Macalester Journal of Philosophy

Volume 8
Issue 1 *Spring 1998*

Article 11

3-25-2011

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Introduction

Can mathematics be reduced to logic? In the beginning of this century, eminent mathematicians and philosophers believed this question should be answered with an emphatic "Yes." Bertrand Russell shared this conviction and created, in collaboration with Alfred North Whitehead, the seminal *Principia Mathematica (PM)*, in which he sought to construct mathematics from logic. In my essay I will survey some of the main influences on Russell's work, and I will give a critical analysis of its success. In particular, I will argue that Russell's program failed due to a variety of philosophical problems, such as the employment of extra-logical axioms. This paper is intended to be a general philosophical survey, and I will therefore ignore some of the deep technicalities that arise when treating the *Principia Mathematica*. I believe, however, that my critique will still be valid, since it is more concerned with Russell's general approach than with the specific implementation of Russell's logicist program. Before we turn our attention to Russell's logicism, we should be clear about the underlying aim of the logicist approach to the foundation of mathematics.

The logicist program

The main thesis of logicism is, of course, that mathematics can be reduced to logic. Logic and mathematics are, therefore, not abstract sciences which are differentiated in kind but are, according to Russell, "identical" (Sainsbury, 272). In fact, Russell wrote:

> . . . logic has become more mathematical and mathematics has become more logical. The consequence is that it has now become wholly impossible to draw a line between the two: in fact, the two are one. They differ as boy and man; logic is the youth of mathematics and mathematics is the manhood of logic . . . (in Tiles, 59).

As should be clear from this quote, in contrast to the intuitionist, the logicist is committed to treat logic as fundamental to mathematics. Mathematics is based on logic and is an outgrowth of that abstract science of "correct reasoning." Thus, logicists treat historical precedence as immaterial to the discussion of the foundations of mathematics. For them it does not matter that logic, when understood as a science of formal languages, was a creation of the late 19th century, whereas mathematics had occupied humans for over two millennia. The claim for the more fundamental nature of a historically more recent theory is *prima facie* not problematic, since we have many cases in which specific instances of a general principle were discovered prior to the general principle itself. Arguably, science often functions in accordance with this claim. Outdated scientific theories can often be seen as approximating, specific instances of newly discovered, more fundamental theories. For instance, notwithstanding Kuhn, Newtonian physics could be viewed as a specific instance of a more general, but historically more recent theory—the theory of relativity.

More difficult to answer is the question, just what is meant by reducing mathematics to logic? The logicist answers by laying out the prospect of axiomatizing mathematics. She would like to create a formal system with logical axioms and rules of inferences which, provably, captures mathematics. More concisely:

> (i) Every mathematical truth can be expressed in a language all of whose expressions are logical: for short, every mathematical truth can be expressed as a true logical proposition.
>
> (ii) Every true logical proposition which translates a mathematical truth is a logical truth.
>
> (iii) Every mathematical truth, once expressed as a logical proposition, is deducible from a small number of logical axioms and rules . . . (Sainsbury, 272).

Right away we must recognize that such an axiomatized system could not be a system logically equivalent to predicate-calculus (to first order logic). Gödel showed the incompleteness of all formal systems powerful enough to express arithmetic. Yet he also showed first-order logic to be complete. Thus, by necessity, the logicist is committed to finding some second-order logic system. Therefore, the question of the validity of a specific logicist approach is a question about the admissibility of certain principles as logical axioms into a second-order logic system. Here the discussion centers around the validity of "Hume's principle" as a logical principle.[1] We, however, will look at some axioms used by Russell, such as the axiom of reducibility, which I will argue to be troublesome for the general logicist approach. Let us now turn to some of the

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[1] For illustrations of the meaning and use of "Hume's principle," see Heck.

main works that have substantially influenced Russell in his creation of the *Principia Mathematica:* the works of Peano and Frege.

