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A Discussion on Estimation of the Best Constant for Spherical Restriction Inequalities

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A DISCUSSION ON ESTIMATION OF THE BEST CONSTANT FOR SPHERICAL RESTRICTION INEQUALITIES

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Abstract

The restriction conjecture asks for a meaningful restriction of the Fourier transform of a function to a sufficiently curved lower dimensional manifold. It then conjectures certain size estimates for this restriction in terms of the size of the original function. It has been proven in 2 dimensions, but it is open in dimensions 3 and larger, and is an area of much recent active effort. In our study, instead of aiming to prove the restriction conjecture, we target understanding its worst-case scenarios within known estimates. Specifically, we investigate the extension operator applied to antipodally concentrating profiles, examining the ratio of the norms of these extensions. This involves understanding how the mass near the north pole compares to the mass near the south pole in terms of magnitude. Initial computational studies confirmed the established dichotomy between $p > 2$ and $1 \leq p \leq 2$. Based on these findings, we propose two conjectures: the first one is that there are 3 cases of the behavior of this constant, and the second one is that there exists a cutoff. We will also present some facts and conjectures related to special values such as the endpoint of t=1.

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Preliminary Concepts and Sharp Inequalities

0.1 Key Concepts in Analysis

Harmonic analysis, a branch of mathematics that studies the representation of functions or signals as the superposition of basic waves, has been a fertile ground for research due to its profound implications across various domains such as quantum mechanics, number theory, and signal processing. At the heart of harmonic analysis lies the Fourier transform, a mathematical tool that decomposes a function into its constituent frequency.

Let us begin the discussion of optimal constants in Fourier restriction hypothesis through explanation of some key ideas and context. As Fourier Transform and Restriction are essentially integrals, we first introduce the concept of **Lebesgue integration**. The Lebesgue measure, denoted as λ, is a mathematical construct used to define the measurement of a set within Euclidean space, which is simply $\mathbb{R}^{\ltimes}$ on our case, in a way that inherits the property of traditional notions of length, area, or volume, and extends beyond it. For a given set A, the Lebesgue measure $\lambda(A)$ can be intuitively thought of as follows:

1. For intervals in real numbers, the Lebesgue measure corresponds to the length of the interval. For an interval $[a, b]$, the Lebesgue measure is $\lambda([a, b]) = b - a$.

2. For higher dimensions, such as areas in 2D or volumes in 3D, the Lebesgue measure extends this idea. For a rectangle in 2D with sides of length l and w, the area (Lebesgue measure) is $\lambda = l \times w$. Similarly, for a box in 3D, the volume (Lebesgue measure) is the product of its length, width, and height.

3. For more complicated sets, the Lebesgue measure can be approximated by covering the set with countable collections of intervals (or rectangles, boxes in higher dimensions) and summing their measures (lengths, areas, volumes). The measure of a set A is essentially the smallest total measure of such coverings.

The Lebesgue integral is closely related to the Lebesgue measure in that it allows for the integration of functions with respect to this measure. We will be primarily working on integrals over $\mathbb{R}^{d+1}$ and its subset, the $d-$dimensional sphere $\mathbb{S}^d$.

Functional Analysis is a branch of mathematics that extends the methods of classical analysis, including calculus and differential equations, to more abstract spaces. Recall that in real analysis, we studied real numbers, the space they inhabit, and functions that take in real numbers as input. Similarly, the primary objects of study in functional analysis are functions the spaces they inhabit, known as function spaces, and functionals which takes in a function as input. These spaces can be thought of as collections of functions that share a common property, such as continuity, differentiability, or integrability, along with a structure that allows for the measurement of distance or convergence.

A metric space is a set of functions X together with a metric $d : X \times X \to \mathbb{R}$, which defines the distance between any two elements of X, and satisfies the following properties: non-negativity, $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$; symmetry, $d(x, y) = d(y, x)$; and the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$. A metric space is said to be complete if the limit of any convergent sequence in the space is also in the space. An L_p space is a special type of metric space that can be defined as a space of measurable functions for which the p-th power of the absolute value is Lebesgue integrable.

The L_p spaces are important in various fields of mathematics and related domains, including analysis, probability, and statistics, because they provide a flexible way to quantify and compare functions' sizes or effects across different contexts. For a given function $f : \mathcal{S} \to \mathbb{C}^n$, a measure $\mu(x)$ (which is often the conventional Lebesgue Measure if not specified), and a real number $p \geq 1$, the L_p norm of f is defined by the following expression:

$$||f||_p = \left(\int_{\mathcal{S}} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

This formula essentially says that we:

1. Take the absolute value of f, raise it to the power of p,
2. Integrate this value over the entire set X with respect to the measure μ,
3. And finally, take the pth root of the integral.

