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The forget time for random walks on trees of a fixed diameter

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Abstract

A mixing measure is the expected length of a random walk on a graph given a set of starting and stopping conditions. We study a mixing measure called the forget time. Given a graph G, the pessimal access time for a target distribution is the expected length of an optimal stopping rule to that target distribution, starting from the worst initial vertex. The forget time of G is the smallest pessimal access time among all possible target distributions. We prove that the balanced double broom maximizes the forget time on the set of trees on n vertices with diameter d. We also give a precise formula for the forget time of a balanced double broom.

Contents

1. Introduction

We can measure the connectedness of a tree T by looking at various measures on random walks on T. These are called *mixing measures*. In this paper we focus on the *forget time* between two vertices.

A random walk is created by starting at a vertex v_0 and picking one of its neighbors v_1 uniformly at random. We then picking one of its neighbors v_2 uniformly at random. We repeat this process until we have a random walk of our desired length.

The *hitting time* $H(i,j)$ from vertex i to vertex j is the expected number of steps in a random walk from from vertex i to vertex j. The *stationary distribution* is the approximate amount of time we spend at each vertex after an infinite random walk. For example, we will end up at vertex with high degree more often than a vertex with low degree. The *access time* to j is the expected number of steps it takes to get from a vertex chosen randomly based on the stationary distribution to vertex j.

In this paper we look at the forget time for a tree which is defined as

$$\mathsf{T}_{\mathsf{forget}} = H(a', b) + \mu_a H(b, a). \tag{1.1}$$

where a, b are the *foci*: the vertices that minimize the maximum hitting time and a', b' are the pessimal partners of a, b respectively. The *pessimal partner* j' of a vertex j is the vertex you start at such that when you end at j, you achieve the maximum hitting time to j. This is defined as

$$H(j', j) = \max_i H(i, j)$$

. Finally, μ_a is the probability that you walk from vertex b to vertex a.

We are specifically focusing on the tree that maximizes Equation 1.1 on n vertices with a fixed diameter d. Previous research on these questions without the restriction of fixing the diameter have proved that the tree that maximizes these measures is a path graph [1]. We find that when fixing the diameter, the double broom, as shown in 1.1, maximizes the forget time.

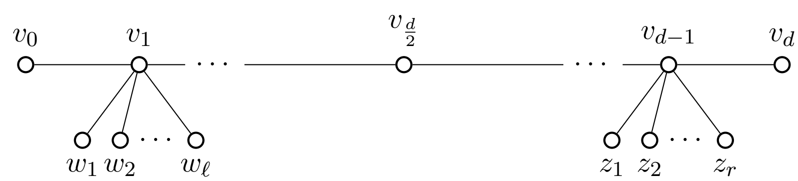

Figure 1.1 A balanced double broom with diameter d where $r = \ell$

2. Preliminaries

2.1 Graphs and trees

We start by defining some graph theory terms that we will use throughout the rest of the paper. A *graph* $G = (V, E)$ is a set of vertices V and a set of edges V that connect pairs of vertices. The *order* of the graph is $|V| = n$ and the *size* of the graph is $|E| = m$. Two vertices u, v are *adjacent* when there exists an edge (u, v) between them. We also call these vertices *neighbors* and denote their adjacency by $u \sim v$. The *degree* $\deg(v)$ of vertex v is the number of vertices adjacent to v.

A *path* between vertices x and y is a unique sequence of vertices $x = v_0, v_1, \ldots, v_d = y$ where $v_i \sim v_{i+1}$ for $0 \leq i < d$. We say a graph is *connected* if there exists a path between any pair of vertices. A *leaf* is a vertex of degree 1, that is, it is only connected to one vertex. A cycle in a graph is a path $(x = v_1, v_2, v_3, \ldots v_k = x)$ such that only the first and last vertices are equal and the rest are distinct. The distance between vertices u and v is the length of the shortest path between them, denoted $\mathrm{dist}(u, v)$. The diameter of the graph is the longest distance between any two vertices, denoted d. A path with length $d = \mathrm{diam}(G)$ is called a geodesic. We note that a graph can have multiple geodesics.

We specifically look at *trees*, which are *acyclic* (meaning they have no cycles) and connected graphs. A tree on n vertices always has $n - 1$ edges.

2.2 Special Trees

In this section we will define some types of trees that we will use in our proofs.

Definition 2.1. Let $\mathcal{T}_{n,d}$ denote the collection of trees on n vertices with diameter d.

Definition 2.2. Let $T \in \mathcal{T}_{n,d}$ be a tree. If T has multiple geodesics, then pick your favorite one and label the vertices on this geodesic $S = \{v_0, v_1, \ldots, v_d\}$ where $d = \mathrm{diam}(T)$. The tree T is a **caterpillar** when every vertex in C is either in S or is a leaf adjacent to a vertex in S. In this case, we refer to the geodesic S as the **spine** S of the tree. For $1 \leq i \leq d-1$, we define

$$W_i = \{v \in V \backslash S : v \sim v_i\}$$

to be the set of leaves in $V \backslash S$ that are adjacent to spine vertex v_i. An example of a caterpillar is shown in Figure 2.1.

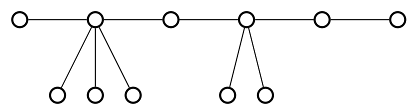

Figure 2.1 An example of a caterpillar graph

Definition 2.3. An **(unbalanced) double broom** $B \in \mathcal{B}_{n,d}$ is a caterpillar such that every vertex is either in S, W_1 or W_{d-1} where S is the spine and W_i are the leaf sets as defined above. We call W_1 the set of left leaves. We denote each vertex in W_1 as w_i where $|W_1| = \ell$. Similarly, we call W_{d-1} the set of right leaves. We denote each vertex in W_{d-1} as z_i where $|W_1| = r$. Without loss of generality, $\ell \geq r$. Note that $n = d + 1 + \ell + r$. An example of an unbalanced double broom is shown in Figure 2.2.

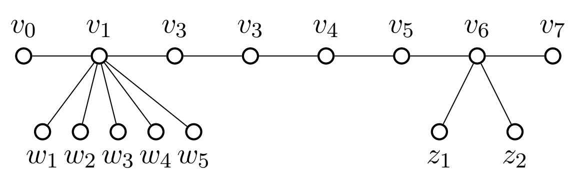

Figure 2.2 An example of an unbalanced double broom

Definition 2.4. A **balanced double broom** is a double broom where the leaves in $V \backslash S$ are divided as evenly as possible between W_1 and W_{d-1}. If the parity of d does not equal the parity of n, then $B \in \mathcal{B}_{n,d}$ is a double broom such that $\ell = r$ as shown in Figure 2.3. If the parity of n equals the parity of d, the balanced double broom $B \in \mathcal{B}_{n,d}$ is the double broom such that $\ell = r+1$ as shown in Figure 2.4. In both cases, we have $n = d+1+\ell+r$.

Definition 2.5. An **appendixed double broom** $C \in \mathcal{C}_{n,d}$ is a caterpillar where every vertex except one, call it v_a, is in either S, W_1 or W_{d-1}. Vertex v_a is a leaf adjacent to some vertex in $S \backslash \{v_0, v_1, v_{d-1}, v_d\}$. An example of an appendixed double broom is shown in Figure 2.5.

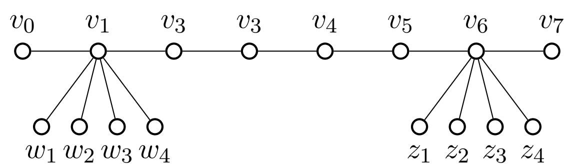

Figure 2.3 An example of a balanced double broom where $\ell = r = 4$

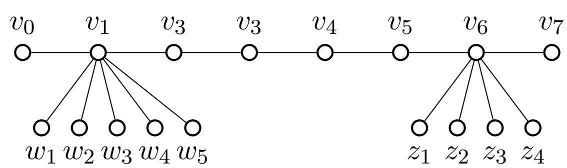

Figure 2.4 An example of a balanced double broom where $\ell = r + 1 = 4 + 1$

2.3 Measuring Random Walks

In this section we'll define some random walk terms and provide some equations we will use in later sections.

Let $T = (V, E)$ be a tree with order n and size $n - 1$. A random walk is a sequence of random vertices where the next u is chosen from the neighbors of the current vertex v with probability $\frac{1}{\deg(v)}$ if $u \sim v$.

The stationary distribution π is defined as the probability that we end up at each of the vertices in our tree after time t as $t \to \infty$. The stationary distribution for each vertex i, π_i is $\frac{\deg(i)}{2|E|}$ because we are more likely to end up at a vertex with high degree than a vertex with low degree. The denominator follows from the handshaking lemma which makes our distribution a probability adding up to 1. The stationary distribution for the whole graph is $\pi = \{\pi_{v_1}, \pi_{v_2}, \ldots, \pi_{v_n}\}$ where $\{v_1, v_2, \ldots, v_n\}$ are the vertices in our graph.

The hitting time $H(i, j)$ from vertex i to vertex j is the expected number of steps to get to vertex j starting at vertex i. If i and j are adjacent then denote $V_{i:j}$ as the subtree acquired by deleting the edge between i and j and only considering the connected component containing i. The hitting time between i and j is

$$H(i, j) = \sum_{p \in V_{i:j}} \deg(p). \tag{2.1}$$

The proof for this equation is explained in [2].

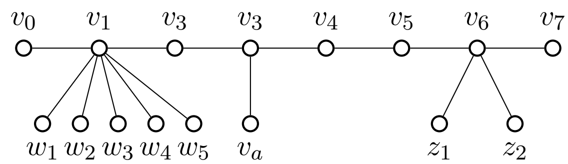

Figure 2.5 An example of an unbalanced double broom

Let i, j be nonadjacent vertices with path $\{i, u_1, u_2, \ldots u_p, j\}$ between them. We can find the hitting time from i to j recursively by first finding the hitting time from i to u_1 and then the hitting time from u_1 to u_2 etc until we get to the hitting time from u_p to j. Note that the vertices counted in $H(i, u_1)$ are also counted in $H(u_1, u_2)$, $H(u_2, u_3)$ and every subsequent hitting time until $H(u_p, j)$. However the vertices first counted in $H(u_{p-1}, u_p)$ are only counted in $H(u_{p-1}, u_p)$ and $H(u_p, j)$ while the vertices first counted in $H(u_p, j)$ are only counted in $H(u_p, j)$. So, the vertices first counted in $H(i, u_1)$ are counted $k+1$ times, while the vertices first counted in $H(u_p, j)$ are only counted once. We can account for this by definition the length of the shared path between i and j :

$$\ell(i, p; j) = \frac{1}{2}(\text{dist}(i, j) + \text{dist}(p, j) - \text{dist}(i, p)).$$

(See Figure 2.6 for example.)

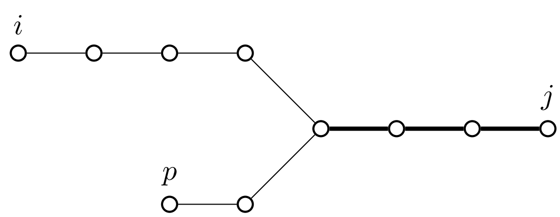

Figure 2.6 The length of the shared path (shown in bold) between the (i, j)-path and (p, j)-path is $\ell(i, p; j) = 3$. [3]

Thus, as shown in [2], the hitting time for any two vertices in a tree is

$$H(i, j) = \sum_{p \in V} \ell(i, p; j). \deg(p) \tag{2.2}$$

The following two examples are taken directly from [3].

Example 2.6. Let $T = (V, E)$ be the tree shown in Figure 2.7.

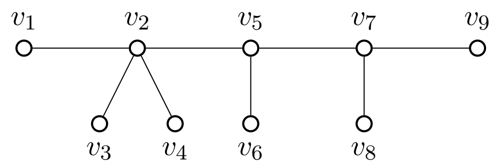

Figure 2.7 A tree on 9 vertices

Consider for example the hitting time $H(v_2, v_5)$ between adjacent vertices. By removing the (v_2, v_5) edge, we get the vertex partition $V_{v_2:v_5} = \{v_1, v_2, v_3, v_4\}$. Now, by Equation 2.1,

$$H(v_2, v_5) = \sum_{p \in V_{v_2:v_5}} \deg(p) = 1 + 4 + 1 + 1 = 7.$$

The following formula gives the hitting time from i to j for any vertices i and j on G:

$$H(i, j) = \sum_{k \in V} \ell(i, p; j) \deg(p). \tag{2.3}$$

Again, this is stated without a formal proof (which can be found in [2]), however, we demonstrate the intuition behind this formula in the following example.

Example 2.7. Let us revisit our tree on on 9 vertices. This time, consider the hitting time $H(v_1, v_9)$, which is equivalent to the sum of the hitting times along each step of the (v_1, v_9)-path, that is,

$$H(v_1, v_9) = H(v_1, v_2) + H(v_2, v_5) + H(v_5, v_7) + H(v_1, v_9).$$

We can visualize the vertex partitions determined by the edges along this path as a set of circles centered at v_1, as suggested in Figure 2.8.