Peano's Axiomatization of Arithmetic

Giuseppe Peano's axiomatic study of arithmetic was a major influence on Russell. Peano postulated the following five axioms and showed them to be sufficient to produce arithmetic. Given the primitive notion of "successor," "zero," and "number," he postulated:

> (1) 0 is a number.
> (2) Every number has a successor which is also a number.
> (3) No two numbers have the same successor.
> (4) 0 is not the successor of any number.
> (5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers. (adapted from Tiles, 74)

Russell assigned great importance to Peano's axiomatization of arithmetic. For Russell, it was a crucially important and powerful result to see arithmetic, with its boundless number of theorems, reduced to a handful of axioms. He was clearly influenced by this method of axiomatization, as well as by Peano's general methodology. (For example, the form of notation we find in *PM* is in part due to Peano.) However, comparable to the later Wittgenstein in relation to the logicist program, Russell formed his own approach to the foundation of arithmetic by a severe critique of the work that had had such a great impact on him.

First of all, he is extremely critical of the mere postulation of the "primitive" terms "zero," "number," and "successor." According to Russell, taking such terms for granted has ". . . all the advantages of theft over honest toil. . . ." (*Ibid.*). Secondly, Peano's approach might be formally, syntactically sufficient to yield arithmetic formulae; yet his axiomatization in no way gives us a satisfying semantic interpretation of arithmetic. According to Peano, "zero" is simply defined as the first element in an infinite series. But zero should really have the semantic interpretation of "emptiness," i.e., be equated to the empty set:[2]

> . . . we want our numbers to be such as can be used for counting common objects, and this requires that our numbers

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[2] This is one of the things that differentiates Russell's program from formalism.

should have a definite meaning, not merely that they should have certain formal properties. This definite meaning is defined by the logical theory of arithmetic. . . (Tiles, 74).

Russell, therefore, takes the production of adequate definitions of Peano's primitive terms ("zero," "number," "successor") to be one of his main tasks in *PM*. Such definitions should make it possible to deduce Peano's axioms from them. Furthermore, such definition should give a sufficient semantic specification to arithmetical terms, so that "zero," "number," and "successor" ". . . have meaning which will give us the right allowance of fingers, eyes and noses. . ." (*Ibid.*, 75). As a result, we see Russell and Whitehead give specific interpretations of Peano's primitive terms from which they try to deduce Peano's axioms (Hochberg, 369). Yet, the adequate definition and deduction of Peano's axiomatization of arithmetic is only one of the aims Russell and Whitehead hoped to achieve with their system. Another aim resulted from contemplating the problems incurred by Frege in his attempt to reduce arithmetic to logic.

The Problem of Frege's Logicism

The precursor of Russell's logicist program was Gottlob Frege's work. In his attempt to reduce arithmetic to logic, Frege was extremely successful: on his own he invented parts of first and second order logic, gave a searching critique of views that held arithmetic to be an *a priori* synthetic enterprise, and, finally, constructed an elaborate system of arithmetic from logic. There was only one problem: his system was inconsistent. This inconsistency was pointed out by Russell, whose argument can be reconstructed as follows (adapted from Hatcher, 110).

Frege defined numbers in terms of classes. But for Frege any condition determines a class. Thus, given the condition $x \notin x$, some class c can be defined in terms of $x \notin x$. This means c is the set of all sets that do not contain themselves. We now ask whether c contains itself or not. If c contains itself, it follows that $c \in \{x | x \notin x\}$. Thus, c must fulfill the specification $\{x | x \notin x\}$, of not belonging to itself, in order to belong to itself, which is a contradiction. Furthermore, suppose that $c \notin \{x | x \notin x\}$. In this case, c does not fulfill the condition of not belonging to itself, which is the specification $\{x | x \notin x\}$. If it does not fulfill this specification, then it must belong to itself, but this is what we just denied in our supposition. We, thus, arrive at a paradox, an unsolvable inconsistency in Frege's system.

