply the technique itself. A brief background of each of these is covered in the subsections below.

1.1 Symmetries

A symmetry of a physical system's action is a transformation that leaves said action invariant [5]. These symmetries are of incredible physical importance, highlighted by Noether's Theorem: "Every continuous symmetry of a physical system's action has a corresponding conservation law." [4][5] Pertinent to this research, it is possible for the symmetries of a system to become spontaneously broken. An example of spontaneous symmetry breaking can be seen by considering the potentials in figure 1.

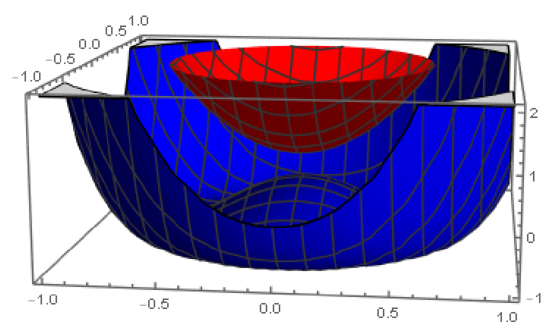

Figure 1: Parabolic (top) vs 'Sombrero' (bottom) potentials

In both the parabolic and 'sombrero' potentials, there is an apparent rotational symmetry. To see more clearly an example of spontaneous symmetry breaking, it is convenient to look at a cross section of these potentials, show below.

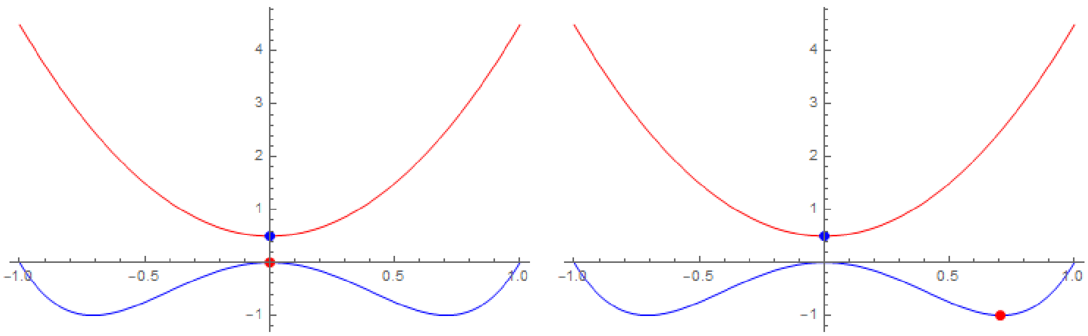

Figure 2: Unbroken rotational symmetry (left), Broken symmetry (right)

On the left are the cross-sections of figure 1 with a ball placed at an equilibrium position of each potential. At this position, the physical system still respects the rotational symmetry of the potential i.e. the symmetry is unbroken. However, the ball can enter a lower-energy state for the bottom potential, shown on the right. This means the ball can "spontaneously break" the rotational symmetry of the physical system by entering its ground state. Note that while the potential itself is still rotationally symmetric, the physical system of the ball and the potential is asymmetric when the ball is in the ground state.

This is just one example of spontaneous symmetry breaking [9]. More generally, when a system in a symmetric state spontaneously ends up in an asymmetric state, the system is said to exhibit spontaneous symmetry breaking. [9]

In the case where the symmetry is not spontaneously broken, a linear realization technique can be used. However, in the case where the symmetry is spontaneously broken, a method of nonlinear realizations must be used.