It is then a straightforward matter to compute the hitting time using Equation 2.1:

$$H(v_1, v_9) = 1 + 7 + 11 + 15 = 34.$$

As we can see, $V_{v_1:v_2} \subset V_{v_2:v_5} \subset V_{v_5:v_7} \subset V_{v_7:v_9} \subset V$ This means that, in our hitting time formula, the degree of each vertex in $V_{v_1:v_2}$ is counted 4 times, the degree of each vertex in $V_{v_2:v_5}$ (but not in $V_{v_1:v_2}$) is counted 3 times, etc. Indeed, for every $p \in V$, the number of times $\deg(p)$ is counted is equal to $\ell(v_1, p; v_9)$, as claimed by Equation 2.3.

Next, we define the pessimal hitting times

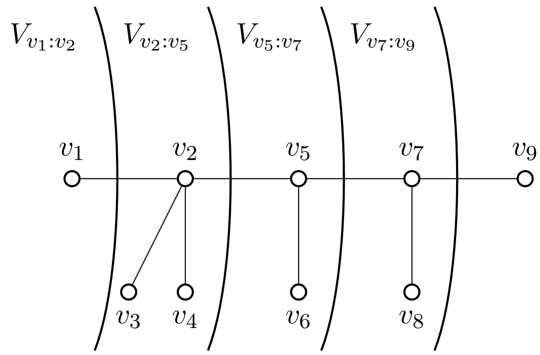

Figure 2.8 The vertex partitions along the (v_1, v_9)-path

Definition 2.8. The pessimal hitting time for any vertex i is the maximal hitting time from any vertex in the graph to i. The vertex that you start at to get this hitting time is denoted i' and is called the *pessimal partner*. In other words,

$$H(i', i) = \max_j H(j, i).$$

The *pessimal vertex in T* is the target vertex z that maximizes this measure, namely

$$H(z', z) = \max_i H(i', i).$$

2.4 The Forget Time

Lovász and Winkler [4] studied a class of parameterless mixing measures that use *stopping rules* to drive the random walk to a desired distribution. Suppose that we are given a *starting distribution* σ and a *target distribution* τ. A (σ, τ) *stopping rule* halts a random walk whose initial state is drawn from σ so that the final state is governed by the distribution τ. The *access time* $H(\sigma, \tau)$ denotes the minimum expected length for a such a (σ, τ) stopping rule to halt. We say that a stopping rule is *optimal* if it achieves this minimum expected length. Using access times, we have two natural mixing measures, the *mixing time* and the *reset time*, given respectively by

$$\mathsf{T}_{\mathsf{mix}} = \max_{v \in V} H(v, \pi), \quad \text{and} \quad \mathsf{T}_{\mathsf{reset}} = \sum_{v \in V} \pi_v H(v, \pi).$$

These are the worst-case and average-case mixing measures. In [5], Lovász and Winkler consider a more exotic mixing measure, called the *forget time*, given by

$$\mathsf{T}_{\mathsf{forget}} = \min_{\tau} \max_{i} H(i\tau)$$

where $i \in V$ and τ is any (target) distribution on the vertices. Remarkably, they prove that for an undirected graph, we always have

$$\mathsf{T}_{\mathsf{forget}} = \mathsf{T}_{\mathsf{reset}}.$$

Furthermore, the distribution μ achieving $\mathsf{T}_{\mathsf{forget}}$ is unique, and is given by the formula

$$\mu_i = \pi_i \left(1 + \sum_{j \sim i} H(j, \pi) - H(i, \pi) \right).$$

Furthermore, they show that if

$$H(x, y) = \max_{i \in V} \max_{j \in V} H(i, j)$$

Then

$$\mathsf{T}_{\mathsf{forget}} = H(x, \mu) = H(y, \mu).$$

2.5 Primary Focus and Secondary Focus of a Tree

In [2], Beveridge characterized the forget time on trees, and showed that the forget distribution μ is always concentrated on one vertex or on two adjacent vertices. First, we define these special, central vertices.

Definition 2.9. Let G be a tree. If $a \in V$ satisfies

$$H(a', a) = \min_{j \in V} \max_{i \in V} H(i, j)$$

then a is a *primary focus* of G. When all of the a-pessimal nodes are contained in a single subtree $G' \subset G \backslash \{a\}$, the unique a-neighbor b in G' is also a focus of G. If $H(b', b) = H(a', a)$ then b is a primary focus. If $H(b', b) < H(a', a)$ then b is a *secondary focus* of G.

When G has a single focus, we say that G is *focal*. When G has two adjacent foci, we say that G is *bifocal*. A bifocal tree may have two primary foci or it may have one primary focus and one secondary focus. $V_{b:a}$.

The primary focus a minimizes the pessimal hitting time $H(i', i)$. The secondary focus b is the vertex that minimizes this measure, second only to

a. Similarly, b has pessimal partner b'. We note that when T is bifocal, we know that $a \sim b$, vertex b is along the (a, a') path and a is along the (b, b') path. Consequenty, we know that the unique (b', a') path has the form

$$b' = v_0, \ldots, v_r = a, v_{r+1} = b, \ldots, v_d, = a'.$$

In other words, in order to get from b' to a', we encounter a immediately followed by b. Many of our results will use this structure of the unique (b', a')-path.

An example of a bifocal tree is shown in 2.9.

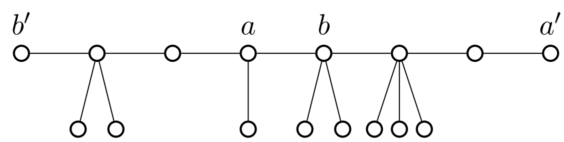

Figure 2.9 Example of bifocal tree

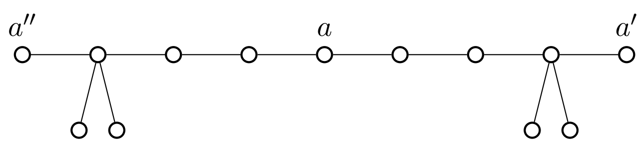

Figure 2.10 Example of focal tree

Beveridge [2] proved the following forget time results for trees. When T is focal with focus a and pessimal partners a' and a'', we have

$$\mathsf{T}_{\mathsf{forget}} = H(a', a) = H(a'', a).$$

When T is bifocal with primary focus a and secondary focus b, we have

$$\mathsf{T}_{\mathsf{forget}} = H(a', \mu) = H(b', \mu)$$

where the forget distribution is given by

$$\begin{aligned} \mu_a &= \pi_i(H(b', b) - H(a', b)), \\ \mu_b &= \pi_i(H(a', a) - H(b', b)), \\ \mu_i &= 0 \quad \text{for } i \notin \{a, b\}. \end{aligned}$$

2.6 Summary of Main Theorem

In this section we give the main result of this paper.

Theorem 2.10. *The quantity*

$$\max_{T\in\mathcal{T}_{n,d}} \mathsf{T}_{\mathsf{forget}}$$

is achieved uniquely by the double broom graph $D_{n,d}$ and is given by the formula

$$\max_{T\in\mathcal{T}_{n,d}} \mathsf{T}_{\mathsf{forget}} = \begin{cases} \dfrac{2(n-d)(d-2)+4}{4} & d \text{ is even}, \\ \dfrac{2(n-d)(d-2)+5}{4} & d \text{ is odd and } n \text{ is even}, \\ \dfrac{2(n-d)(d-2)+5}{4} - \dfrac{d-2}{2(n-1)} & d \text{ is odd and } n \text{ is odd}. \end{cases}$$

3. General Methods and Techniques

3.1 Commonly Used Moves

In this section, we will define some moves used commonly throughout the paper. Each move corresponds to one simple change between two trees G and G^* such that $\mathsf{T}_{\mathsf{forget}}(G^*) > \mathsf{T}_{\mathsf{forget}}(G)$. Note that we will not prove these results in this section, we will only be defining the moves. We will then prove that $\mathsf{T}_{\mathsf{forget}}(G^*) > \mathsf{T}_{\mathsf{forget}}(G)$ in Section 4.

Definition 3.1. Caterpillarize: Let G be a tree on n vertices with diameter d and spine $S = \{v_0, v_1, \ldots v_d\}$. Let $T_j = (W_j, E_j)$ be the subtree rooted at v_j for $j \in \{1, d-1\}$. Let G^* be a caterpillar on n vertices with diameter d. Let G^* be a tree on n vertices with diameter d, spine $S = \{v_0, v_1, \ldots v_d\}$ and every $v \in W_j$ is adjacent to v_j. Intuitively, $G^* = G$ but you take every subtree T_j and push the leaves up so that we have $e \sim v_j$ for all $e \in T_j$. This is $G^* = \text{caterpillarize}(G)$ and $G^* \in \mathcal{C}_{n,d}$. Example of caterpillarization is shown in Figure 3.1.

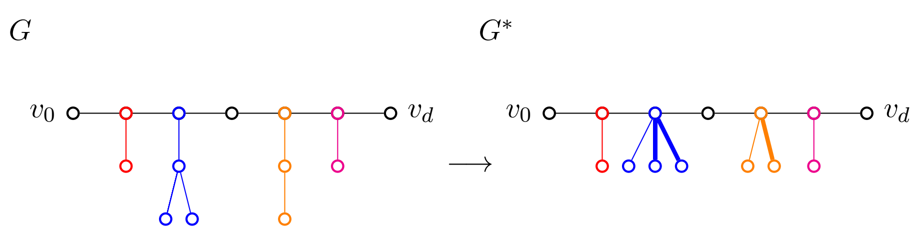

Figure 3.1 The caterpillarization of a tree: push vertices "up" so that they are adjacent to the path [3].

Definition 3.2. Moving leaves in pairs: Let G be a caterpillar on n vertices with diameter d. If there exists $2 \leq i \leq j \leq d-2$ and leaves $v_x, v_y \in V \backslash S$ such that $v_x \sim v_i$ and $v_y \sim v_j$ then we can move a pair of leaves such that $G^* = (V, E^*)$ is the caterpillar where

$$E^* = E - (v_i, v_x) - (v_j, v_y) + (v_{i-1}, v_x) + (v_{j+1}, v_y).$$

This is $G^* = \text{pairpush}(G, i, j)$. Here $G \in \mathcal{C}_{n,d}$ and $2 \leq i \leq j \leq d-2$. An example of pairpush is shown in Figure 3.2.

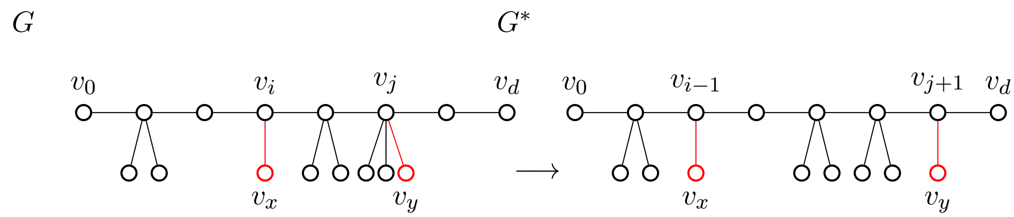

Figure 3.2 Example of pairpush.

Definition 3.3. Push: Let G be a caterpillar on n vertices with diameter d. If there exists $k+1 \leq j \leq d-2$ such that there exists v_a with $v_a \sim v_j$, then we can move v_a such that $G^* = (V, E^*)$ is the caterpillar where

$$E^* = E - (v_i, v_a) + (v_{i+1}, v_a)$$

This is $G^* = \text{push}(G, i)$. Here $G \in \mathcal{C}_{n,d}$ and $2 \leq i \leq d-2$. An example of push is shown in Figure 3.3.

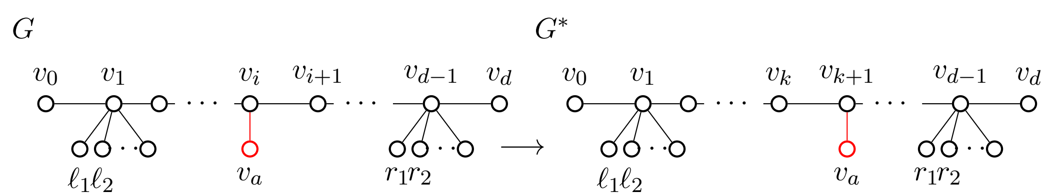

Figure 3.3 Example of push

Definition 3.4. Throw: Let $D_{d,\ell,r}$ be an unbalanced double broom. We can move one leaf v_x from W_1 to W_{d-1} such that $D^*_{d,\ell,r}$ is the double broom where

$$E^* = E - (v_1, v_x) + (v_{d-1}, v_x).$$

This is $G^* = \text{throw}(G, x)$. Here $G^* \in \mathcal{B}_{n,d}$. An example of throw is shown in Figure 3.4.

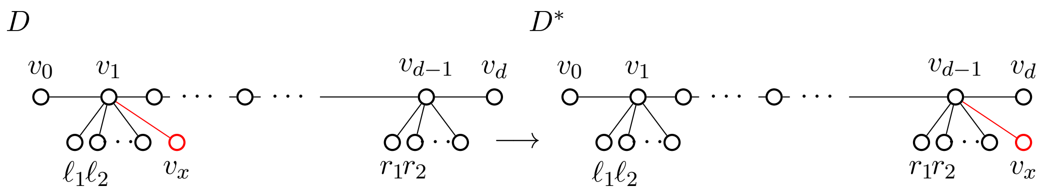

Figure 3.4 Example of throw

3.2 Commonly Used Lemmas

We also provide some commonly used lemmas that will help us prove our theorem in Section 4.