The result is a single number, $||f||_p$, which is the L_p norm of f. For our study, we will be looking at integrals on **spheres**, with $\mathcal{S} = \mathbb{S}$, the d-dimensional unit sphere in $\mathbb{R}^{d+1}$, which is not very different from (Riemann) sphere integral in a familiar setting.

Here are some special values of p that is of particular interest:

- For $p = 1$, functions in L_1 space are often used in computational geometry and statistics. Consider the function $f(x) = e^{-x^2}$, which is an example of a Gaussian function.

$$\int_{-\infty}^{\infty} |e^{-x^2}|\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

  The integral converges to a finite value, which is known to be $\sqrt{\pi}$, demonstrating that $f(x) = e^{-x^2}$ is indeed an L_1 function.

- For $p = 2$, the L_2 space is of particular interest because it relates to the concept of Euclidean distance in an infinite-dimensional space, and $||f||_2$ corresponds to the "energy" or "power" of the function f. It is also the only L_p space associated with an inner product.

- While p can be any positive real number, we focus on $p \geq 1$ because for these values of p, the space becomes a complete metric space.

0.2 Introduction to Fourier Analysis

Fourier Analysis is a branch of mathematical analysis that deals with expressing a function as a sum of periodic components and recovering the function from those components. It is a powerful tool for analyzing functions and signals in various domains, such as time or space, by decomposing them into frequencies. Grounded in the study of Fourier series and Fourier transforms, Fourier Analysis helps us to understand functions using sines and cosines.

For a function $f : \mathbb{R}^n \to \mathbb{C}$, the Fourier Transform and its inverse are defined as follows:

- The Fourier Transform of $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, into the frequency domain is given by:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \xi \cdot \mathbf{x}} d\mathbf{x}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ represents the frequency variables, and $\xi \cdot \mathbf{x}$ is the dot product of ξ and $\mathbf{x}$.

- The function $f(\mathbf{x})$ can be recovered from its Fourier Transform $\hat{f}(\xi)$ using the inverse Fourier Transform, which is:

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot \mathbf{x}} d\xi$$

After reviewing several important concepts related to our work, we will now discuss the concept of sharp inequality.

0.3 Sharp Inequalities

An inequality is said to be sharp if it cannot be relaxed and still be valid in general. Defined formally the inequality $A \leq B$ is called sharp if B is the least upper bound for all values of A under consideration.This criterion guarantees that sharp inequalities are the most stringent conditions under which a mathematical statement remains true. This motivates the question of "When does an inequality become an equality?"

The equality conditions in sharp inequalities are crucial as they reveals the extremal configurations or scenarios where the mathematical bounds are exactly met, offering deeper insights into the structure and behavior of mathematical entities under study. In our case, the entities are transformations of functions, which we will elaborate in the coming sections. To illustrate the special properties in structure of the conditions where equality holds, we discuss the following well-known inequalities.

0.3.1 Example: Hölder's Inequality

Hölder's Inequality, a fundamental theorem in functional analysis, generalizes the concept of the Cauchy-Schwarz Inequality to a broader range of spaces, including L_p spaces. Hölder's Inequality states that

$$\left| \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx \right| \leq \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g(x)|^q \, dx \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p, q \leq \infty$. This inequality studies the magnitudes of functions and their integrals, asserting that the integral of the product of two functions is bounded by the product of their L_p and L_q

norms. Equality holds if and only if f^p and g^q are linearly dependent almost everywhere, meaning that there exist real numbers $\alpha, \beta \geq 0$, not both of them zero, such that $\alpha|f|^p = \beta|g|^q$ -almost everywhere.