1.2 Lie Groups and Algebras

While there are both discrete and continuous symmetries, only the latter is considered in this paper. An example of a continuous symmetry is the continuous rotational symmetry of a sphere in three dimensions. Regardless of what angle the sphere is rotated by, it will retain its shape i.e. it is left invariant. Thus, the angles are over a continuous range and the symmetry is likewise continuous. In other words, a continuous rotational symmetry means that an object is left invariant under transformations of a continuous group. For the purposes of this paper, it is sufficient to define a *Lie group* as a continuous group. Lie groups have associated *Lie algebras* which encode the fundamental properties of the group. In general, Lie algebras are constructed by introducing a particular product that satisfies the following conditions:

Bilinearity

$$[x, ay + bz] = a[x, y] + b[x, z], \quad [ax + by, z] = a[x, z] + b[y, z] \tag{1}$$

Alternativity

$$[x, x] = 0 \tag{2}$$

Jacobi Identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{3}$$

In this paper, the product is the *Commutator*, and is defined as

$$[x, y] = xy - yx$$

Lie algebras can be described by an explicit choice of basis for its generators, or by writing down commutation relations of its generators; the latter choice is used in this paper. The generators of the Lie algebra can be exponentiated to obtain elements of the Lie group, i.e. the transformations themselves. The specific Lie groups/algebras discussed in this paper will be presented in the methodology section.

1.3 Covariance and Invariance

Constructing invariant quantities from covariant building blocks is essential to forming the action for a particle invariant under the Maxwell algebra.

A covariant object is simply something that transforms with the generators of the algebra and the elements of the group. A simple example of a covariant object in the three dimensional continuous rotation group (i.e. the Lie group SO(3)) is a vector; the generators and group elements of SO(3) are 3x3 matrices, which transform vectors and with other 3x3 matrices. Interestingly, vectors can be used to construct an invariant quantity: the dot product.

Consider the vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ and the rotation matrix $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Taking the dot product of $\vec{v}$ with itself, what's found is

$$\vec{v} \cdot \vec{v} = {v_x}^2 + {v_y}^2 + {v_z}^2$$

Now, transforming this vector by applying the rotation matrix:

$$\vec{v}' = R\vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \cos\theta - v_z \sin\theta \\ v_y \sin\theta + v_z \cos\theta \end{pmatrix}$$

And taking the dot product of $\vec{v}'$ with itself:

$$\vec{v}' \cdot \vec{v}' = {v_x}^2 + {v_y}^2(\cos^2\theta + \sin^2\theta) + {v_z}^2(\cos^2\theta + \sin^2\theta) = {v_x}^2 + {v_y}^2 + {v_z}^2$$

Upon examination, it is clear that $\vec{v} \cdot \vec{v} = \vec{v}' \cdot \vec{v}'$, i.e. the dot product of covariant vectors is invariant under rotations.

Furthermore, the action for a particle under the SO(3) group can be constructed from these covariant/invariant objects: the Lagrangian (integrand of the action) takes the form

$$L = \frac{1}{2}m(\vec{v} \cdot \vec{v}) - \frac{1}{2}k(\vec{r} \cdot \vec{r}) \tag{4}$$

Where $\vec{r}$ is the position vector associated with the particle.
From this point, dynamics of the particle can be obtained by solving the associated Euler-Lagrange equations.

2 Notation

In an effort to keep the paper as self-contained as possible, some relevant notation will be discussed.

2.1 Tensors and the Einstein Summation Convention

Previously, vectors were denoted by an arrow atop a letter, and were defined as a 3-component column vector. In 3-dimensional Euclidean space, it is possible (and useful) to instead denote a vector by an index a, which ranges from 1 to 3 and is defined like before:

$$v^a = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Fixing the free index a to some number in the range 1-3 picks the corresponding component. For example, $v^1 = v_x$.
While this object is still covariant, to construct an invariant quantity, another tensor is needed: the *metric tensor*.

In the previous section, rotations in 3-dimensional Euclidean space were considered. The metric tensor for 3D Euclidean space is

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, to construct an invariant quantity, two of the same covariant building blocks are contracted with this metric using the Einstein summation convention:

$$v^a g_{ab} v^b = v^1 g_{11} v^1 + v^2 g_{22} v^2 + v^3 g_{33} v^3 = {v_x}^2 + {v_y}^2 + {v_z}^2 = \vec{v} \cdot \vec{v}$$

Note particularly that while the Einstein summation convention indicates that terms like $v^1 g_{10} v^0$ should also be considered, the metric tensor contains only diagonal components, and so if the indices on the covariant vectors v^a and v^b are not equal, the term evaluates to zero.