Lemma 3.5. *For any tree $G \in T_{n,d}$, for all $i \in V$, its pessimal partner i' is a leaf.*

Proof. Let i' be the pessimal partner for i. That means that $\max_{k\in V} H(k,i) = H(i',i)$. Suppose for the sake of contradiction that i' is not a leaf. Then there exists some vertex $j \sim i'$ such that i' is on the (j,i)-path. Then $H(j,i) = H(j,i') + H(i',i) > H(i',i)$, so i' is not actually the pessimal partner for i, a contradiction. □

Lemma 3.6. *Let $G \in G_{n,d}$ be a caterpillar on n vertices with spine $S = \{v_0, v_1, \dots, v_d\}$ and leaf sets $W_1, W_2, \dots W_{d-1}$. Then the pessimal partner for any vertex on the caterpillar is either v_0 or v_d, the vertices on the endpoints of the spine.*

Proof. By Lemma 3.5, we know that a' and b' must both be leaves. To show that the pessimal partner of any vertex on the caterpillar is either v_0 or v_d, we must consider two cases.

Case 1: $v_i \in S$. Without loss of generality, say the pessimal partner of v_i is $v_i' \in W_j$ where $j \le i$. If $v_i' \in W_1$, then $H(v_0, v_i) = H(v_i', v_i)$ because the hitting time from any leaf in W_1 to v_i is the same. If $v_i' \in W_j$ with $1 < j \le i$, then $H(v_0, v_i) > H(v_i', v_i)$ because

$$H(v_0, v_i) = H(v_0, v_j) + H(v_j, v_i),$$

and

$$\begin{aligned} H(v_i', v_i) &= H(v_i', v_j) + H(v_j, v_i) \\ &= 1 + H(v_j, v_i) \end{aligned}$$

because v'_i and v_j are adjacent since $v'_i \in W_j$.

Since $H(v_0, v_j) > 1$, we know that

$$H(v_0, v_i) > H(v'_i, v_i)$$

which is a contradiction and v'_i is not actually the pessimal partner of v_i. Thus, if $v_i \in S$, then $v'_i = v_0$.

Case 2: $u \in W_i$. If instead, the vertex was a leaf, we know that the hitting time from any vertex $w \neq u$ to u is

$$H(w, u) = H(w, v_i) + H(v_i, u),$$

so for u' we have

$$H(u', u) = H(u', v_i) + H(v_i, u).$$

We know that

$$H(u', v_i) \leq H(v'_i, v_i),$$

so

$$\begin{aligned} H(u', u) &= H(u', v_i) + H(v_i, u) \\ &\leq H(v'_i, v_i) + H(v_i, u) \end{aligned}$$

so $u' = v_i$. In case 1 we showed that $v'_i = v_0$ so $u' = v_0$ and in any case, the pessimal partner of a vertex on the caterpillar is v_0. Note that without loss of generality we said that $j \leq i$, so the same argument applies for $j \geq i$, and we get that the pessimal partner is v_d. So, pessimal partner of any vertex on the caterpillar is either v_0 or v_d.

□

4. Forget Time

In this chapter, we prove that the balanced double broom $B_{n,d}$ is the unique tree in $\mathcal{T}_{n,d}$ that maximizes the forget time. In Section 4.1 we calculate the forget time for balanced double brooms.

In Section 4.2, we show that any unbalanced double broom will have a forget time larger than a caterpillar on the same number of vertices with the same diameter.

In Section 4.3, we show that the balanced double broom maximizes the forget time among all double brooms.

Finally, in section 4.4, we show how to transform a tree with diameter d into a caterpillar with diameter d with a larger forget time.

4.1 Forget Time Formulas for Balanced Double Brooms

First, we provide the equations for the forget time of a balanced double broom depending on the diameter d and number of vertices n.

Lemma 4.1. *Let $B_{n,d}$ be the balanced double broom with n vertices and odd diameter d. Then $B_{n,d}$ is bifocal with $a = \frac{d-1}{2}$ and $b = \frac{d+1}{2}$.*

Proof. Recall that the foci are given by

$$\min_j \max_i H(i, j),$$

the hitting time to some vertex v_q on the spine from v_0 is

$$H(v_0, v_q) = q^2 + 2\ell(q - 1),$$

and the hitting time to q from v_d is

$$H(v_d, v_q) = (d - q)^2 + 2r(d - q - 1).$$

$\min_j \max_i H(i, j) = min_j H(j', j)$ is achieved when $q^2 + 2\ell(q - 1) = (d - q)^2 + 2r(d - q - 1)$.

We then have two cases: when n is even and $\ell = r$ and when n is odd and $\ell = r + 1$:

1. When n is even and $\ell = r$, we have

$$\begin{aligned} q^2 + 2\ell(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q^2 + 2r(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q &= \frac{d}{2}. \end{aligned}$$

Since d is odd, $\frac{d}{2}$ is not an integer, so the foci are at $\frac{d-1}{2}$ and $\frac{d+1}{2}$, the two closest integers. Since $\ell = r$, $H(v_0, \frac{d+1}{2}) = H(v_d, \frac{d-1}{2})$ so without loss of generality, we will say that $a = \frac{d-1}{2}$ and $b = \frac{d+1}{2}$.

2. When n is odd and $\ell = r+1$ we have

$$\begin{aligned} q^2 + 2\ell(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q^2 + 2(r+1)(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q &= \frac{d^2 + 2rd + 2}{2d + 4r + 2} \\ &= \frac{d}{2} - \frac{d+2}{2(d+2r+1)} \geq \frac{d}{2} - \frac{1}{2}. \end{aligned}$$

Since d is odd, this means that $a = \frac{d-1}{2}$. Since $H(v_d, \frac{d-1}{2}) > H(v_0, \frac{d-1}{2})$, the pessimal partner for a is v_d, so the pessimal partner for b is v_0 and thus $b = \frac{d+1}{2}$ because if it was less than $\frac{d-1}{2}$, its pessimal partner would be v_d.

□

Lemma 4.2. *Let $B_{n,d}$ be the balanced double broom with n vertices and even diameter d. If n is odd, then $B_{n,d}$ is focal with focus $a = \frac{d}{2}$. If n is even, then $B_{n,d}$ is bifocal with $a = \frac{d}{2}$ and $b = \frac{d}{2} - 1$.*

Proof. Recall that the foci are given by

$$\min_j \max_i H(i,j),$$

the hitting time to some vertex v_q from v_0 is

$$H(v_0, v_q) = q^2 + q\ell(k-1),$$

and the hitting time to v_q from v_d is

$$H(v_d, v_q) = (d-q)^2 + 2r(d-q-1).$$

To find $\min_j \max_i H(i,j)$, we solve $H(v_0, v_q) = H(v_d, v_q)$ for q and round to $\lceil q \rceil$ or $\lfloor q \rfloor$.

We then have two cases: when n is odd and $\ell = r$ and when n is even and $\ell = r + 1$:

1. When n is odd and $\ell = r$, we have
$$\begin{aligned} q^2 + 2\ell(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q^2 + 2r(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q &= \frac{d}{2}. \end{aligned}$$
Since d is even, $\frac{d}{2}$ is an integer and thus our only focus, so $a = \frac{d}{2}$.

2. When n is odd and $\ell = r + 1$ we have
$$\begin{aligned} q^2 + 2\ell(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q^2 + 2(r+1)(q-1) &= (d-q)^2 + 2r(d-q-1) \\ q &= \frac{d^2 + 2rd + 2}{2d + 4r + 2} \\ &= \frac{d}{2} - \frac{d+2}{2(d+2r+1)} \geq \frac{d}{2} - \frac{1}{2}. \end{aligned}$$
Since d is even, this means that $a = \frac{d}{2}$. Since $H(v_d, \frac{d-1}{2}) > H(v_0, \frac{d-1}{2})$, the pessimal partner for a is v_d, so the pessimal partner for b is v_0 and thus $b = \frac{d+1}{2}$ because if it was less than $\frac{d-1}{2}$, its pessimal partner would be v_d.

□

Lemma 4.3. *The value of the forget time of the balanced double broom $B_{n,d}$ is given by the formula*

$$\max_{T \in \mathcal{T}_{n,d}} \mathsf{T}_{\mathsf{forget}} = \begin{cases} \dfrac{2(n-d)(d-2)+4}{4} & d \text{ is even}, \\ \dfrac{2(n-d)(d-2)+5}{4} & d \text{ is odd and } n \text{ is even}, \\ \dfrac{2(n-d)(d-2)+5}{4} - \dfrac{d-2}{2(n-1)} & d \text{ is odd and } n \text{ is odd}. \end{cases}$$

Proof. We proceed with cases because when d is even, the pessimal partner for a is v_0 and when d is odd, the pessimal partner for a is v_d so we must use both forget time equations.

Our first case is when d is even.

We start with the forget time equation

$$\mathsf{T}_{\mathsf{forget}} = H(a', a) + \mu_b H(a, b)$$

where

$$\mu_b = \frac{1}{2|E|}\Big(H(a', a) - H(b', a)\Big).$$

By definition,

$$H(b', a) = (d-a)^2 + 2r(d-a-1),$$
$$H(a', a) = a^2 + 2\ell(a-1),$$

and

$$H(a, b) = 2(d-a) + 2r + 1.$$

The forget distribution at b is

$$\begin{aligned}
\mu_b &= \frac{1}{2|E|}\Big(H(a', a) - H(b', a)\Big) \\
&= \frac{1}{2(d+\ell+r)}\bigg(\Big(a^2 + 2\ell(a-1)\Big) - \Big((d-a)^2 + 2r(d-a-1)\Big)\bigg) \\
&= \frac{-2r(-a+d-1) + 2ad + 2\ell(a-1) - d^2}{2(d+\ell+r)} \\
&= a - 1 - \frac{(d-2)(d+2r)}{2(d+r+\ell)}.
\end{aligned}$$

We can plug in all of these equations to get

$$\mathsf{T}_{\mathsf{forget}} = (d-a)^2 + 2r(d-a-1) + \left(a - 1 - \frac{(d-2)(d+2r)}{2(d+r+\ell)}\right)(2(d-a)+2r+1). \tag{4.1}$$

We then proceed with subcases:

n even: When both n and d are even, we have an odd number of leaves, so without loss of generality say $\ell = r+1$. There are $n-d-1$ additional leaves. We have $\ell = (n-d)/2$ and $r = (n-d-2)/2$ and $b = d/2+1$. Additionally, by lemma 4.2, we know that $a = \frac{d}{2}$. Plugging these values into Equation 4.1 gives

$$\mathsf{T}_{\mathsf{forget}} = \frac{2d(n+1) - d^2 - 4n + 4}{4}.$$

n odd: When n is odd and d is even, $\ell = r$ and by 4.2 we know that $a = \frac{d}{2}$ and B is a focal double broom, so we use

$$\mathsf{T}_{\mathsf{forget}} = H(a', a) = \left(\frac{d}{2}\right)^2 + 2\frac{n-d}{2}\left(\frac{d}{2} - 1\right) = \frac{2d(n+2) - d^2 - 4n}{4}.$$

Next, we have the case where d is odd and we use the following forget time equation:

$$\mathsf{T}_{\mathsf{forget}} = H(b', b) + \mu_a H(b, a)$$

where

$$\mu_a = \frac{1}{2|E|}\Big(H(b', b) - H(a', b)\Big).$$

By definition,

$$H(a', b) = (d-b)^2 + 2r(d-b-1),$$
$$H(b', b) = b^2 + 2\ell(b-1),$$

and

$$H(b, a) = 2(d-b) + 2r + 1.$$

The forget distribution at a is

$$\begin{aligned}
\mu_a &= \frac{1}{2|E|}\Big(H(b', b) - H(a', b)\Big) \\
&= \frac{1}{2(d+\ell+r)}\Big(\Big(b^2 + 2\ell(b-1)\Big) - \Big((d-b)^2 + 2r(d-b-1)\Big)\Big) \\
&= \frac{-2r(-b+d-1) + 2bd + 2\ell(b-1) - d^2}{2(d+\ell+r)} \\
&= b - 1 - \frac{(d-2)(d+2r)}{2(d+r+\ell)}.
\end{aligned}$$

We can plug in all of these equations to get

$$\mathsf{T}_{\mathsf{forget}} = (d-b)^2 + 2r(d-b-1) + \left(b - 1 - \frac{(d-2)(d+2r)}{2(d+r+\ell)}\right)(2(d-b)+2r+1). \tag{4.2}$$

We then proceed with subcases:

n even: When n is even and d is odd, we have $\ell = r$ and $b = \frac{d+1}{2}$. We have $\ell = r = (n-d-1)/2$ and by 4.1, $b = (d+1)/2$. Plugging into Equation 4.2 gives

$$\mathsf{T}_{\mathsf{forget}} = \frac{2d(n+1) - d^2 - 4n + 5}{4}.$$

n odd: When both n and d are odd we have $\ell = r + 1$ and by 4.1, $b = \frac{d+1}{2}$. We have $\ell = (n - d)/2$ and $r = (n - d - 2)/2$ and $b = (d + 1)/2$. Plugging into equation Equation 4.2 gives

$$\begin{aligned} \mathsf{T}_{\mathsf{forget}} &= \frac{2dn^2 + d^2 - d^2n - 4n^2 + 9n - 4d - 1}{4(n-1)} \\ &= \frac{5 - d^2 - 4n + 2d + 2dn}{4} + \frac{2 - d}{2(n-1)}. \end{aligned}$$

□

Thus, we now have all the equations for the forget time on a balanced double broom.