More technically, Frege assumed a certain principle of abstraction defined as

$$\forall x: (x \in \{y | A(y)\} \leftrightarrow A(x))$$

where A(y) is a wff,[3] containing y as free variable, x is free for y in A(x), and A(x) results from A(y) upon replacing y by x in all the free occurrence of y in A(y). This principle, together with the assumption that $x \notin x$ defines a class, makes it possible to arrive at a contradiction in Frege's system (by first assuming $\{x \mid x \notin x\} \notin \{x \mid x \notin x\}$ and deriving $\{x \mid x \notin x\} \in \{x \mid x \notin x\}$, and then assuming $\{x \mid x \notin x\} \in \{x \mid x \notin x\}$ which leads to $\{x \mid x \notin x\} \notin \{x \mid x \notin x\}$).[4]

Russell, who had found this inconsistency, sought to fix this paradox in Frege's system. He agreed with Frege's care about the definition of number and the operations of arithmetic. But he found Frege's postulation of the unbridled correspondence of classes with definitions extremely problematic. For Russell, the question now arose: ". . . How can we set up a system that allows us to develop something like Frege's construction of mathematics but which is not contradictory? . . ." (Hatcher, 111).

Diagnosis: The Vicious Circle Principle (VCP)

The paradox found in Frege's system was not the only type of paradox that Russell wanted to banish from the logicist approach. Generally, according to Ramsey (in Hatcher, 118), we have to differentiate between two types of paradoxes—the syntactic paradoxes and the semantic paradoxes. The former are paradoxes, such as Russell's paradox, which can be *derived* in a set-theoretic formal system. These paradoxes are syntactic results of the formal system. The semantic paradoxes, on the other hand, make references to notions such as "truth" and "definability." They consist of a statement or statements that refer back to themselves in virtue of their meaning. For instance, the famous liar's paradox, encapsulated in the statement "This sentence is false," is a prime example of semantic paradoxes.[5]

Russell gave a concise definition of the source of the semantic and syntactic paradoxes. His thoughts, however, were greatly influenced by Poincaré's analysis of the paradoxes. Poincaré noticed the "circularity," some inherent self-reference, that seems to underlie the above-mentioned paradoxes. He introduced the notion of impredicativity:

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[3] Wff stands for "well formed formula."

[4] For more detail, see Appendix A with the full inconsistency proof for Frege's system.

[5] The distinction between syntactic and semantic to some degree breaks down in the proof methods of Gödel, i.e., the arithmetization of a formal language and with it the arithmetization of truth checking procedures.

[Impredicative definitions are] definitions by a relation between the object to be defined and all the objects of a certain kind of which the object to be defined is itself supposed to be a part (or at least some objects which depend for their definition on the object to be defined). . . (Hatcher, 115).

The important idea here is a critique of those definitions which contain themselves as part of their definition. Poincaré holds that such objects, which can only be defined in terms of themselves, should be regarded as meaningless. He postulates that a meaningful definition of an object must presuppose those objects which are part of the defined object's definition. This means that the objects used in the definition have to be already well-defined. If we think of definitions reaching only over a set of primitive objects, then such difficulties could not arise. Yet, when we allow definitions of objects unrestricted access to non-primitive objects, thus to include objects which themselves stand in need of clear definition, then paradoxes such as the one in Frege's system can easily appear. A result of this insight leads to the conclusion that not all definitions which seem to create a concept also create an extension with a determinate membership of objects. Russell agreed with Poincaré's analysis that the semantic and syntactic paradoxes are based on illicit self-references.[6]

As a result, Russell formulated a restrictive principle, which was supposed to banish the semantic and syntactic paradoxes from the logicist program. This principle is the vicious circle principle: ". . . No entity can be defined in terms of a totality of which it is itself a possible member. . ." (Tiles, 72). This vicious circle principle is a solution of the paradoxes, but it is itself not a mechanism to evade the paradoxes in the logicist program. What had to be created was a theory that would implement the restrictions of the vicious circle principle in logic, being itself nothing more than purely logical machinery.

> . . . It is necessary, that is to say, to construct a theory of expressions containing apparent variables which will yield the vicious-circle principle as an outcome. It is for this reason that we need a reconstruction of logical first principles, and cannot rest content with the mere fact that the paradoxes are due to vicious circles. . . (Russell, in Sainsbury, 327).

As such a logical reconstruction, Russell proposed the theory of types.