Notably, equation 4 can be rewritten in tensor notation as

$$L = \frac{1}{2} m v^a g_{ab} v^b - \frac{1}{2} k r^a g_{ab} r^b \tag{5}$$

Where r^a is the position vector.

3 Methodology

With the fundamentals mostly described, the actual method of nonlinear realizations is available. The method begins by constructing a coset element. This is done by taking the Lie group of interest and "dividing out" those symmetries which can be linearly realized, leaving only the generators corresponding to nonlinearly realized symmetries [3]. Once the coset is constructed, the Maurer-Cartan form can be calculated. This form delivers the basic information about the structure of our group, and more tangibly the covariant building blocks that will be used to construct the action [1][2]. Once the building blocks are retrieved and the action constructed, all that's left is to solve the corresponding Euler-Lagrange equations and physically interpret them.

3.1 Lorentz and Poincare Group/Algebra

It is helpful to present the Poincare group, which the Maxwell group extends. Previously considered was the 3D rotation group, SO(3). For a relativistic theory, it is essential to extend this group temporally. This is known as the Lorentz group, SO(1,3). This includes transformations involving both Lorentz rotations as well as boosts. However, to include spacetime translations (which must be nonlinearly realized), the group must be extended again; the Poincare group

ISO(1,3) contains transformations involving spacetime translations, Lorentz rotations, and Lorentz boosts [3][6]. The Poincare algebra is shown below.

$$[P^m, P^n] = 0$$

$$[M^{mn}, P^k] = -\eta^{km} P^n + \eta^{kn} P^m$$

$$[M^{mn}, M^{kl}] = -\eta^{km} M^{nl} + \eta^{kn} M^{ml} + \eta^{lm} M^{nk} - \eta^{ln} M^{mk}$$

Above, the indices m, n, k, l range from 0 to 3, with 0 corresponding to time t, and 1,2,3 to the usual x, y, z. The P^m are spacetime translation generators, the M^{mn} are Lorentz generators, and the η^{mn} is the Minkowski metric.

3.2 Maxwell Group and Algebra

The Maxwell group is relatively new, and notation for it is not as consistent as the groups previously discussed. However, the Maxwell algebra is a clear extension of the Poincare algebra. The algebra is defined by the following commutation relations:

$$[P^m, P^n] = Z^{mn}, \qquad [M^{mn}, P^k] = -\eta^{km} P^n + \eta^{kn} P^m$$

$$[Z^{mn}, P^k] = 0 \qquad [Z^{mn}, Z^{kl}] = 0$$

$$[M^{mn}, Z^{kl}] = -\eta^{km} Z^{nl} + \eta^{kn} Z^{ml} - \eta^{lm} Z^{kn} + \eta^{ln} Z^{km}$$

$$[M^{mn}, M^{kl}] = -\eta^{km} M^{nl} + \eta^{kn} M^{ml} - \eta^{lm} M^{kn} + \eta^{ln} M^{km}$$

The extension is galvanized by making the spacetime translation generators no longer commute. Instead, their commutator results in an antisymmetric tensor Z^{mn} [7][8]. This new generator requires more commutation relations to fully describe the algebra, those relation being with the Lorentz generator, the spacetime translation generator, and itself.

3.3 Coset and Maurer Cartan Form

With the Maxwell algebra now defined, it is possible to truly begin the method of nonlinear realizations. First, the coset element must be defined. This is done by dividing the Maxwell group by the Lorentz group, and takes the form

$$\Omega \equiv e^{x^m P_m} e^{\Theta^{mn} Z_{mn}} \tag{6}$$

Here, the linearly realized symmetries have been "divided out" and only the nonlinearly realized symmetries remain, their corresponding generators being P_m and Z_{mn}. Note as Z_{mn} is antisymmetric, Θ^{mn} must also be antisymmetric.