4.2 Caterpillar to Unbalanced Double Broom

In this section, we prove that among all the caterpillars on n vertices with diameter d, the one that maximizes the forget time is a double broom.

Let $C \in \mathcal{C}_{n,d}$ be a caterpillar on n vertices with diameter d. Let $\mathsf{T}_{\mathsf{forget}}(C)$ be the forget time of C.

If C is bifocal, let a, b be the foci and a', b' be their pessimal partners respectively. By Lemma 3.6, a', b' will be the two endpoints of the caterpillar. Without loss of generality, say

$$b' = v_0, \quad a = v_k, \quad b = v_{k+1}, \quad a' = v_d.$$

If C is focal, let a be the focus and a', a'' be its pessimal partners. By Lemma 3.6, a', a'' will be the two endpoints of the caterpillar. Without loss of generality, say

$$a'' = v_0, \quad a = v_k, \quad a' = v_d.$$

Note that in our proof, we use a to mean both the vertex v_j that is the focus and to reference the integer j where a is the focus. We do the same with the letter b. It will be clear based on context whether we are using it to mean the vertex or the index of the vertex.

By [2] we know that for a bifocal caterpillar,

$$\begin{aligned} \mathsf{T}_{\mathsf{forget}}(C) &= H(b', a) + \mu_b \cdot H(a, b) && (4.3) \\ &= H(a', b) + \mu_a \cdot H(b, a) && (4.4) \end{aligned}$$

and for a focal caterpillar,

$$\mathsf{T}_{\mathsf{forget}}(C) = H(a'', a) = H(a', a). \tag{4.5}$$

Recall Equation 2.2:

$$H(i,j) = \sum_{p \in V} \ell(i, p; j) \deg(p).$$

Lemma 4.4. *Let $C \in \mathcal{C}_{n,d}$ and $C^* \in \mathcal{C}_{n,d}$ be two caterpillars on n vertices with diameter d, and let $1 \leq i < j \leq d$. Define*

$$P_t = \{p \in V : \ell(v_i, p; v_j) = t\}$$

to be the set of vertices such that the shared distance from v_i to v_j is t. Let

$$\deg(P_t) = \sum_{p \in P_t} \deg(p) \quad \textit{and} \quad \deg(P_t^*) = \sum_{p \in P_t^*} \deg^*(p).$$

for $0 \leq t \leq j - i$. If

$$\deg(P_t) = \deg(P_t^*) \quad \textit{for} \quad 0 \leq t \leq j - i,$$

then $H(x, y) = H^*(x^*, y^*)$.

Proof. By Equation 2.2, we have

$$\begin{aligned} H(v_i, v_j) &= \sum_{p \in V} \ell(v_i, p; v_j) \deg(p) \\ &= \sum_{t=0}^{j-i} t \, \deg(P_t) = \sum_{t=0}^{j-i} t \, \deg(P_t^*) \\ &= \sum_{p \in V} \ell(v_i, p; v_j) \deg^*(p) = H^*(v_i, v_j). \end{aligned}$$

□

Our next lemma shows how the push move increases forget time.

Lemma 4.5. *Let $C \in \mathcal{C}_{n,d}$ be a caterpillar on n vertices with diameter d. Let $C^* = \text{push}(C, j)$, the push caterpillar as defined in 3.3. Then, $H^*(v_d, v_i) = H(v_d, v_i) + 2$ for any $j < i < d$. Similarly, if we redefine the push move such that $0 \leq j \leq k$ instead of $k + 1 \leq j \leq d$, then $H^*(v_0, v_i) = H(v_0, v_i) + 2$ for any $0 < i < j$.*

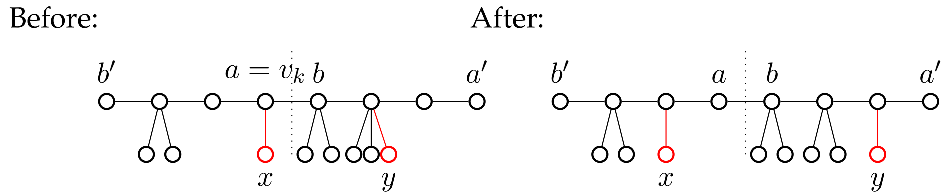

Figure 4.1 Example of Lemma 4.6 on a bifocal caterpillar.

Proof. When measuring $H^*(v_d, i)$, we add degrees recursively based on how close they are to i. (See equation Equation 2.2). Since $x \sim j \in C$ and $x \sim j+1 \in C^*$, it is one vertex farther away from i in C^*, so its degree has to be counted one more time. Similarly, the degree of vertex v_j is counted one less time because it is not adjacent to x, but the degree of v_{j+1} is counted twice because it is adjacent to x which adds a degree and is one vertex farther away from i which adds another degree. So we have

$$H^*(v_d, v_i) = H(v_0, v_i) + 1 - 1 + 2 = H(v_d, v_i) + 2$$

An analogous proof can be used to show that $H^*(v_0, v_i) = H(v_0, v_i) + 2$ if the push move is defined in the opposite direction. □

Our next lemma concerns a bifocal caterpillar with primary focus a and secondary focus b.

Lemma 4.6. *Let $C \in \mathcal{C}_{n,d}$ be a bifocal caterpillar on n vertices with spine $S = \{v_0, v_1, \dots, v_d\}$, primary focus $a = v_k$ and secondary focus $b = v_{k+1}$ Suppose that there exists $2 \le i \le k$ and $k+1 \le j \le d-2$ such that $x \in W_i$ and $y \in W_j$. Let $C^* = \text{pairpush}(C, i, j)$ the pairpush caterpillar as defined in Definition 3.2. Then C^* is a bifocal caterpillar with the same foci as C and $\mathsf{T}_{\mathsf{forget}}(C) < \mathsf{T}_{\mathsf{forget}}(C^*)$.*

Figure 4.1 shows an example of this lemma.

Proof. Recall that by Equation 4.3

$$\mathsf{T}_{\mathsf{forget}}(C) = H(b', a) + \mu_b\, H(a, b).$$

We make the following observations. First, $H^*(b', a) = H(b', a) + 2$ by Lemma 4.5. Second, $H(a, b) = H^*(a, b)$ since $a \sim b$, and $H(a, b)$ is the sum of degrees from b' to a which does not change. Third, μ_b does not change because

$$\begin{aligned}\mu_b^* &= \frac{1}{2(n-1)}(H^*(b',b) - H^*(a',b)) \\ &= \frac{1}{2(n-1)}(H(b',b) + 2 - (H(a',b) + 2)) \\ &= \frac{1}{2(n-1)}(H(b',b) - H(a',b)) \\ &= \mu_b.\end{aligned}$$

Thus

$$\begin{aligned}\mathsf{T}_{\mathsf{forget}}(C^*) &= H^*(b',a) + \mu_b^*\, H^*(a,b) \\ &= H^*(b',a) + \mu_b\, H(a,b) \\ &= H(b',a) + 2 + \mu_b\, H(a,b) \\ &> \mathsf{T}_{\mathsf{forget}}(C).\end{aligned}$$

Note that in C, we have $H(b',b) > H(b',a)$ and in $C^*, H^*(b',b) = H(b',b) + 2$ and $H^*(b',a) = H(b',a) + 2$. Thus, in C^*, we have $H(b',b) > H(b',a)$ so C^* is bifocal. □

We now prove an analogous lemma for a focal caterpillar with focus a.

Lemma 4.7. *Let $C \in \mathcal{C}_{n,d}$ be a focal caterpillar on n vertices with spine $S = \{v_0, v_1, \ldots, v_d\}$, focus $a = v_k$. Suppose that there exists $2 \le i \le k \le j \le d-2$ such that the leaf sets W_i and W_j are nonempty. Let $C^* = \mathrm{pairpush}(C, i, j)$ the pairpush caterpillar as defined in Definition 3.2. Then C^* is a focal caterpillar with the same foci as C and $\mathsf{T}_{\mathsf{forget}}(C) < \mathsf{T}_{\mathsf{forget}}(C^*)$.*

Proof. Recall that by Equation 4.5, $\mathsf{T}_{\mathsf{forget}} = H(a'',a)$. We make the following observations. First,

$$H^*(a'',a) = H(a'',a) + 2$$

by Lemma 4.5. Similarly,

$$H^*(a',a) = H(a',a) + 2.$$

Thus

$$T^*(a'',a) = T^*(a',a),$$

so C^* is still focal and

$$\mathsf{T}_{\mathsf{forget}}(C^*) = H^*(a'',a) = H(a'',a) + 2 > \mathsf{T}_{\mathsf{forget}}(C).$$

Note that in C, we have $H(a', a) = H(a'', a)$ and in C^*, we have $H^*(a', a) = H(a', a) + 2$ and $H^*(a'', a) = H(a'', a) + 2$, so in C^*, we have $H * (a', a) = H(a'', a)$ so C^* is also focal. □

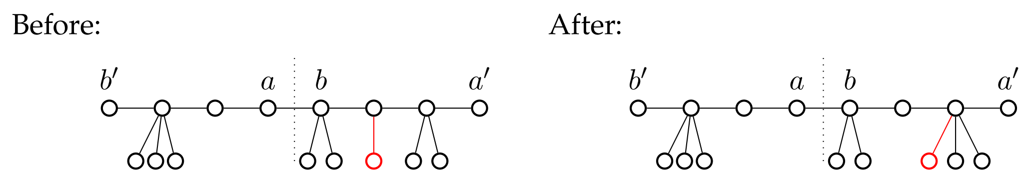

Figure 4.2 Example of Lemma 4.8 on a bifocal caterpillar.

We now prove a lemma about a bifocal caterpillar.

Lemma 4.8. *Let $C \in \mathcal{C}_{n,d}$ be a bifocal caterpillar on n vertices with spine $S = \{v_0, v_1, \ldots, v_d\}$, primary focus $a = v_k$ and secondary focus $b = v_{k+1}$ Suppose that there exists $k + 1 \leq j \leq d - 2$ such that the leaf set W_j is nonempty. Let $C^* = \text{push}(C, j)$ the push caterpillar as defined in Definition 3.3. Then C^* is a focal caterpillar with the same foci as C and $\mathsf{T}_{\mathsf{forget}}(C) < \mathsf{T}_{\mathsf{forget}}(C^*)$.*

Figure 4.2 shows an example of this lemma.

Proof. Recall that by Equation 4.3

$$\mathsf{T}_{\mathsf{forget}}(C) = H(b', a) + \mu_b \cdot H(a, b).$$

We will proceed with cases:

- **Case 1:** the foci don't change. We have

  $$H^*(a', a) = H(a', a) + 2$$

  by Lemma 4.5. Similarly, we have

  $$H^*(a', b) = H(a', b) + 2.$$

  Meanwhile, we have $H^*(b', a) = H(b', a)$ and $H^*(a, b) = H(a, b)$ by

Lemma 4.4. Thus,

$$\begin{aligned}
\mu_b^* &= \frac{1}{2|E|}\big(H^*(a',b) - H^*(b',a)\big) \\
&= \frac{1}{2|E|}\big((H(a',b)+2) - H(b',a)\big) \\
&> \frac{1}{2|E|}\big(H(a',b) - H(b',a)\big) \\
&= \mu_b.
\end{aligned}$$

So, overall

$$\begin{aligned}
\mathsf{T}_{\mathsf{forget}}(C^*) &= H^*(b',a) + \mu_b^* \cdot H^*(a,b) \\
&= H(b',a) + \mu_b^* \cdot H(a,b) \\
&> H(b',a) + \mu_b \cdot H(a,b) \\
&= \mathsf{T}_{\mathsf{forget}}(C).
\end{aligned}$$

- **Case 2:** The foci change.

  We have two subcases.

  1. C^* is still bifocal. Figure 4.3 shows an example of this case. If the foci change but C^* is still bifocal, then moving x increased the hitting time enough such that

     $$H^*(a',b) > H^*(b',b)$$

     so now the pessimal partner of b is a'. Vertex a is no longer a focus; since C^* is bifocal, vertex $c = v_{a+2}$ is now a focus. Observe that $H^*(b',b) = H(b',b)$ by Lemma 4.4, and we have

     $$\begin{aligned}
     \mathsf{T}_{\mathsf{forget}}(C^*) &= H^*(b',b) + \mu_c^*\, H^*(b,c) \\
     &= H(b',b) + \mu_c^*\, H^*(b,c) \\
     &> H(b',b) \\
     &= H(b',a) + H(a,b) \\
     &= H(b',a) + \mu_b\, H(a,b) \\
     &= \mathsf{T}_{\mathsf{forget}}(C)
     \end{aligned}$$

     because $0 < \mu_c^*$ and $\mu_b < 1$.