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[6] For example, see Russell's paper, "Le Paradoxes de la Logique" (1906).

Remedy: The Ramified Theory of Types

In order to avoid the semantic as well as the syntactic paradoxes in his system, Russell introduced the ramified theory of types. This theory consists of two main parts: the simple theory of types, onto which the theory of orders is superimposed. In 1903 Russell first enunciated the theory of types, which he claimed to be a solution for the syntactic paradoxes. He attempts to deal with the semantic paradoxes through his theory of orders.

Throughout *PM* Russell applies the theory of types to propositional functions, which are the primitive logical concept to which Russell attempted to reduce not only arithmetic, but also language itself. Propositional functions are generalizations over propositions:

> . . . By a 'propositional function' we mean something which contains a variable *x*, and expresses a proposition as soon as a value is assigned to *x*. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned. . . (*PM*, 38).

So, for example, both "x is a philosophy professor" and "tan x = 1" are propositional functions. Let us now define the simple theory of types (ST) for the propositional functions.

Russell proposes a stratification of logical objects[7] into different types. A type is defined in such a way that a property or propositional function can apply significantly only to objects of a single type, which is lower than the type of the property or propositional function. Thus, type 0 objects are called individuals and are all those objects which are neither propositions nor propositional functions. For example, 'Janet Folina' might be a type 0 object. Type 1 objects, however, are all those objects which apply significantly to type 0 objects. This means that all those propositional functions which have as their arguments only individuals, viz. type 0 objects, are considered to be type 1 objects. For example, the propositional function "x is an individual" would be among type 1 objects because it applies significantly only to type 0 objects. Type 2 objects are those propositional functions which apply significantly to type 1 objects. In general, type n propositional functions apply significantly only to objects of type n-1. For

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[7] "Logical objects" here mean propositional functions and their arguments for objects of type >0 and individuals form objects of type 0. These are the only structures Russell wanted RT to refer to. I don't attempt to smuggle in an existential claim (i.e., objects are things that exist), but I use this terminology for a more succinct and convenient way of writing about Russell's system.

a propositional function to apply significantly to some argument means that the propositional function must yield a truth value when the argument is inserted. If that is not the case, Russell points out, we are confronted with a non-significant application of a propositional function. Such an application Russell declares to be meaningless. An example for this case is the following attempt to apply 'the property of being a beautiful individual' for x in "x is a man." Here as an argument, an object of type 1 is used in place of an object of type 0. Consequently, the use of the above argument renders this application of the propositional function "x is a man" meaningless. How does this stratification of types banish the syntactic paradoxes?

Reconsider the way in which the inconsistency in Frege's system could be derived: in his system, a specification of the form $\{x \mid x \notin x\}$ was possible. But this means that the argument x that specifies the class x is of the same type as the class itself. By Russell's provision of types, however, such a specification of a totality would be illegitimate since the definiens is of the same type as the definiendum. This, translated into Russell's usage of propositional functions,[8] would be a non-significant argument use, which would be declared meaningless. In Russell's logic with ST, "Russell's paradox" cannot be formulated. ST is thus a restriction on the formation of meaningful sentences. It is a construction check eradicating illicit formal self-references in the arguments of propositional functions.

Russell was, however, not content with ST since this theory could not prevent the semantic paradoxes. For this purpose Russell amended ST with the theory of orders, which yields the ramified theory of types (RT). What problems did Russell envision?

Consider, for instance, the properties of a given type (from Sainsbury, 317), let's say type 1. Russell argued that this is not a legitimate totality, i.e., that the vicious circle principle will apply. For among those properties will be found the sentence $\forall\phi(\phi x)$ (the property of having *all* type 1 properties). This seems to be a type 1 object since the type of an object was defined on the domain of its arguments (the domain of x which is of type 0 in our case) and we did not take quantification into account. But the above statement quantifies over itself and thus defines itself in terms of itself. According to the VCP, such a totality could not be formed; thus, $\forall\phi(\phi x)$ must be ruled meaningless, and must be prevented.