With the coset element clearly defined, the Maurer-Cartan form can also be defined. This form will eventually deliver the covariant building blocks required for the action, so it is central to this paper. It is defined as the inverse of the coset element multiplied by the coset's exterior derivative [1][2][3]:

$$\text{Maurer-Cartan form} \equiv \Omega^{-1}d\Omega$$

3.4 Calculating the Maurer-Cartan form

To actually calculate the Maurer-Cartan form, additional formulae are helpful. These are known as the Baker-Campbell-Hausdorf formulae. While there are several of them, only the relevant equations are listed below:

$$e^{-A}de^{A} = dA + \frac{1}{2!}[-A, dA] + \frac{1}{3!}[-A, [-A, dA]] + ...$$

And

$$e^{A}Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + ...$$

Then the calculation for the Maurer-Cartan form is rigorous but straightforward:

$$\Omega^{-1}d\Omega = e^{-\frac{1}{2}\Theta^{mn}Z_{mn}}e^{-x^m P_m}(de^{x^m P_m}e^{\frac{1}{2}\Theta^{mn}Z_{mn}} + e^{x^m P_m}de^{\frac{1}{2}\Theta^{mn}Z_{mn}})$$

becomes

$$\Omega^{-1}d\Omega = dx^m P_m + \frac{1}{2}Z_{mn}(d\Theta^{mn} - x^m dx^n)$$

This can be rewritten in "building-block" form:

$$\Omega^{-1}d\Omega = \omega_P^m P_m + \omega_Z^{mn} Z_{mn} + \omega_M^{mn} M_{mn} \tag{7}$$

where

$$\omega_P^m = dx^m, \quad \omega_Z^{mn} = \frac{1}{2}d\Theta^{mn} - \frac{1}{4}(x^m dx^n - x^n dx^m), \quad \omega_M^{mn} = 0$$

4 Results

With the covariant building blocks obtained from the Maurer-Cartan form, it is time to construct an action for a particle invariant under the Maxwell group. Subsequently the associated Euler-Lagrange equations are to be solved and physically interpreted.

4.1 Lagrangian

The Lagrangian is the integrand of the action, and is sufficient for particle mechanics. Note that proper time is invariant, so using the building blocks

$$\omega_P^m/d\tau = \dot{x}^m, \qquad \omega_z^{mn} = \frac{1}{2}d\Theta^{mn} - \frac{1}{4}(x^m dx^n - x^n dx^m)$$

The Lagrangian takes the form

$$L = \omega_Z^{mn}\eta_{mk}\eta_{nl}\omega_Z^{kl} + \dot{x}^m\eta_{mn}\dot{x}^n \tag{8}$$

Rewriting ω_Z^{mn} as:

$$\omega_Z^{mn} = \left(\frac{1}{2}\dot{\Theta}^{mn} - \frac{1}{4}(x^m\dot{x}^n - x^n\dot{x}^m)\right)dt$$

Gives the full Lagrangian:

$$L = \frac{1}{4}\left(\dot{\Theta}^2 - \dot{\Theta}^{mn}(x_m\dot{x}_n - x_n\dot{x}_m) + \frac{1}{2}(x^2\dot{x}^2 - (x_m\dot{x}^m)(x^n\dot{x}_n)\right) + \dot{x}^2 \tag{9}$$

4.2 Equations of Motion

The components of the Euler Lagrange equations are:

$$\frac{\partial\mathcal{L}}{\partial\Theta^{mn}} = 0, \qquad \frac{\partial\mathcal{L}}{\partial x^k} = f^{mk}\dot{x}_m$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\Theta}^{mn}} = f^{mn} = \frac{1}{2}\dot{\Theta}^{mn} - \frac{1}{4}(x^m\dot{x}^n - x^n\dot{x}^m), \qquad \frac{\partial\mathcal{L}}{\partial\dot{x}^k} = f^{km}x_m + 2\dot{x}^k$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\Theta}^{mn}} = \dot{f}^{mn} = 0, \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^k} = f^{km}\dot{x}_m + 2\ddot{x}^k$$

So the equations of motion are:

$$\dot{f}^{mn} = 0, \qquad \ddot{x}^n = f^{mn}\dot{x}_m \tag{10}$$

Where f^{mn} is manifestly antisymmetric.