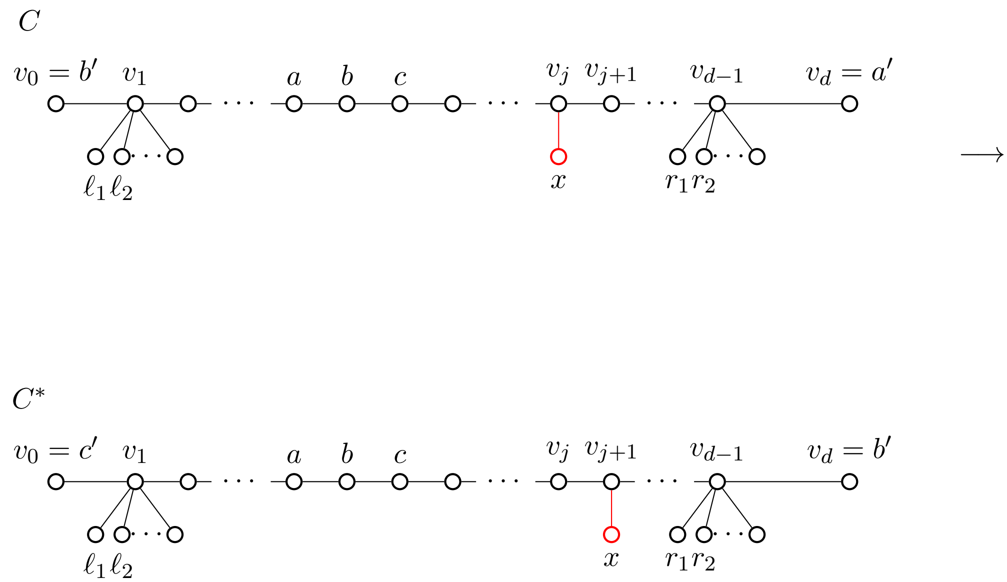

Figure 4.3 Case where the foci change and C^* is still bifocal

2. C^* is focal.
   If C^* is focal, moving leaf x made it so that
   $$H^*(a', b) = H^*(b', b) = \mathsf{T}_{\mathsf{forget}}(C^*).$$
   After this change, vertex b has both a' and b' as pessimal partners and is the only focus.
   Observe that C and C^* are identical between b' and b so
   $$\mathsf{T}_{\mathsf{forget}}(C^*) = H^*(b', b) = H(b', b) > H(b', a) + \mu_b H(a, b) = \mathsf{T}_{\mathsf{forget}}(C)$$
   because $\mu_b < 1$.

Thus, in all cases, $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$ □

We now prove the analogous lemma for a focal caterpillar.

Lemma 4.9. *Let $C \in \mathcal{C}_{n,d}$ be a focal caterpillar on n vertices with spine $S = \{v_0, v_1, \ldots, v_d\}$, focus $a = v_k$, $a'' = v_0$ and $a' = v_d$. Suppose that there exists $k \leq j \leq d - 2$ such that the leaf set W_j is nonempty. Let $C^* = \mathrm{push}(C, j)$ the push caterpillar as defined in Definition 3.3. Then C^* is a bifocal caterpillar and $\mathsf{T}_{\mathsf{forget}}(C) < \mathsf{T}_{\mathsf{forget}}(C^*)$.*

Note: when $x \sim v_i$ where $2 \leq i \leq k$, then we can reindex the spine of C by $w_j = v_{d-j}$ and then apply this lemma.

Proof. Recall that by Equation 4.5,

$$\mathsf{T}_{\mathsf{forget}}(C) = H(a'', a) = H(a', a).$$

Note that C^* will be bifocal because

$$H^*(a'', a) = H(a'', a) < H(a'a) + 2 = H^*(a', a).$$

by Lemma 4.5. The forget time of caterpillar C is

$$\mathsf{T}_{\mathsf{forget}}(C) = H(a'', a)$$

while the forget time of caterpillar C^* is

$$\begin{aligned}\mathsf{T}_{\mathsf{forget}}(C^*) &= H^*(a'', a) + \mu_b^* H^*(a, b) = H(a'', a) + \mu_b^* H^*(a, b) \\ &> H(a'', a) = \mathsf{T}_{\mathsf{forget}}(C)\end{aligned}$$

because $\mu_b^* > 0$. □

We prove the following lemma for bifocal caterpillars.

Lemma 4.10. *For any bifocal caterpillar $C \in \mathcal{C}_{n,d}$ on n vertices with diameter d, there exists some double broom $D \in \mathcal{C}_{n,d}$ such that $\mathsf{T}_{\mathsf{forget}}(C) \leq \mathsf{T}_{\mathsf{forget}}(D)$.*

Proof. Let C be a bifocal caterpillar. We can apply Lemma 4.6 to push a pair of leaves to either side of the caterpillar and the forget time will increase. Since C^* is still bifocal, we can repeat Lemma 4.6 until we cannot anymore, that is to say, until we have either

1. Leaves adjacent to only either v_1 or v_{d-1}.
2. Leaves adjacent to only vertices v_i such that $2 \leq i \leq a$
3. Leaves adjacent to only vertices v_j such that $a + 1 \leq j \leq d - 2$

In case one, we now have a double broom and have maximized the forget time. If we are in either case 2 or 3, we then apply Lemma 4.8 to move leaves towards the ends of the caterpillar and increase the forget time. After applying Lemma 4.8 once, we have two cases: C^* is bifocal, or it is focal. If it is bifocal, we can repeat Lemma 4.8, and if it is focal we can use Lemma 4.9. Keep repeating either Lemma 4.8 or Lemma 4.9 until C^* is a double broom. The forget time will have only increased from C to C^*.

□

Now we prove the analogous lemma for focal caterpillars.

Lemma 4.11. *For any focal caterpillar $C_{n,d}$ on n vertices with diameter d, there exists some double broom $D_{n,d}$ such that $\mathsf{T}_{\mathsf{forget}}(C_{n,d}) \leq \mathsf{T}_{\mathsf{forget}}(D_{n,d})$.*

Proof. Let C be a focal caterpillar. We can apply Lemma 4.7 to push a pair of leaves to either side of the caterpillar and the forget time will increase.

Since C^* is still focal, we can repeat Lemma 4.7 until we have

1. Leaves adjacent to only either v_1 or v_{d-1}.
2. Leaves adjacent to only vertices v_i such that $2 \leq i \leq a$
3. Leaves adjacent to only vertices v_j such that $a \leq j \leq d-2$

In case one, we now have a double broom and have maximized the forget time. If we are in either case 2 or 3, we then apply Lemma 4.9 to move leaves towards the ends of the caterpillar and increase the forget time.

As shown in Lemma 4.9, C^* is now bifocal, so if there are any appendices left to move, we can apply Lemma 4.8 to move leaves towards the ends of the caterpillar and increase the forget time. We end up in two cases: C^* is still bifocal, or it is not focal. If it is still bifocal, we can repeat Lemma 4.8, and if it is focal we can use Lemma 4.9.

Keep repeating either Lemma 4.8 or 4.9 until C^* is a double broom. The forget time will have only increased from C to C^*.

□

Theorem 4.12. *For any caterpillar $C_{n,d}$ on n vertices with diameter d, there exists some double broom $D_{n,d}$ such that $\mathsf{T}_{\mathsf{forget}}(C_{n,d}) \leq \mathsf{T}_{\mathsf{forget}}(D_{n,d})$.*

Proof. By Lemma 4.10 and Lemma 4.11, whether C is a bifocal caterpillar or a focal caterpillar, there exists a double broom for which the forget time is greater than that of the caterpillar. □

Thus, of all caterpillars, the one that maximizes the forget time is some double broom.

4.3 Unbalanced Double Broom to Balanced Double Broom

In this section we prove that of all double brooms on n vertices with diameter d, the one that maximizes the forget time is the balanced double broom on n vertices with diameter d.

Let $B \in \mathcal{B}_{n,d}$ be an unbalanced double broom with diameter d, ℓ vertices in W_1 and r vertices in W_{d-1}. Without loss of generality, say $\ell > r$.

Lemma 4.13. *Let $B \in \mathcal{B}_{n,d}$ be a bifocal double broom with $\ell = |W_1|$ and $r = |W_{d-1}|$ where $\ell \geq r+2$. Let a, b be the foci of B. Now let $B^* = \mathrm{throw}(B)$ be a bifocal broom. If the foci of B^* are different from the foci of B then $b \leq d/2$.*

Proof. Recall that $v_0 = b', v_k = a, v_{k+1} = b$ and $v_d = a'$. If the foci are different between B and B^*, since we are throwing a leaf from left to right, $H^*(a', b) > H(a', b)$ and $H^*(b', b) < H(b', b)$. The only way the foci could change is if in B, the pessimal partner of b is v_0 and in B^*, the pessimal partner of b is v_d. This means that in $B, H(b', b) > H(a', b)$ and in $B^*, H(a', b) > H(b'b)$.

We begin with cases.

d is odd: If d is odd, we show that $b \leq \frac{d-1}{2}$. Assume for contradiction that $b \geq \frac{d+1}{2}$.

In B, we have

$$\begin{aligned} H(b', b) &\geq \left(\frac{d+1}{2}\right)^2 + 2\ell\left(\frac{d+1}{2} - 1\right) \\ &\geq \frac{d^2 + 2d + 1}{4} + d\ell + \ell - 2\ell \\ &\geq \frac{d^2 + 1}{4} + \frac{d}{2} + (d-1)\ell. \end{aligned}$$

We also have

$$\begin{aligned} H(a', b) &\leq \left(d - \left(\frac{d+1}{2}\right)\right)^2 + 2r\left(d - \frac{d+1}{2} - 1\right) \\ &\leq \left(\frac{d-1}{2}\right)^2 + 2r\left(\frac{d-3}{2}\right) \\ &\leq \frac{d^2 - 2d + 1}{4} + rd - 3r \\ &\leq \frac{d^2 + 1}{4} - \frac{d}{2} + r(d-3). \end{aligned}$$

Since $\ell \geq r + 2$, we know that

$$H(b', b) > H(a', b)$$

and thus, in B is the pessimal partner of b is $b' = v_0$.

Now, in B^*, we have

$$H^*(b',b) \geq \left(\frac{d+1}{2}\right)^2 + 2(\ell-1)\left(\frac{d-1}{2}\right)$$
$$\geq \frac{d^2+1}{4} + \frac{d}{2} + (\ell-1)(d-1)$$

and

$$H^*(a',b) \leq \left(d - \left(\frac{d+1}{2}\right)\right)^2 + 2(r+1)\left(d - \frac{d+1}{2} - 1\right)$$
$$\leq \left(\frac{d-1}{2}\right)^2 + 2(r+1)\left(\frac{d-3}{2}\right)$$
$$\leq \frac{d^2+1}{4} - \frac{d}{2} + (r+1)(d-3).$$

Since $\ell - 1 \geq r+1$, we have

$$H^*(b',b) > H^*(a',b)$$

which is a contradiction because we assumed that in $B^*, H^*(b',a) > H^*(b',b)$.

d is even: If d is even, we show that $b \leq \frac{d}{2}$. Assume for contradiction that $b \geq \frac{d}{2}+1$.

In B, we have

$$H(b',b) \geq \left(\frac{d}{2}+1\right)^2 + 2\ell\left(\frac{d}{2}+1-1\right)$$
$$\geq \frac{d^2}{4} + d + 1 + \ell d$$
$$\geq \frac{d^2}{4} + d(\ell+1) + 1.$$

We also have

$$H(a',b) \leq \left(d - \left(\frac{d}{2}-1\right)\right)^2 + 2r\left(d - \left(\frac{d}{2}+1\right) - 1\right)$$
$$\leq \left(\frac{d}{2}-1\right)^2 + 2r\left(\frac{d}{2}-2\right)$$
$$\leq \frac{d^2}{4} - d + 1 + rd - 4r$$
$$\leq \frac{d^2}{4} + d(r-1) - 4r + 1.$$

Since $\ell \geq r+2$, we know that

$$H(b', b) > H(a', b)$$

and thus, in B' is the pessimal partner of b is b'.

Now, in B^*, we have

$$\begin{aligned} H^*(b', b) &\geq \left(\frac{d}{2}+1\right)^2 + 2(\ell-1)\left(\frac{d}{2}+1-1\right) \\ &\geq \frac{d^2}{4} + d + 1 + d\ell - d \\ &\geq \frac{d^2}{4} + d\ell + 1. \end{aligned}$$

We also have

$$\begin{aligned} H^*(a', b) &\leq \left(d - \left(\frac{d}{2}-1\right)\right)^2 + 2(r+1)\left(d - \left(\frac{d}{2}+1\right) - 1\right) \\ &\leq \left(\frac{d}{2}-1\right)^2 + 2(r+1)\left(\frac{d}{2}-2\right) \\ &\leq \frac{d^2}{4} - d + 1 + dr - 4r + d - 4 \\ &\leq \frac{d^2}{4} + dr - 4r - 3. \end{aligned}$$

Since $\ell \geq r+2$, we have

$$H^*(b', b) > H^*(a', b)$$

which is a contradiction because we assumed that in $B^*, H^*(b', a) > H^*(b', b)$.

