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The Realist Account of Axiomatic Truth

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When Ernst Zermelo wrote "Investigations in the Foundations of Set Theory I" in 1908, great advances were being made in the area of set theory, but the actual notion of set remained unclear. There was a great need to increase the rigor of the discipline, not only to relieve the philosophical uneasiness present, but also (as became quite clear as the Burali-Forti paradox and Russell's paradox were revealed) to serve the needs of the working mathematician. It is in his paper that Zermelo provides the first axiomatic basis for set theory. This revived old, and created new, discussion on the nature of set-theoretic truths. Disagreements over which axioms, if any, should be accepted in the formal system were brought out. Further, the question was posed whether this axiomatization is merely a formal theory or actually expresses some deeper truth. The very notion of what a set is (if sets do in fact exist) is still in doubt. It is the aim of this paper to examine the realist answers to these questions.

Essential to the debate over axiomatic truth is the connection between truth and provability. There is a great desire to think that mathematical truths and mathematically provable statements coincide. But even if we accept the idea that all things provable are true (which, as will be brought up, is highly questionable), how can we be sure all things true are provable? The distinction between truth and provability does not pose any problems until we run into statements which cannot be proven to be true or false. I do not just mean statements which we have not devised a way to prove, or that we, as humans with limited capacities and resources, will never prove, but rather formally undecidable statements: those which can be shown to be neither provably true nor provably false in our system. The Continuum Hypothesis (CH) has been shown to be just such a statement. Neither it nor its negation can be proven. Both, in fact, are perfectly consistent with the rest of set theory. This means that no matter how much we develop our proof techniques, we will still never arrive at an answer in our formal, axiomatic system. This is quite a result, and the possible conclusions that can be drawn from it are a good way to examine the difference between the main schools of thought on the nature of axiomatic truth.

The realist believes, basically, in the existence of mathematical (in this case, set-theoretical) objects. Even though I am generalizing here, I am not saying there is some agreement between the realists on what the term "exists" means. The viewpoint of some extreme Platonist would probably scare the "average" realist (and even the "average" realists would not totally agree). But at the center of this is the idea that there actually is a set-theoretic universe that they are trying to describe. An immediate consequence of this is the idea that set-theoretical statements have a definite truth value. In talking of objects that exist, statements about them are either true or false. It would, of course, be nice if we could prove the truth or falsity; but for a realist, a proof is merely a confirmation or a demonstration of the truth or falsity. As Crispin Wright wrote, "[For some mathematical theorem] The theorem is not made true by its proof; it is merely conclusively shown to be true" (Wright, 8). The realist, then, sees a definite separation between truth and provability. Things which are provable are true if our axioms are true (because, of course, accepting the truth of our axioms automatically means we must accept the truth of the conclusions), but not all the things which are true are provable. Both parts of this must be examined; I will start with the latter.

The realist notion of the separation of truth and provability locks one into a view that the formal system with which we prove things is in some way inadequate. That is not to say that the axioms are incorrect (though I would think the realist has to accept this as a possibility); it just means that our axioms do not fully capture the set-theoretic universe. It was never stated by the realist that we know all the properties of a system in advance (Frege made the analogy between discovering a mathematical truth and geology). Going back to the Continuum Hypothesis, we may, through our continued study of the field, come upon an axiom we feel we can justifiably add to our system that yields CH (or its negation), thus exposing the truth value that it always held.

The realist can perhaps be seen, then, as striving for some ideal axiomatization which perfectly expresses the system in question. In this ideal, all truths would be provable (in the sense that the formal system allows their proof, not that we, with our finite time constraints, could actually prove every truth). But there may be truths that cannot be expressed in any formal language. Hao Wang calls them absolutely undecidable, which I take to mean propositions which are undecidable regardless of which or how many new axioms are added. He says of this realist view, which he calls the objectivist viewpoint: "Perhaps this does leave room for the possibility [of absolutely undecidable propositions]" (Wang, 545). I think the realist would have to leave open this possibility; nonetheless, Wang states, "But nobody knows how to work with the concept of absolute undecidability" (ibid.). Presumably he does not see it as essential to the realist position, but I think it is an important point to look at. The way I see it, the realist claims an absolute distinction between truth and provability: not just that truths do not necessarily correspond to provable statements in our formal system, but that they do not necessarily correspond to provable statements in any language; or to put it another way, truth runs deeper than expressability.

Realists accept that the axioms of set theory (or of any mathematical theory) may not be complete, but they would like to think they are true. The process of coming to know which axioms to hold as true, however, is no easy matter. Wang wrote, "Sometimes we accept an axiom once stated, sometimes it takes a fairly long time before an axiom is accepted (e.g., the axiom of choice), but at the end we reach an agreement" (Wang, 545). Accepting this claim as true may add to the strength of the realist argument. The fact that a viewpoint gains almost universal agreement goes in favor of the idea that there is a right or wrong answer to the questions being posed. But this does nothing to help us explain by what process we arrive at the agreement. It only tries to decrease the value of the question by saying that a conclusion will be reached. The process itself may be unique for each decision, and may rely on psychological factors that are considered outside of the realm of mathematics. But I still think we need to look at the central question, "By what criteria do we judge the truth of the axioms?"

One idea is that of intuition. As the often-quoted Gödel statement goes, "[The axioms] force themselves upon us as being true." This is a very appealing viewpoint in many respects. There does seem to be a certain obviousness about many mathematical claims.[1] A problem that quickly arises, however, is the fact that intuitions can lead us astray. Truths deemed "intuitive" have turned out to be incorrect. An obvious example is that of the naive conception of set. It seems perfectly obvious at fundamentals, but as we examine the full ramifications, we see that it leads to contradictions. I do not think this means that all intuitions are faulty and can never be relied on, but I think we need to look more closely at the nature of intuition.