4.3 Physical Interpretation

To guide the physical intepretation of these equations of motion, it is helpful to consider the well-studied covariant relativistic Lorentz force:

$$m\ddot{x}^m = -q(F^{nm}\dot{x}_n) \tag{11}$$

Here, F^{nm} is the electromagnetic field strength tensor, which contains the components of the electric and magnetic fields.

This equation looks incredibly similar to equation (10) in the previous section, and taking $f^{mn} = \frac{q}{m}F^{mn}$, it is identical. Therefore the dynamics of a relativistic, charged, massive free particle in an electromagnetic field have been retrieved. Furthermore, the penultimate equation in the previous section indicates that this EM field is constant, as it is not changing in time.

Thus the objective of this project has been fulfilled and some interesting particle mechanics have been retrieved.

5 Afterword and Future work

In conclusion, the useful method of nonlinear realizations has been applied to retrieve the action for a particle invariant under the Maxwell group. The resulting equations of motion are those of a relativistic, charged, massive free particle in a constant electromagnetic field.

My application of this technique was made possible by first studying previous applications of it in a field theory context. This was chiefly done by working through my research advisor, Tonnis ter Veldhuis's paper on Brane dynamics from nonlinear realizations[1].

Future work using this method involve a variety of possibilities:

- Extending the Maxwell algebra to retrieve multipole dynamics
- Considering a time-dependent electromagnetic field strength tensor
- Conducting a super-symmetric extension of the Maxwell algebra
- Considerations of gravity and spin in writing the coset element

These are all invigorating possibilities, and there are even more possibilities considering that this method can be applied to other contexts as well.

Bibliography

1. Clark, T., Nitta, M., Ter Veldhuis, T. (n.d.). Brane dynamics from non-linear realizations. Retrieved from https://core.ac.uk/download/pdf/25356163.pdf

2. Attard, J. (n.d.). Conformal Gauge Theories, Cartan Geometry and Transitive Lie Algebroids. Retrieved from https://tel.archives-ouvertes.fr/tel-01944804/document

3. Delacrétaz, L., Endlich, S., Monin, A., E. (n.d.). (Re-)inventing the relativistic wheel: Gravity, cosets, and spinning objects. Retrieved from https://arxiv.org/pdf/1405.7384.pdf

4. Eichman, G. 2014, QCD and hadron physics, lecture notes,Universities of Giessen and Graz. Appendix B. Retrieved from http://cftp.ist.utl.pt/ gernot.eichmann/lecture-notes

5. Bertschinger, E. 2002. Symmetry Transformations, the Einstein-Hilbert Action, and Gauge Invariance. Retrieved from http://web.mit.edu/edbert/GR/gr5.pdf

6. Nisticò, G 2019 J. New representations of Poincare group for consistent Relativistic Particle Theories. Phys.: Conf. Ser. 1275 012034

7. Barducci, A., Casalbuoni, R., amp; Gomis, J. (2019, April 01). Contractions of the Maxwell algebra. Retrieved from https://core.ac.uk/display/200819668

8. De Azcarraga, J., Kamimura, K., amp; Lukierski, J. (2012, January 13). Maxwell symmetries and some applications. Retrieved from https://arxiv.org/abs/1201.2850

9. Arodz, H., Dziarmaga, J., amp; Zurek, W. H. (2003). Patterns of symmetry breaking. Dordrecht: Kluwer Academic.