0.3.2 Example: The Isoperimetric Inequality

In the realm of geometry, the Isoparametric Inequality links the volume of a domain to the surface area of its boundary, with spheres giving the equality condition. The inequality states that for a bounded set $S \subset \mathbb{R}^n$ with surface area $\operatorname{per}(S)$ and volume $\operatorname{vol}(S)$, the inequality is given by

$$\operatorname{per}(S) \geq n \operatorname{vol}(S)^{\frac{n-1}{n}} \operatorname{vol}(B_1)^{\frac{1}{n}},$$

where $B_1 \subset \mathbb{R}^n$ is a unit ball. Equality holds when S is a ball in $\mathbb{R}^n$. Also a well-studied functional inequality, the Isoparametric Inequality in $\mathbb{R}^2$ has an elegant proof which involves harmonic analysis. We will give a proof due to Hurwitz in 1901, mentioned in Stein and Shakarchi[1] of the problem in 2 dimension:

Lemma 1 (Isoparametric Inequality in 2D)**.** *Suppose that Γ is a simple closed curve in $\mathbb{R}^2$ of length ℓ, and let A denote the area of the region enclosed by this curve. Then*

$$A \leq \frac{\ell^2}{4\pi},$$

with equality if and only if Γ is a circle.

Proof. For this proof, we will use the **bilinear form of Parseval's Identity**, which states that

$$\sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \frac{1}{2\pi} \int_0^{2\pi} f(s)\overline{g(s)}\, ds$$

where $\{a_n\}$ is the Fourier coefficient of f and $\{b_n\}$ is the Fourier coefficient of g. We can always map any simple closed curve(a curve with no break points that does not intersect anywhere with itself) to one with arbitrary length. It suffices to prove that if $\ell = 2\pi$ then $A \leq \pi$, with equality only if Γ is a circle.

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ with $\gamma(s) = (x(s), y(s))$ be a parametrization by arc-length of the curve Γ, that is, $x'(s)^2 + y'(s)^2 = 1$ for all $s \in [0, 2\pi]$. This implies that

$$\frac{1}{2\pi} \int_0^{2\pi} (x'(s)^2 + y'(s)^2)\, ds = 1.$$

Since the curve is closed (it can be drawn with no break points), the functions $x(s)$ and $y(s)$ are 2π-periodic, so we may consider their Fourier series

$$x(s) = \sum_{n=-\infty}^{\infty} a_n e^{ins} \quad \text{and} \quad y(s) = \sum_{n=-\infty}^{\infty} b_n e^{ins}.$$

When we differentiate $x(s)$ with respect to s, we obtain

$$x'(s) = \frac{d}{ds}\left(\sum_{n=-\infty}^{\infty} a_n e^{ins}\right) = \sum_{n=-\infty}^{\infty} \frac{d}{ds}\left(a_n e^{ins}\right).$$

Therefore,

$$x'(s) \sim \sum_{n=-\infty}^{\infty} ina_n e^{ins}.$$

Similarly, for $y(s)$ we have

$$y'(s) \sim \sum_{n=-\infty}^{\infty} inb_n e^{ins}.$$

For the squared derivatives, Parseval's identity can be written as

$$\int_0^{2\pi} |x'(s)|^2 \, ds = \sum_{n=-\infty}^{\infty} |ina_n|^2,$$

and for $y(s)$,

$$\int_0^{2\pi} |y'(s)|^2 \, ds = \sum_{n=-\infty}^{\infty} |inb_n|^2.$$

Combining these two and considering the original condition from equation (2) that $x'(s)^2 + y'(s)^2 = 1$, we get

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} (x'(s)^2 + y'(s)^2) \, ds \\ &= \sum_{n=-\infty}^{\infty} (|ina_n|^2 + |inb_n|^2) \\ &= \sum_{n=-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2). \end{aligned}$$

Applying the given condition yields

$$\sum_{n=-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2) = 1,$$

which is the equation given by applying Parseval's identity to the squared derivatives of the functions $x(s)$ and $y(s)$.

We now apply the bilinear form of Parseval's identity to the integral defining A due to Green's Theorem. Since $x(s)$ and $y(s)$ are real-valued, we have $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$, so we find that

$$A = \frac{1}{2}\left|\int_0^{2\pi} x(s)y'(s) - y(s)x'(s)\,ds\right| = \pi \sum_{n=-\infty}^{\infty} \left|n(a_n\overline{b_n} - b_n\overline{a_n}).\right|$$

Note that

$$|a_n\overline{b_n} - b_n\overline{a_n}| \quad \leq |a_n\overline{b_n}| + |b_n\overline{a_n}| \leq 2|a_n||b_n| \quad \leq |a_n|^2 + |b_n|^2$$

where the first inequality comes from the triangle inequality for complex numbers, and the second inequality is an application of the arithmetic-geometric mean inequality.