Thus, if the foci are different between B and B^* where $B^* = \text{throw}(B)$, then $b \leq \frac{d}{2}$. □

Lemma 4.14. *Let $B \in \mathcal{B}_{n,d}$ be a focal unbalanced double broom on n vertices with diameter d, with ℓ left leaves and r right leaves where $\ell = |W_1| \geq |W_{d-1}| + 2 = r+2$. Let a be the focus. Let $B^* = \text{throw}(B)$ the throw broom as defined in Definition 3.4. Then, $\mathsf{T}_{\mathsf{forget}}(B) < \mathsf{T}_{\mathsf{forget}}(B^*)$.*

Proof. Note that since B is focal, we have

$$\mathsf{T}_{\mathsf{forget}} = H(a', a) = H(a'', a)$$

where $a'' = v_0$ and $a' = v_d$

In B^*, we have

$$H^*(a', a) > H^*(a'', a)$$

because $H^*(a', a)$ increases and $H^*(a'', a)$ decreases. Note that this means that B^* is bifocal with $b = a + 1$. Since $H^*(a', a) > H(a', a)$, vertex a is the primary focus of B^* with pessimal partner $a' = v_0$. This means that the secondary focus b has pessimal partner a'' and $b = a + 1$.

Thus,

$$\mathsf{T}^*_{\mathsf{forget}} = H^*(a', b) + \mu^*_a H^*(b, a) \tag{4.6}$$

where $b = a + 1$. Since $r^* = r + 1$,

$$H^*(a', b) = H(a', b) + 2(d - b - 1) \tag{4.7}$$

and

$$H^*(b, a) = H(b, a) + 2. \tag{4.8}$$

Recall that

$$\mu^*_a = \frac{1}{2|E|}\Big(H^*(b', b) - H^*(a', b)\Big) \tag{4.9}$$

and note that $b' = v_0 = a''$. Since $\ell^* = \ell - 1$, we have

$$H^*(b', b) = H(a'', b) - 2(b - 1). \tag{4.10}$$

We can plug Equation 4.8 and Equation 4.10 into Equation 4.9 to get

$$\begin{aligned}
\mu^*_a &= \frac{1}{2|E|}\Big(H^*(b', b) - H^*(a', b)\Big) \\
&= \frac{1}{2|E|}\Big(\Big(H(a'', b) - 2(b - 1)\Big) - \Big(H(a', b) + 2(d - b - 1)\Big)\Big) \\
&= \frac{1}{2|E|}\Big(\Big(H(a'', b) - H(a', b)\Big) - \Big(2(b - 1) + 2(d - b - 1)\Big)\Big) \\
&= \mu_a - \frac{1}{2|E|}\Big(2(b - 1) + 2(d - b - 1)\Big) \\
&= \mu_a - \frac{1}{2|E|}\Big(2(d - 2)\Big) \\
&= \mu_a - \frac{(d - 2)}{|E|}.
\end{aligned} \tag{4.11}$$

Next, we plug Equation 4.7, Equation 4.9 and Equation 4.8 into Equation 4.6 to get

$$\begin{aligned}
\mathsf{T}^*_{\mathsf{forget}} &= H^*(a', b) + \mu^*_a H^*(b, a) \\
&= \Big(H(a', b) + 2(d - b - 1)\Big) + \Big(\mu_a - \frac{(d-2)}{|E|}\Big)\Big(H(b, a) + 2\Big) \\
&= \Big(H(a', b) + \mu_a H(b, a)\Big) + 2(d - b - 1) + 2\mu_a - \frac{(d-2)}{|E|}\Big(H(b, a) + 2\Big) \\
&= \mathsf{T}_{\mathsf{forget}} + 2(d - b - a) + 2\mu_a - \frac{(d-2)}{|E|}\Big(H(b, a) + 2\Big).
\end{aligned}$$

Note that $\mu_a = 1$ and $H(b, a) = 2(d - b) + 2r + 1$ so we can further simplify:

$$\begin{aligned}
\mathsf{T}^*_{\mathsf{forget}} &= \mathsf{T}_{\mathsf{forget}} + 2(d - b - a) + 2\mu_a - \frac{(d-2)}{|E|}\Big(H(b, a) + 2\Big) \\
&= \mathsf{T}_{\mathsf{forget}} + 2d - 2b - 2 + 2(1) - \frac{(d-2)}{|E|}\Big(2(d - b) + 2r + 1 + 2\Big) \\
&= \mathsf{T}_{\mathsf{forget}} + 2d - 2b - \frac{(d-2)(2d - 2b + 2r + 3)}{|E|}.
\end{aligned}$$

Then, we want to check whether

$$0 < 2d - 2b - \frac{2d^2 - 2db + 2dr + 3d - 4d + 4b - 4r - 6}{|E|}.$$

Note that $b = a + 1$, so we want to check

$$\begin{aligned}
0 &< 2d - 2(a + 1) - \frac{2d^2 - 2d(a + 1) + 2dr - d + 4(a + 1) - 4r - 6}{|E|} \\
0 &< 2d - 2a - 2 - \frac{2d^2 - 2da - 3d + 2dr + 4a - 4r - 2}{|E|}. \qquad (4.12)
\end{aligned}$$

Then, since T is focal, we have

$$\begin{aligned}
H(a'', a) &= H(a', a) \\
a^2 + 2\ell(a + 1) &= (d - a)^2 + 2r(d - a - 1) \\
2\ell a + 2\ell &= d^2 - 2da + 2rd - 2ar - 2r + (-2r + 2r) \\
d^2 - 2da + 2dr - 4r &= 2\ell a + 2\ell + 2ra - 2r.
\end{aligned}$$

So we can substitute it into Equation 4.12 to get

$$0 < 2d - 2a - 2 - \frac{\left(d^2 - 2da + 2dr - 4r\right) + d^2 - 3d + 4a - 2}{|E|}$$
$$0 < 2d - 2a - 2 - \frac{\left(2\ell a + 2\ell + 2ra - 2r\right) + d^2 - 3d + 4a - 2}{|E|}.$$

Next, we rearrange our desired inequality noting that $|E| = n - 1$:

$$\frac{2\ell a + 2\ell + 2ra - 2r + d^2 - 3d + 4a - 2}{n-1} < 2d - 2a - 2$$
$$2\ell a + 2\ell + 2ra - 2r + d^2 - 3d + 4a - 2 < (2d - 2a - 2)(n-1)$$
$$2\ell a + 2\ell + 2ra - 2r + d^2 - 3d + 4a - 2 < 2dn - 2an - 2n - 2d + 2a + 2$$
$$d(d+2) + 2a(\ell + r + 2 + n) + 2\ell + 2n < d(n+3) + dn + 2r + 2a + 2.$$

Now, we note that $n = d + 1 + \ell + r$. So our desired inequality becomes

$$d(d+2) + 2a(\ell + r + 2 + n) + 2\ell + 2n < d(d + 1 + \ell + r + 3) + dn + 2r + 2a + 2$$
$$d(d+2) + 2a(\ell + r + 2 + n) + 2\ell + 2n < d(d+2) + d(\ell + r + 2) + dn + 2r + 2a + 2$$
$$2a(\ell + r + 2 + n) + 2\ell + 2n < d\ell + dr + 2d + dn + 2r + 2a + 2$$
$$2a(\ell + r + 2 + n) + 2\ell + 2n < (d-2)\ell + 2\ell + dr + 2d + (d-2)n + 2n + 2r + 2a + 2$$
$$2a(\ell + r + 2 + n) < (d-2)\ell + dr + 2d + (d-2)n + 2r + 2a + 2$$
$$2a(\ell + r + 1 + n) < (d-2)\ell + dr + 2d + (d-2)n + 2r + 2.$$

Observe that $a \leq \frac{d-2}{2}$ because $\ell^* \geq r^*$ and $H^*(a', a) > H^*(b, a)$. And we finally see that the stronger condition

$$2\left(\frac{d-2}{2}\right)(\ell + r + 1 + n) < (d-2)\ell + dr + 2d + (d-2)n + 2r + 2$$
$$(d-2)\ell + (d-2)r + (d-2) + (d-2)n < (d-2)\ell + dr + 2d + (d-2)n + 2r + 2$$
$$(d-2)r + (d-2) < dr + 2d + 2r + 2$$
$$dr - 2r + d - 2 < dr + 2d + 2r + 2$$
$$0 < d + 4r + 4$$

holds because $r \geq 0$ and $d > 0$. This proves that

$$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$$

and the proof is complete. □

Lemma 4.15. *Let $B \in \mathcal{B}_{n,d}$ be a bifocal unbalanced double broom on n vertices with diameter d, primary focus a and secondary focus b. Suppose there exists $x \in W_1$ and $\ell = |W_1| \geq |W_{d-1}| + 2 = r + 2$. Let $B^* = \text{throw}(B)$ the throw broom as defined in Definition 3.4. Then $\mathsf{T}_{\mathsf{forget}}(B) < \mathsf{T}_{\mathsf{forget}}(B^*)$.*

Proof. Recall Equation 4.3:

$$\mathsf{T}_{\mathsf{forget}}(T) = H(a', b) + \mu_a H(b, a).$$

We use cases:

Case 1: T^* is bifocal and has the same foci as T. First we have that

$$H^*(b, a) = H(b, a) + 2$$

and

$$H^*(a', b) = H(a', b) + 2(d - b - 1)$$

because $r^* = r + 1$. Then, by the same steps as Equation 4.11

$$\mu_a^* = \mu_a - \frac{(d-2)}{|E|}.$$

Finally,

$$H(b, a) = 2(d - b) + 2r + 1.$$

Thus,

$$\begin{aligned}
\mathsf{T}^*_{\mathsf{forget}} &= H^*(a', b) + \mu_a^* H^*(b, a) \\
&= \Big(H(a', b) + 2(d - b - 1)\Big) + \left(\mu_a - \frac{(d-2)}{|E|}\right)\Big(H(b, a) + 2\Big) \\
&= \mathsf{T}_{\mathsf{forget}} + 2(d - b - 1) + 2\mu_a - \frac{(d-2)}{|E|}(2(d - b) + 2r + 1 + 2).
\end{aligned}$$

Then, we want to see if all of the terms added and subtracted to $\mathsf{T}_{\mathsf{forget}}$ are greater than or equal to 0 so that the overall forget time increases. We are trying to prove that

$$0 < 2(d - b - 1) + 2\mu_a - \frac{(d-2)}{|E|}(2(d - b) + 2r + 1 + 2).$$

We know that $|E| = n - 1$ and $n = d + \ell + r + 1$ so we can use algebra to get

$$
\begin{aligned}
0 &< 2(d-b-1) + 2\mu_a - \frac{(d-2)}{|E|}(2(d-b) + 2r + 1 + 2) \\
0 &< 2d - 2b - 2 + 2\mu_a - \frac{(d-2)}{|E|}(2d - 2b + 2r + 1 + 2)
\end{aligned}
$$

and rearranging yields

$$
\begin{aligned}
\frac{(d-2)}{n-1}(2d - 2b - 2 + 2\mu_a) + 2b + 2 &< 2d + 2\mu_a \\
(d-2)(2d - 2b + 2r + 3) + 2b + 2 &< (2d + 2\mu_a)(n-1) \\
2d^2 - 2db + 2dr + 3d - 4d + 4b - 4r - 6 + 2b + 2 &< 2dn - 2d + 2n\mu_a - 2\mu_a \\
2d^2 - 2db + 2dr + d + 6b - 4r - 4 &< 2dn + 2n\mu_a - 2\mu_a \\
2d^2 + 2dr + d + 6b + 2\mu_a &< 2dn + 2n\mu_a + 2db + 4r + 4.
\end{aligned}
$$

Next, we substitute $n = d + \ell + r + 1$ to get

$$
\begin{aligned}
2d^2 + 2dr + d + 6b + 2\mu_a &< 2d(d + \ell + r + 1) + 2(d + \ell + r + 1)\mu_a + 2db + 4r + 4 \\
2d^2 + 2dr + d + 6b + 2\mu_a &< 2d^2 + 2dr + 2d\ell + 2d + 2\mu_a(d + r + \ell) + 2\mu_a + 2db + 4r + 4 \\
6b &< 2d\ell + 2db + 4 + r.
\end{aligned}
$$

Since $d \geq 3$, we can instead check that

$$
\begin{aligned}
6b &< 6\ell + 2\mu_a(3 + r + \ell) + 6b + 4r + 4 \\
0 &< 6\ell + 2\mu_a(3 + r + \ell) + 4r + 4.
\end{aligned}
$$

Since $\ell, r, \mu_a \geq 0$, this is true and thus

$$
\mathsf{T}_{\mathsf{forget}} < \mathsf{T}^*_{\mathsf{forget}}.
$$

Case 2: B^* is focal

$$
\mathsf{T}_{\mathsf{forget}} = H(a', b) + \mu_a H(b, a) < H(a', b)
$$

because $\mu_a < 1$.

If B^* is focal, then the focus must be b because $H^*(b', a) < H^*(a', a)$. So

$$\mathsf{T}^*_{\text{forget}} = H^*(b', b) = H^*(a', b) = H(a', b) + 2(d - b - 1).$$

Thus,

$$\begin{aligned}\mathsf{T}^*_{\text{forget}} &= H(a', b) + 2(d - b - 1) \\ &> H(a', b) \\ &> \mathsf{T}_{\text{forget}}.\end{aligned}$$

Case 3: B^* is bifocal and has different foci from B.

Since the foci in B are different from the foci in B^*, if B has foci a, b and pessimal partners a', b' respectively, then B^* has pessimal partners $b, c = b + 1$ with pessimal partners $c'^* = b' = v_0$ and $b'^* = a' = v_d$ respectively. This is because if moving one leaf makes it such that $H^*(a', b) > H(b', b)$, then the second focus must be to the right of b with pessimal partner b'.