Russell's solution is the introduction of the concept of order. The objects within a type are divided into orders. We begin with properties in a given type, for whose specification no quantification is needed. Inductively, we are then allowed only to quantify over objects that have already been specified. Thus,

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8 For a technical treatment of how ST avoids Russell's paradox, see Appendix B.

given all objects of type 1 that have no quantification as part of their definition, we are then allowed to quantify over these objects (see also *PM* 167). However, the class of objects we quantify over can contain only those objects already constructed; thus, it could not contain the type 1 object that does the quantification.

Arguably, RT prevents the semantic[9] and syntactic paradoxes, yet its introduction leads to deep and, as I will show, insurmountable problems for Russell's logicist program.

Failure: The Extra-Logical Axioms

The ramified theory of types is quite successful in eradicating the paradoxes that Russell meant to prevent from entering into his system. Yet, the mechanism of RT created tremendous problems for Russell's reductionist attempt to show arithmetic to be nothing more than an outgrowth of logic.

Remember that Russell's methodology of achieving this reduction was to prove that Peano's axioms could be deduced in his system of logic. Now consider Peano's third axiom, "No two numbers have the same successor." To show this to be a logical truth, Russell agrees that he would have to prove that if x and y are numbers and $x+1=y+1$ then $x=y$ (Sainsbury, 305). But

> . . . the definition of '+1' has the consequence that n+1 has members only if there is a class consisting of the union of some class, α, belonging to n, with some class, ß, whose sole member is an individual lying outside α. If there are exactly n individuals, there is no such ß, and so no such α∪β, and so $n+1=\Lambda$ [where L is the empty set]. Hence we [arrive at the result] that n and n+1 are numbers, $n+1=(n+1)+1$, yet $n \neq n+1$, contradicting what we had to prove. . . (*Ibid.*, 305).

This proof seems to show that presupposing a finite set will be insufficient to prove Peano's third axiom. This seems to imply that Russell needs an infinity of individuals in order to prove Peano's third axiom to be a logical truth. He solves it by theft, as opposed to honest toil; he simply stipulates his axiom of infinity, which states that there exist an infinite number of individuals. Why couldn't Russell get the infinity needed for his proof of the successor relation from a conglomerations of classes, classes of classes, and so on?

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[9] It does not block all semantic paradoxes, as was to be expected. For more detail, see Sainsbury, 321.

The reason for this is simple. Frege could put together logical classes, regardless of type restrictions, to form new classes. He proved the successor axiom by collecting together all the preceding classes corresponding to k numbers into some new class containing k + 1 members. But,

> . . . this will not work for Russell because a class of numbers is of higher type than any of the numbers of which it is made up and so will not be able to form part of their number series. Russell has to start from individuals and define n_{0} the class of classes X such that if X is the extension of $F(x_0)$, then $\exists_n F(x_0)$ . . .
> (Tiles, 72)

Here we see the necessity for the existence of an infinite number of individuals in Russell's system. Russell's type restriction requires him to define numbers as a series of objects of the same type, since otherwise quantification (as well as making property claims about the whole number series) would be obstructed by the type and order restrictions. This is so since, for example, any property claim about the whole of the natural numbers would require the domain over which the claim is made to be of lower type than the property claim itself. If we define 'number' as a conglomeration of higher and higher typed objects, we have no bound on the type of the argument of the property claim. But if we cannot determine the argument type, we cannot meaningfully assert the property claim and find it impossible to make statements about the property of all natural numbers. Since this is necessary for any theory of arithmetic, we either have to reject ST or define number in terms of individuals, which is what Russell did, of course. If we define number in terms of individual objects, and we want to work with an infinite amount of numbers (which is necessary if we want to have a full-fledged theory of arithmetic), then we also need an infinite amount of building blocks out of which the natural numbers are constructed. But how do we know that there are an infinite number of these "building blocks"? We don't, but can only assert it.

Russell's "solution" to the question of proving the successor relation leads to dire consequences: it invalidates the claim that his system represents a solution to the logicist program. Why? Because I claim that the axiom of infinity (necessary for his proof of the third axiom of Peano) is not a logical axiom. The reason for this is simple.