The problem of bad intuitions can be partially explained away, I think, by distinguishing between intuition and something like first reaction. Upon first exposure to a concept, we may hold an incorrect idea; but as we really examine the concept and how we understand it, we may drop our first instinct for some other intuition. (Further, I do not think it is even necessary for us to explicitly realize the inconsistency of our first opinion for us to reject it. We may just disregard it out of an idea that the other intuition explains the concept in a more desirable way.) But even if this idea of "first instinct" is not explored, the fact that we can have differing intuitions may be perfectly acceptable. Our reliance on intuition as a criterion becomes not that we need the intuition, but rather an intuition.

For example, in "The Iterative Conception of Set," George Boolos tries to give an intuition of the axioms of set theory. He does not deny that the naive conception of set is an intuition that people can hold or that the iterative conception is somehow more intuitive; he is just offering a new intuition. He argues that the fact that the naive conception came along first is merely an accident of history. There is no reason to assume it is prior, in some deeper sense, to the iterative conception. The iterative conception, he writes, "strikes people as entirely natural, free from artificiality, not at all ad hoc, and one they might perhaps have formed themselves" (Boolos, 489). By showing, as he does, that the axioms of set theory (with the exception of the axioms of replacement and choice) follow from this intuition of set, he is supporting the claim that the axioms can be intuited.

But these exceptions need to be addressed. If intuition is to be the sole criterion on which we judge the truth value of the axioms, then we need to find some intuition (that does not contradict the iterative conception of set) that shows them to be true. While Boolos states that the axiom of choice is "obvious" (Boolos, 502), he makes the following comment about the axioms of replacement: "the reason for adopting [them] is quite simple: they have many desirable consequences and (apparently) no undesirable ones" (Boolos, 500). This is another possible criterion. Measuring the consequences and judging the fruitfulness that the adoption of an axiom will have is a big part of how actual decisions about this sort of thing get made. As Gödel wrote:

[1] To me, set-theoretic claims specifically seem a little less obvious than some other elementary mathematical statements. I think that can be partially attributed to my lack of past exposure to set theory.

> even disregarding the [intuitiveness] of some new axiom, and even in case it has no [intuitiveness] at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its 'success.' There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field and yielding such powerful methods for solving problems. . .that, no matter whether or not they are [intuitive], they would have to be accepted at least in the same sense as any well-established physical theory (Gödel, 32-33).

When taken in addition to his statement about the obviousness of some axioms, an attractive theory is suggested.

Gödel draws an analogy between mathematics and the natural sciences. He compares intuition to perception. We intuit mathematical truths in the same way that we perceive physical truths. But, just as we cannot perceive all of the truths about physical objects, there are mathematical truths we cannot intuit. In both cases we rely on theories which explain what we do perceive or intuit. In this sense, if a certain axiom is in itself unintuitable but yields many fruitful consequences which strike us as being intuitively true, then we have just cause to accept this axiom as true. With this set of criteria, we manage both to explain the obviousness of many mathematical truths and to justify the acceptance of those axioms which are not always obvious, but which yield desirable consequences. At the same time, however, there are ramifications to this view which should be investigated.

By linking truth to this idea of consequences, we are coming very close to making an *a posteriori* claim about the nature of mathematics. That is not to say that the realist position as I have described it is inherently *a priori,* but many realists would support that claim. Gödel and others allow fruitfulness in areas other than mathematics (e.g., physics) as a criterion of truth in mathematics. This does not sit well with many mathematicians, who would like to believe that their work is justified by something higher than the activities in some laboratory. They would make the objection that by basing acceptance of axiomatic truth on consequences the theory has in other areas, mathematics is reduced to being based on our experiences in those other areas. But we must take a moment to clear up an important point. There is a distinction that must be made between what makes something true and evidence that something is true. This is at the heart of the realist position. Just as proofs do not make theorems true, evidence and criteria (for the acceptance of axioms) don't make the axioms true for the realist. I dismissed a quote from Wang earlier, but I think we are now in a better position to examine it. As I see it, his claim is not that (as I used it earlier) we should disregard the question of how we accept axioms as true, just that the agreement reached upon is not what makes the axioms true.

In seeing this distinction, I think, Gödel's analogy escapes the earlier criticisms. He does not neglect the fact that there is a difference between evidence and underlying truth. In fact, this difference is just as true in physics as mathematics, so I feel the analogy grows even stronger. There still are other questions to be asked, however. For example, how do we determine the philosophical and mathematical importance of an axiom's consequences? Adding a powerful axiom will no doubt lead to many new theorems and proofs, but how are we to decide which consequences are desirable? Also, we can never know all the possible consequences of all the different axioms we might propose. On a pragmatic level, then, problems still remain.

I think the largest problem with Gödel's analogy is not pragmatic (in fact, for the most part Gödel's analogy is a description of how the working mathematician gets things done). Instead, I think the biggest concern is a philosophical one. The idea of desirable consequences being a criterion for an axiom's acceptance relies on the assumption that mathematics should be applicable to other areas of study. I am sure almost all mathematicians would say that it is; but I do not see this as a trivial assumption to make if your only starting point is the realist position. In other words, I do not think the realist point of view necessarily entails the applicability of mathematics. Why should the results of some abstract mathematical universe coincide with some physical reality? We would like to believe they do, but I think the realist position has a very difficult time defending that. If we are not positive that applicability should result, there seems no reason to accept desirable consequences as evidence of the truth or falsity of an axiom.

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