In B, we have

$$\mathsf{T}_{\text{forget}} = H(a', b) + \mu_a H(b, a)$$

and in T^* we have

$$\mathsf{T}^*_{\text{forget}} = H^*(a', c) + \mu^*_b H^*(c, b).$$

We know that

$$H^*(a', c) = H(a', c) + 2(d - c - 1) \tag{4.13}$$

because we now have one additional leaf on the right hand side. Additionally,

$$H(a', c) = H(a', b) - H(c, b). \tag{4.14}$$

Putting together Equation 4.13 and Equation 4.14 gives us

$$H^*(a', c) = H(a', b) - H(c', b) + 2(d - c - 1). \tag{4.15}$$

Then, we also have that

$$H^*(c, b) = H(c, b) + 2 \tag{4.16}$$

because we have one more leaf to count. Also, since

$$H(c, b) = \sum_{p \in V} \ell(c, p; b) \deg(p),$$

we know that

$$H(b,a) - H(c,b) = \deg(b) = 2,$$

so

$$H(c,b) = H(b,a) - 2 \tag{4.17}$$

because we are traveling along one less edge. Thus, using Equation 4.16 and Equation 4.17 we get

$$H^*(c,b) = H(b,a) - 2 + 2 = H(b,a) \tag{4.18}$$

Then,

$$H^*(b',c) = H(b',c) - 2(c-1) \tag{4.19}$$

because we lost one leaf on the left side and

$$H(b',c) = H(b',b) + H(b,c) \tag{4.20}$$

by definition of hitting times. Thus, by Equation 4.19 and Equation 4.20 we get

$$H^*(b',c) = H(b',b) + H(b,c) - 2(c-1). \tag{4.21}$$

By definition,

$$\mu_b^* = \frac{1}{2|E|}\Big(H^*(b',c) - H^*(a',c)\Big).$$

Note that

$$H(x,y) + H(y,x) = 2|E| \tag{4.22}$$

for any neighbors $x, y \in B(V)$ because, by Equation 2.1, we are simply counting the degree of every vertex in the tree.

Recall that

$$H(b,a) = 2(d-b) + 2r + 1. \tag{4.23}$$

Using Equation 4.21, Equation 4.15 and Equation 4.22, we get

$$\begin{aligned}
\mu_b^* &= \frac{1}{2|E|}\Big(H^*(b',c) - H^*(a',c)\Big) \\
&= \frac{1}{2|E|}\Big(\Big(H(b',b) + H(b,c) - 2(c-1)\Big) - \Big(H(a',b) - H(c,b) + 2(d-c-1)\Big)\Big) \\
&= \frac{1}{2|E|}\Big(H(b',b) - H(a'b)\Big) + \frac{1}{2|E|}\Big(H(b,c) - 2c + 2 + H(c,b) - 2d + 2c + 2\Big) \\
&= \mu_a + \frac{1}{2|E|}\Big(H(b,c) - 2c + 2 + H(c,b) - 2d + 2c + 2\Big) \\
&= \mu_a + \frac{1}{2|E|}\Big(2|E| - 2d + 4\Big) \\
&= \mu_a + 1 - \frac{(d-2)}{|E|}. && (4.24) \\
& && (4.25)
\end{aligned}$$

Using Equation 4.15, Equation 4.24 and Equation 4.18, we have

$$\begin{aligned}
\mathsf{T}^*_{\mathsf{forget}} &= H^*(a',c) + \mu_b^* H^*(c,b) \\
&= H(a',b) - H(c,b) + 2(d-c-1) + \left(\mu_a + 1 - \frac{d-2}{|E|}\right)\Big(H(b,a)\Big) \\
&= H(a',b) + \mu_a H(b,a) - H(c,b) + 2(d-c-1) + H(b,a) - H(b,a)\cdot\frac{d-2}{|E|} \\
&= \mathsf{T}_{\mathsf{forget}} - H(c,b) + 2(d-c-1) + H(b,a) - H(b,a)\cdot\frac{d-2}{|E|}.
\end{aligned}$$

We want to show that

$$0 < -H(c,b) + 2(d-c-1) + H(b,a) - H(b,a)\cdot\frac{d-2}{|E|}.$$

We'll use Equation 4.17, Equation 4.23 and the fact that $c = b+1$, $n = d + 1 + \ell + r$ and that $b \leq \frac{d}{2}$ by Lemma 4.13. We want to show

$$0 \leq -H(c,b) + 2(d-c-1) + H(b,a) - H(b,a)\cdot\frac{d-2}{|E|}$$

which is equivalent to

$$
\begin{aligned}
H(c,b) + 2c + 2 + H(b,a) \cdot \frac{d-2}{|E|} &\leq 2d + H(b,a) \\
H(b,a) - 2 + 2c + 2 + H(b,a) \cdot \frac{d-2}{|E|} &\leq 2d + H(b,a) \\
2c + H(b,a) \cdot \frac{d-2}{|E|} &\leq 2d \\
(2d - 2b + 2r + 1)\frac{d-2}{|E|} &\leq 2d - 2c \\
(2d - 2b + 2r + 1)(d-2) &\leq (2d - 2c)(n-1) \\
2d^2 - 2db + 2dr + d - 4d + 4b - 4r - 2 &\leq (2d - 2c)(d + \ell + r) \\
2d^2 - 2db + 2dr - 3d + 4b - 4r - 2 &\leq 2d^2 - 2dc + 2d\ell - 2\ell c + 2dr - 2rc \\
4b + 2dc + 2\ell c + 2rc &\leq 2db + 3d + 4r + 2 + 2d\ell \\
4b + 2d(b+1) + 2\ell(b+1) + 2r(b+1) &\leq 2db + 3d + 4r + 2 + 2d\ell \\
4b + 2db + 2d + 2\ell b + 2\ell + 2rb + 2r &\leq 2db + 3d + 4r + 2 + 2d\ell \\
4b + 2\ell b + 2\ell + 2rb &\leq d + 2r + 2 + 2d\ell \\
4b + 2\ell b + 2rb &\leq d + 2r + 2 + 2d\ell - 2\ell \\
b(4 + 2\ell + 2r) &\leq d + 2r + 2 + 2d\ell - 2\ell \\
b &\leq \frac{d + 2r + 2 + 2d\ell - 2\ell}{(4 + 2\ell + 2r)}
\end{aligned}
$$

Our two extremes are when $r = \ell - 2$ and when $\ell = r$ so we'll plug in for both of these cases.

Plugging in for $r = \ell - 2$ gives us:

$$
\begin{aligned}
b &\leq \frac{d + 2(\ell - 2) + 2 + 2d\ell - 2(\ell - 2)}{4 + 2\ell + 2(\ell - 2)} \\
&\leq \frac{d + 2d\ell + 2}{4\ell} \\
&\leq \frac{d}{2} + \frac{d+2}{4\ell}
\end{aligned}
$$

which holds because we know that $b \leq d/2$.

Plugging in for $r = \ell$ gives us:

$$
\begin{aligned}
b &\leq \frac{d + 2\ell + 2 + 2d\ell - 2\ell}{4 + 2\ell + 2\ell} \\
&\leq \frac{d + 2 + 2d\ell}{4\ell + 4} \\
&\leq \frac{d(\ell + 2)}{2(\ell + 2)} + \frac{d\ell + 2 - d}{2(\ell + 2)} \\
&\leq \frac{d}{2} + \frac{d(\ell - 1) + 2}{2(\ell + 2)}
\end{aligned}
$$

which holds because $b \leq d/2$ and $\ell > 1$.

Thus, at our two extremes, it is true that $b \leq \frac{d}{2}$, so we'll take the derivative to show that there are no extremal points and that $b \leq \frac{d}{2}$ at all points between our two extremes.

The derivative of the right hand side of the inequality with respect to r gives

$$
\begin{aligned}
&\frac{2}{2l + 2r + 4} - \frac{2(d(2l + 1) - 2l + 2r + 2)}{(2l + 2r + 4)^2} \\
&= \frac{-2(d - 2)(1 + 2l)}{(2l + 2r + 4)^2} < 0
\end{aligned}
$$

for $0 \leq r \leq \ell - 2$. So we have no extremal points and thus $b \leq \frac{d}{2}$ for all values of ℓ and r.

Since we assumed that $b \leq \frac{d}{2}$, it is true that the forget time has increased.

□

Theorem 4.16. *For any unbalanced double broom $D_{d,\ell,r}$ on n vertices with diameter d, the forget time of the balanced double broom $B_{n,d}$ is greater than that of the unbalanced double broom*

Proof. We have two cases: If D is a focal unbalanced double broom and if D is a bifocal unbalanced double broom.

D is focal: Let D be a focal unbalanced double broom. We can apply Lemma 4.14 throw one leaf from W_1 to W_{d-1} and the forget time will increase.

Note that D^* will be bifocal. If D^* is unbalanced, we can use Lemma 4.15 to throw one leaf from L to R and the forget time will increase.

If D^{**} is still unbalanced, depending on whether is is focal or bifocal, we can repeat either Lemma 4.14 or Lemma 4.15 until we have a balanced double broom and the forget time will only have increased.

D is bifocal: Let D be a bifocal unbalanced double broom. We can apply Lemma 4.15 throw one leaf from W_1 to W_{d-1} and the forget time will increase.

If D^* is unbalanced, depending on whether is is focal or bifocal, we can use either Lemma 4.14 or Lemma 4.15 and keep repeating until we have a balanced double broom and the forget time will only have increased.

□

Thus, we have proved that of all the double brooms, the one that maximizes the forget time is the balanced double broom.

4.4 Tree to Caterpillar

In this section, we prove that of all trees on n vertices with diameter d, the one that maximizes the forget time is some caterpillar on n vertices with diameter d.

We start by showing that on balanced double brooms on n vertices, the longer the diameter, the greater the forget time.

Lemma 4.17. *Let $B = B_{n,d'}$ and $B^* = B_{n,d}$ where $d' < d$. Then, $\mathsf{T}_{\mathsf{forget}}(B) < \mathsf{T}_{\mathsf{forget}}(B^*)$.*

Proof. We fix the number n of vertices and prove by induction on d.

Base Case: Let $d = 3$. First, we observe that when $d = 3$, the tree must be a caterpillar. Next, by 4.2 and 4.3, we know that the tree that maximizes the forget time is the balanced double broom. Figure 4.4 shows what this tree could look like.

Suppose that G is a bifocal tree with diameter 3, so that the vertices $b' = v_0, a = v_1, b = v_2, a' = v_3$ form a path of length 3. Without loss of generality, $\deg(v_1) = \ell + 2$ and $\deg(v_2) = r + 2$. Using Equation 4.3, we have

$$\mathsf{T}_{\mathsf{forget}}(G) = H(a', b) + \mu_a H(b, a) = H(v_3, v_2) + \mu_a H(v_2, v_1).$$

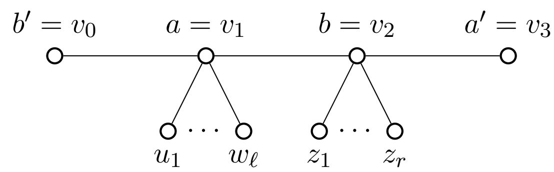

Figure 4.4 A tree $G \in T_{n,3}$

Then, to find μ_a we have

$$\begin{aligned}
\mu_a &= \frac{1}{2|E|}\Big(H(b',b) - H(a',b)\Big)\\
&= \frac{1}{2(n-1)}\Big(H(v_0,v_2) - H(v_3,v_2)\Big)\\
&= \frac{1}{2(d+\ell+r)}\Big((2^2 + 2(2)\ell - 2\ell) - (3^2 - 2(3)(2) + 2^2 + 2(3)r - 2(2)r - 2r)\Big)\\
&= \frac{1}{2(3+\ell+r)}\Big(4 + 4\ell - 2\ell - (9 - 12 + 4 + 6r - 4r - 2r)\Big)\\
&= \frac{1}{2(3+\ell+r)}\Big(3 + 2\ell\Big)\\
&= \frac{3+2\ell}{2(3+\ell+r)}. \qquad (4.26)
\end{aligned}$$

Plugging it all in we get

$$\begin{aligned}
\mathsf{T}_{\mathsf{forget}}(G) &= H(v_3, v_2) + \frac{3+2\ell}{2(3+\ell+r)} H(v_2, v_1) \\
&= \left(3^2 - 2(3)(2) + 2^2 + 2(3)r - 2(2)r - 2r\right) + \left(\frac{3+2\ell}{2(3+\ell+r)}\right)\left(2(3) - 2(2) + 2r + 1\right) \\
&= \left(9 - 12 + 4 + 6r - 4r - 2r\right) + \left(\frac{3+2\ell}{2(3+\ell+r)}\right)\left(6 - 4 + 1 + 2r\right) \\
&= 1 + \left(\frac{3+2\ell}{2(3+\ell+r)}\right)\left(3 + 2r\right) \\
&= 1 + \frac{(3+2\ell)(3+2r)}{2(3+\ell+r)} \\
&= 1 + \frac{9 + 6r + 6\ell + 4\ell r}{6 + 2\ell + 2r} \\
&= \frac{6 + 2\ell + 2r}{6 + 2\ell + 2r} + \frac{9 + 6r + 6\ell + 4\ell r}{6 + 2\ell + 2r} \\
&= \frac{15 + 8\ell + 8r + 4\ell r}{6 + 2\ell + 2r}.
\end{aligned}$$

We have two cases: When $\ell = r$ and n is even, or when $\ell = r + 1$ and n is odd.