The axiom of infinity asserts that there is an infinity of individuals. This, however, is an existential claim which might be negated without contradiction. As Leibniz already pointed out, any logical truth must necessarily be an assertion that, when negated, leads to a contradiction. This means that for some assertion to be a logical truth, it is a necessary condition that this assertion must be necessarily true. But such is not the case with the axiom of infinity. It is not necessarily true that there has to be an infinity of individuals. We can negate this claim without contradiction. Its truth is at best verifiable by empirical means, although I would not know how that would be done. The axiom of infinity therefore appears to be an extra-logical claim, which pollutes Russell's system and, in fact, turns the system postulated in *PM* into a system which is not strictly logic (although it is certainly logical). Russell himself was aware of this difficulty, but found it impossible to dispense with it. This is the first grave point of critique we have to bring forward to the claim that the *PM* system embodies the solution to the logicist program.

A different problem occurred to Russell and Whitehead in connection with the introduction of the theory of orders. RT turned out to restrict the quantification over properties of arithmetic entities in such a way that important results of classical mathematics could not be derived. For instance, let us consider the concept of inductive number. Russell points out that an inductive number ought to be defined as ". . . one possessing *every* hereditary property of 0. . ." (Sainsbury, 333). But remember the theory of order: we are only allowed to quantify over properties that have already been defined. Thus, the above definition would form an illegitimate quantification over the totality of differently ordered properties, possibly including itself. Yet, this problem cannot easily be solved:

> . . . Nor is there an easy remedy, for if we insert a restriction to even a very high order there would still be the possibility that some yet higher order property is possessed by 0 and is hereditary for +1 and yet is lacked by some entity which, on this modified definition, is an inductive number. In short, it seems that however we restrict the order of relevant properties we shall not have a definition answering to our purposes. . . (*Ibid.*).

Yet, without the concept of inductive number, the concept of arithmetic induction would be impossible, which means that a great many mathematical results could not be produced in *PM*. Another problem that occurs due to the restrictions of RT is the difficulty of defining the identity of two objects. Russell saw identity to be the occurrence of exactly the same properties in the objects compared. Yet RT, and here the theory of order in particular, restricts the use of quantification over all properties of an object. In RT there does not exist an unrestricted quantification over all properties, which, in a sense, was precisely the aim of introducing the theory of orders in the first place. The above problems all indicate that it is difficult to have it both ways in Russell's system: it is difficult to avoid the semantic and syntactic paradoxes *as well* as retaining all of classical arithmetic. Russell proposed the following solution to the over restrictiveness of the theory of orders. He postulated the axiom of reducibility.

The axiom states that for every property of some given order and type, there exists a coextensive predicative property, where a property is predicative if ". . . it has the lowest order consistent with its type. . ." (*Ibid.*, 334). This means, given some propositional function ϕ of order n, we are assured that there exists a propositional function ψ applying to the same set of arguments as ϕ, but with order smaller than n: ". . . The axiom of reducibility lets us reduce the order of any term to the lowest possible order consistent with well-formedness. . ." (Hatcher, 146). The well-formedness criterion in effect means that we can assume ψ to have as the value of its order the value of the order of its highest order variable (Wilder, 241).[10] Thus, the axiom of reducibility both gives a determinate reduction of order and postulates that there exists a propositional function whose order corresponds to the reduced order value. But by the very definition of order, as being dependent on the quantification in terms, the axiom of reducibility allows us to ignore quantification. It therefore runs directly against the very intentions of Russell's order theory. This might not be surprising, since some self-referential definitions seem necessary in the definition of terms in classical mathematics (something that should become obvious when considering Weyl's reconstruction of analysis under the exclusion of non-specifiable and impredicative entities). If one accepts the axiom of reducibility, one can get classical mathematics and avoid the specific problems mentioned above. For example,

> . . . in the case of induction, [the axiom of reducibility] is supposed to have the consequence that the scruples we felt about the adequacy of the definition of an inductive number, once a restriction is placed on the order of the hereditary properties, can be swept away: and not by selecting some very high order of hereditary property, but rather the lowest order consistent with the type of numbers. . . (Sainsbury, 335).