1. When $\ell = r$ we have

$$\begin{aligned}
\mathsf{T}_{\mathsf{forget}}(G) &= \frac{15 + 8r + 8r + 4r^2}{6 + 2r + 2r} \\
&= \frac{15 + 8r + 8r + 4r^2}{6 + 2r + 2r} \\
&= \frac{15 + 16r + 4r^2}{6 + 4r}.
\end{aligned}$$

2. When $\ell = r + 1$ we have

$$\begin{aligned}
\mathsf{T}_{\mathsf{forget}}(G) &= \frac{15 + 8(r+1) + 8r + 4(r+1)r}{6 + 2(r+1) + 2r} \\
&= \frac{15 + 8r + 8 + 8r + 4r^2 + 4r}{6 + 2r + 2 + 2r} \\
&= \frac{23 + 20r + 4r^2}{8 + 4r}.
\end{aligned}$$

Inductive hypothesis: For $3 \leq d' < d$,, we have $\mathsf{T}_{\mathsf{forget}}(B_{n,d'}) < \mathsf{T}_{\mathsf{forget}}(B_{n,d})$.

Inductive Step: Let $B = (V, E)$ be a balanced double broom with order n, diameter d, and ℓ left leaves and r right leaves. Recall that Lemma 4.3 provides the formulas for the forget time for all balanced double brooms. We will show that increasing the diameter of T will increase the forget time. Have two cases:

1. Our first case is when the diameter of our final balanced double broom B^* has the same parity as B so we can move leaves in pairs. In this case we have three subcases:

   (a) Let B have even d. Let B^* be the balanced double broom such that $d^* = d + 2$, $\ell^* = \ell - 1$ and $r^* = r - 1$. Note that $n^* = n$. Using Lemma 4.3, we want to show that

   $$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

   We have

   $$\mathsf{T}_{\mathsf{forget}} = \frac{4 - d^2 - 4n + 2d + 2dn}{4}$$

   and

   $$\mathsf{T}^*_{\mathsf{forget}} = \frac{4 - (d+2)^2 - 4n + 2(d+2) + 2(d+2)n}{4}.$$

   We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

   $$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} &= \frac{4 - (d+2)^2 - 4n + 2(d+2) + 2(d+2)n}{4} \\ &\quad - \frac{(4 - d^2 - 4n + 2d + 2dn)}{4} \\ &= n - d > 0.\end{aligned}$$

   Thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

   (b) Let B have odd d and even n. Let B^* be the balanced double broom such that $d^* = d + 2$, $\ell^* = \ell - 1$ and $r^* = r - 1$. Note that $n^* = n$. Using lemma 4.3, we want to show that

   $$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

We have that

$$\mathsf{T}_{\mathsf{forget}} = \frac{5 - d^2 - 4n + 2d + 2dn}{4}$$

and

$$\mathsf{T}^*_{\mathsf{forget}} = \frac{5 - (d+2)^2 - 4n + 2(d+2) + 2(d+2)n}{4}.$$

We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

$$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} = \frac{5 - (d+2)^2 - 4n + 2(d+2) + 2(d+2)n}{4} \\ - \frac{(5 - d^2 - 4n + 2d + 2dn)}{4} \\ = n - d > 0.\end{aligned}$$

Thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

(c) Let B have odd d and odd n. Let B^* be the balanced double broom such that $d^* = d + 2$, $\ell^* = \ell - 1$ and $r^* = r - 1$. Note that $n^* = n$. Using lemma 4.3, we want to show that

$$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

We have that

$$\mathsf{T}_{\mathsf{forget}} = \frac{2dn^2 + d^2 - d^2n - 4n^2 + 9n - 4d - 1}{4(n-1)}.$$

and

$$\mathsf{T}^*_{\mathsf{forget}} = \frac{2(d+2)n^2 + (d+2)^2 - (d+2)^2n - 4n^2 + 9n - 4(d+2) - 1}{4(n-1)}.$$

We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

$$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} &= \frac{2(d+2)n^2 + (d+2)^2 - (d+2)^2n - 4n^2 + 9n - 4(d+2) - 1}{4(n-1)} \\ &\quad - \frac{2dn^2 + d^2 - d^2n - 4n^2 + 9n - 4d - 1}{4(n-1)} \\ &= \frac{n(n-d-1) + d - 1}{n-1} \\ &= \frac{n(\ell + r) + d - 1}{n-1} \\ &> 0.\end{aligned}$$

Thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

2. Our second case is when the diameter of B^* has a different parity from the diameter of B so we can only move one leaf. We have four subcases:

   (a) Let B have even d and even n so $\ell = r + 1$. Let B^* be the balanced double broom such that $d^* = d + 1$, $\ell^* = \ell - 1$ and $r^* = r$. Note that $n^* = n$. Using lemma 4.3, we want to show that

   $$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

   We have

   $$\mathsf{T}_{\mathsf{forget}} = \frac{4 - d^2 - 4n + 2d + 2dn}{4}$$

   and

   $$\mathsf{T}^*_{\mathsf{forget}} = \frac{5 - (d+1)^2 - 4n + 2(d+1) + 2(d+1)n}{4}.$$

   We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

   $$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} = \frac{5 - (d+1)^2 - 4n + 2(d+1) + 2(d+1)n}{4} \\ -(\frac{4 - d^2 - 4n + 2d + 2dn)}{4} \\ = \frac{n-d}{2}.\end{aligned}$$

   $n > d$ by definition, so $n - d > 0$ and thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

   (b) Let B have even d and odd n so $= r$. Let B^* be the balanced double broom such that $d^* = d + 1$, $\ell^* = \ell - 1$ and $r^* = r$. Note that $n^* = n$. Using lemma 4.3, we want to show that

   $$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

   We have

   $$\mathsf{T}_{\mathsf{forget}} = \frac{4 - d^2 - 4n + 2d + 2dn}{4}$$

   and

   $$\mathsf{T}^*_{\mathsf{forget}} = \frac{2(d+1)^2n^2 + (d+1)^2 - (d+1)^2 n - 4n^2 + 9n - 4(d+1) - 4}{4(n-1)}.$$

We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

$$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} &= \frac{2(d+1)^2n^2 + (d+1)^2 - (d+1)^2n - 4n^2 + 9n - 4(d+1) - 4}{4(n-1)} \\ &\quad - (\frac{4 - d^2 - 4n + 2d + 2dn)}{4} \\ &= \frac{n(n-d)}{2(n-1)} > 0.\end{aligned}$$

Thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

(c) Let B have odd d and even n so $\ell = r$. Let B^* be the balanced double broom such that $d^* = d+1$, $\ell^* = \ell - 1$ and $r^* = r$. Note that $n^* = n$. Using lemma 4.3, we want to show that

$$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

We have

$$\mathsf{T}_{\mathsf{forget}} = \frac{5 - d^2 - 4n + 2d + 2dn}{4}$$

and

$$\mathsf{T}^*_{\mathsf{forget}} = \frac{4 - (d+1)^2 - 4n + 2(d+1) + 2(d+1)n}{4}.$$

We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

$$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} &= \frac{4 - (d+1)^2 - 4n + 2(d+1) + 2(d+1)n}{4} \\ -(\frac{5 - d^2 - 4n + 2d + 2dn)}{4} \\ &= \frac{n-d}{2}.\end{aligned}$$

$n > d$ by definition, so $n - d > 0$ and thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

(d) Let B have odd d and odd n so $\ell = r + 1$. Let B^* be the balanced double broom such that $d^* = d+1$, $\ell^* = \ell - 1$ and $r^* = r$. Note that $n^* = n$. Using lemma 4.3, we want to show that

$$\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}.$$

We have

$$\mathsf{T}_{\mathsf{forget}} = \frac{2dn^2 + d^2 - d^2n - 4n^2 + 9n - 4d - 1}{4(n-1)}$$

and

$$\mathsf{T}^*_{\mathsf{forget}} = \frac{4 - (d+1)^2 - 4n + 2(d+1) + 2(d+1)n}{4}.$$

We'll subtract $\mathsf{T}_{\mathsf{forget}}$ from $\mathsf{T}^*_{\mathsf{forget}}$ and show that it's greater than zero:

$$\begin{aligned}\mathsf{T}^*_{\mathsf{forget}} - \mathsf{T}_{\mathsf{forget}} &= \frac{4 - (d+1)^2 - 4n + 2(d+1) + 2(d+1)n}{4} - \\ &\qquad \frac{(2dn^2 + d^2 - d^2n - 4n^2 + 9n - 4d - 1)}{4(n-1)} \\ &= \frac{n(n-d-1) + 2d - 2}{2(n-1)} \\ &= \frac{n(\ell + r) + 2d - 2}{2(n-1)} > 0.\end{aligned}$$

Thus $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$.

Thus, in all cases $\mathsf{T}^*_{\mathsf{forget}} > \mathsf{T}_{\mathsf{forget}}$. □

Thus we have proved that of all balanced double brooms on n vertices, the longer the diameter, the greater the forget time.

Lemma 4.18. *Let $G \in \mathcal{T}_{n,d}$ be any tree on n vertices with diameter d. Let $B_{n,d}$ be the balanced double broom of diameter d on n vertices. Then $\mathsf{T}_{\mathsf{forget}}(B_{n,d}) > \mathsf{T}_{\mathsf{forget}}(G)$.*

Proof. Given $G \in \mathcal{T}_{n,d}$, let $z \in V$ be such that $H(z', z) = \max_{v \in V} H(v', v)$ and let $\delta = \mathrm{dist}(z, z')$. We know that the focus (or foci) are on the (z, z') path P. We have three cases of what this path looks like

1. The path P is $\{z = v_0, v_1, \dots v_k = a, \dots v_\delta = z'\}$ and the tree is focal.

2. The path P is $\{z = v_0, v_1, \dots v_k = a, v_{k+1} = b, \dots v_\delta = z'\}$ and the tree is bifocal.

3. The path P is $\{z' = v_0, v_1, \dots v_k = a, v_{k+1} = b, \dots v_\delta = z\}$ and the tree is bifocal.

Note that Case 2 and 3 end up being the same case because it does not matter where z and z', so they combine into the one bifocal case.

Then, we use Move 3.1 to caterpillarize G into the δ path such that every vertex in C is either on the δ path or is a leaf adjacent to a vertex on the δ path.

By Theorem 4.12, $\mathsf{T}_{\mathsf{forget}}(D) > \mathsf{T}_{\mathsf{forget}}(C)$ and by Theorem 4.16, $\mathsf{T}_{\mathsf{forget}}(B) > \mathsf{T}_{\mathsf{forget}}(D)$. Recall that the diameter of B is δ so we have proved that $\mathsf{T}_{\mathsf{forget}}(C) \leq \mathsf{T}_{\mathsf{forget}}(B_{n,\delta})$.

Finally, by Lemma 4.17, $\mathsf{T}_{\mathsf{forget}}(B_{n,\delta}) \leq \mathsf{T}_{\mathsf{forget}}(B_{n,d})$. Thus, we have

$$\mathsf{T}_{\mathsf{forget}}(C) < \mathsf{T}_{\mathsf{forget}}(B_{n,\delta}) \leq \mathsf{T}_{\mathsf{forget}}(B_{n,d})$$

and the balanced double broom maximizes the forget time. □

All of these sections together prove that of all trees on n vertices with diameter d, the tree that maximizes the forget time is the balanced double broom on n vertices with diameter d.

5. Future Work

This paper only gives results on the *maximum* forget time, so future work could focus on finding what trees *minimize* the forget time. Similarly, there are other mixing measures to be explored, such as mixing time, which finds the expected number of steps it takes to get from the pessimal vertex to the stationary distribution π. We could ask which tree maximizes the mixing time as well as minimizing it.

Bibliography

[1] A. Beveridge and M. Wang, “Exact mixing times for random walks on trees.,” *Graphs & Combinatorics*, vol. 29, no. 4, pp. 757–772, 2013.

[2] A. Beveridge, “Centers for random walks on trees,” *SIAM Journal on Discrete Mathematics*, vol. 23, no. 1, pp. 300–318, 2009.

[3] A. Holcombe-Pomerance, “Mixing measures for trees of fixed diameter,” *Macalester Honors*, pp. 1–60, 2023.

[4] L. Lovász and P. Winkler, “Efficient stopping rules for Markov chains,” in *Proc. 27th ACM Symp. Theory Comput.*, pp. 76–82, 1995.

[5] L. Lovász and P. Winkler, “Reversal of Markov chains and the forget time,” *Combin., Probab. Comput.*, vol. 7, pp. 189–204, 1998.

[6] L. Lovasz and P. Winkler, “Exact mixing in an unknown markov chain,” *The Electronic Journal of Combinatorics*, vol. 2, no. R15, pp. 1–12, 1995.

[7] A. Beveridge and J. Youngblood, “The best mixing time for random walks on trees.,” *Graphs & Combinatorics*, vol. 32, no. 6, pp. 2211 – 2239, 2016.