Yet the acceptance of the axiom of reducibility is by no means a trivial step. It introduces new, serious philosophical problems for Russell's logicist aim.

First of all, it must be pointed out that, comparable to the axiom of infinity, the axiom of reducibility is an existential claim. It is an existential claim about the existence of a potentially infinite number of propositional functions

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10 The technical reasons for this claim are given in Hatcher, 147.

which have extensions that can be correlated to propositional functions of a higher order. But Russell's program so far has purposefully excluded any realist concept of class or conglomeration in general. His *PM*, with the exception of the axioms of reducibility and infinity, is emphatically constructivistic. This it must be since it purports to be a constructive proof of the logicist aim, namely showing, not arguing, that mathematics can be reduced to logic. The most straightforward, and perhaps only, proof for the truth of the logicist thesis is a construction which yields mathematics from simple notions *of* logic. But the axiom of reducibility makes this impossible because it is *extra*-logical in at least two senses. First, it simply postulates the existence of certain logical entities (propositional functions, properties) without showing or even requiring the construction of these entities. The predicative properties and propositional functions are simply assumed to exist. This is at least an *ad hoc* method and appears to me a return to a realist foundation (which, arguably, was already presupposed by the axiom of infinity).

But, furthermore, not only the content of the axiom of reducibility, but its very form renders it an extra-logical axiom. It makes an existential claim about logical entities. But we can negate this existential claim without any contradiction. We can assume that there are no predicative properties or propositional functions as postulated by the axiom of reducibility. What we get is a system that is not strong enough to express all of classical mathematics, but that is not a contradiction and not a logical problem as such. Thus, as in the case of the axiom of infinity, the axiom of reducibility also appears to be an extra-logical claim. Russell defended the axiom on pragmatic grounds: ". . . the reasoning it permits and the results to which it leads are all such as appear valid. . . ." (*PM* 59). But usefulness in the production of seemingly valid results is itself not a proof for the logical nature of an axiom. In fact, we should recall Russell's criticism of Peano's methodology. Russell criticized Peano for taking concepts that stand in need of proper justification for granted. He called it the benefit of "theft over honest toil." But the axiom of reducibility appears to me to be itself a prime case of assuming what we would like to have, instead of gaining it from adequate definitions and deductions. The axiom is not only not a logical axiom, but in its content highly disputable. The axiom of infinity had some empirical credibility, yet the justification for the axiom of reducibility seems to lie exclusively in the reasons for which it was assumed: namely, that it yields classical mathematics. It thus turns out that this axiom represents another part of *PM* which invalidates the logicist aim of reducing mathematics to logic via *PM*. Russell later came to recognize this:

> . . . Viewed from [a] strictly logical point of view, I do not see any reason to believe that the axiom of reducibility is logically necessary, which is what would be meant by saying that it is

> true in all possible worlds. The admission of this axiom into a system of logic is therefore a defect, even if the axiom is empirically true. . . (1919, in *Introduction to Mathematical Philosophy*) (Kline, 224).

The impossibility of abandoning the axioms of reducibility and infinity in *PM* without throwing important results of classical mathematics out of the window, led Russell to abandon his logicist program and declare in frustration:

> . . . I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable. . . (*Ibid.*, 230).

This completes our journey through Russell's logicism, and it is time to recapitulate some of the important notions introduced in this paper.

Conclusion

Can mathematics be reduced to logic? When one considers the strenuous effort of Russell and Whitehead, especially their failure to retain the results of classical mathematics in combination with the attempt to expel semantic and syntactic paradoxes from their system, one is tempted to conclude that what *PM* comes to prove is the opposite of Russell's aim: namely, that mathematics cannot be reduced to logic. In his attempt to prove the possibility of reducing mathematics to logic, Russell was motivated by the work of Frege and Peano. The latter restricted the task of proving logicism correct to gaining his axioms in a logical fashion. The inconsistencies in Frege's system, on the other hand, brought Russell to formulate both the source of, and a remedy for, the paradoxes afflicting the early logicist and set-theoretic approaches to the foundation of mathematics. Russell identified the source of the paradoxes to be an illegitimate application of the definiendum in the definiens which he encapsulated in his Vicious Circle Principle. Russell then attempted to avoid the paradoxes in his *PM* by using his ramified theory of types. This theory stratifies logical propositions and propositional functions into types and orders, making illicit self reference impossible. Yet, the ramified theory of types introduced the necessity for the inclusion of two extra-logical axioms—the axiom of infinity and the axiom of reducibility. The adherence to both axioms stifles the logicist aims underlying *PM*. *PM* is a logical system, but *PM* is not strictly logic. It follows that Russell was successful in formalizing mathematics and developing logic, but he failed in his main goal: to show logicism to be the correct view about the foundations of mathematics.

Appendix A

Deriving Russell's paradox (from Hatcher, 110)

Given Frege's principle of abstraction:

$$\text{Abstr}\ {}_{df}\ \forall x:(x\in\{y\,|\,A(y)\}\quad A(x))$$

where A(y) is a wff, containing y as free, x is free for y in A(x), and A(x) results from A(y) upon replacing y by x in all the free occurrence of y in A(y),

we can derive the contradiction $\vdash \{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}\,\&\,\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$ as follows

$\vdash \{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$

(1) $\{x\,|\,x\notin x\}\in\{x\,|\,x\notin x\}$	H
(2) $\{x\,|\,x\notin x\}\in\{x\,|\,x\notin x\}\quad\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$	Abstr.,UE
(3) $\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$	1,2 Taut,MP
(4) $\{x\,|\,x\notin x\}\in\{x\,|\,x\notin x\}\cdot\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$	1,3 Hyp. elem.
(5) $\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$	4, Taut,MP

$\vdash \{x\,|\,x\notin x\}\in\{x\,|\,x\notin x\}$

(1) $\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}$	(just proven)
(2) $\{x\,|\,x\notin x\}\notin\{x\,|\,x\notin x\}\quad\{x\,|\,x\notin x\}\in\{x\,|\,x\notin x\}$	Abstr.,UE
(3) $\{x\,|\,x\notin x\}\in\{x\,|\,x\notin x\}$	1,2 Taut,MP

Thus $\vdash \{x|x \notin x\} \in \{x|x \notin x\} \& \{x|x \notin x\} \notin \{x|x \notin x\}$ from the above two proofs.

Appendix B

How exactly does ST block Russell's paradox? (Sainsbury, 315)

'$a \notin a$' is meaningful only if '$a \in a$' is meaningful. A class variable, like 'a' in this context , abbreviates a class-abstract variable (*PM* 190) , so '$a \in a$' is meaningful only if some expression of the form '$z(\psi z) \in z(\psi z)$' is meaningful. Now let us modify *20.01 so as to eliminate the '!' which derives from RT: the result is $f\{z(\psi z)\} =_{df} (\exists\phi)((\forall x)(\phi x \mid \psi x) \& f\{\phi z\})$. Putting $z(\psi z) \in$ for 'f' gives:

$$z(\psi z) \in z(\psi z) =_{df} (\exists\phi)((\forall x)(\phi x \mid \psi x) \& (\exists\chi)((\forall x)(\chi x \psi x) \& \chi z \in \phi z))$$

Now let us modify *20.02 so as to eliminate the '!' which arises from RT. The result is $x \in \phi z =_{df} \phi x$. Applying this to the definiens, and putting χz for x we have:

$$=_{df} (\exists\phi)((\forall x)(\phi x \mid \psi x) \& (\chi)((\forall x)(\chi x \mid \phi) \& \phi(\chi z)))$$

But this expression is illegitimate. Since ϕ and χ are coextensive, they are of the same type , and so χz cannot occur in the argument place of ϕ . So the definiendum $z(\psi z) \in z(\psi z)$, and thus $a \in a$ and $a \notin a$ are illegitimate. Hence the Russell paradox cannot be significantly formulated if one adheres to ST.

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