Thus,

$$\begin{aligned} A &= \pi \sum_{n=-\infty}^{\infty} |n(a_n\overline{b_n} - b_n\overline{a_n})| \\ &\leq \pi \sum_{n=-\infty}^{\infty} |n|(|a_n|^2 + |b_n|^2) \\ &\leq \pi \sum_{n=-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2) \\ &\leq \pi \end{aligned}$$

Now, we will demonstrate that the inequality is sharp by studying the special case of conditioned equality. While the study of the Isoperimetric Inequality dated back significantly longer than the invention of Fourier Analysis, and the curve that should achieve inequality was hypothesized to be a circle, the hypothesis relied on heuristics, not proofs.

For the inequality to become an equality, we must reach the identity

$$\sum_{n=-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2) = \sum_{n=-\infty}^{\infty} n(a_n\overline{b_n} - b_n\overline{a_n})$$

which is possible if and only if all $\{a_n\}, \{b_n\}$ to vanish apart from $\{a_{-1}, a_0, a_1\}$ and $\{b_{-1}, b_0, b_1\}$ because $|n| < |n^2|$ as soon as $|n| \geq 2$.

When $A = \pi$, the area enclosed by the curve, we deduce from the previous argument that

$$x(s) = a_{-1}e^{-is} + a_0 + a_1e^{is} \quad \text{and} \quad y(s) = b_{-1}e^{-is} + b_0 + b_1e^{is},$$

We know that $x(s)$ and $y(s)$ are real-valued, so $a_{-1} = \overline{a_1}$ and $b_{-1} = \overline{b_1}$. Since we have equality, we must have $|a_1| = |b_1| = 1/2$. We write

$$a_1 = \frac{1}{2}e^{i\alpha} \quad \text{and} \quad b_1 = \frac{1}{2}e^{i\beta}.$$

The fact that $1 = 2|a_1b_{-1} - a_{-1}b_1|$ implies that $|\sin(\alpha - \beta)| = 1$, hence $\alpha - \beta = k\pi/2$ where k is an odd integer. From this we find that

$$x(s) = a_0 + \cos(\alpha + s) \quad \text{and} \quad y(s) = b_0 \pm \sin(\alpha + s),$$

where the sign in $y(s)$ depends on the parity of $(k-1)/2$.

□

While this proof only studies the curves in $\mathbb{R}^2$ that can be parameterized, it offers valuable insight on our ability to analyze a problem with periodic functions.

Introduction to Fourier Restriction Theory

0.4 Introduction to Restriction Theory

With the definition of Sharp Inequalities in mind, we now come to explore which sharp inequality we are expected to observe in particular. Despite its classical origins and widespread application, the Fourier transform's behavior, particularly its interaction with lower-dimensional manifolds, remains an area of active investigation. This inquiry is epitomized by the Restriction Conjecture, which has spurred much of the research in harmonic analysis over the last fifty years.

The Restriction Conjecture is concerned with understanding the conditions under which the Fourier transform of a function can be restricted to a sufficiently curved lower-dimensional manifold and conjectures certain size estimates for this restriction in terms of the size of the original function.

As a key problem in harmonic analysis for over five decades, he conjecture inquires about the conditions under which the Fourier transform of a function can be meaningfully restricted to a sufficiently curved lower-dimensional manifold, proposing size estimates for this restriction. Proven in two dimensions but open in higher, it has implications beyond harmonic analysis, such as in differential equations[**?**] and number theory. We explore the restriction operator $\mathcal{R}$ associated with the unit sphere $\mathbb{S}^d$, defined for a function $g : \mathbb{R}^{d+1} \to \mathbb{C}$ as

$$\mathcal{R}g(\omega) = \int_{\mathbb{R}^{d+1}} e^{-ix\omega} g(x)\, dx$$

for $\omega \in \mathbb{S}^d$.

This transform takes it in $g : \mathbb{R}^{d+1} \to \mathbb{C}$ computes it's Fourier transform, which is a new function $\hat{g} : \mathbb{R}^{d+1} \to \mathbb{C}$. The fact that $g \in L_1 \to \hat{g}$ implies that the restriction operator is well-defined. Finally we have the function $\mathcal{R}g(\omega) : \mathbb{S}^d \to \mathbb{C}$. This transform is denoted as "restriction" because the new function has a more reduced domain of definition. For $g \in L_1(\mathbb{R}^{d+1})$, $\hat{g}$ is continuous and bounded, yielding the restriction inequality $\|\mathcal{R}g\|_{L^\infty(\mathbb{S}^{d+1})} \leq \|g\|_{L_1(\mathbb{R}^{d+1})}$. $\hat{g}$ is bounnded by $\|g\|_{L_1}$, and such inequalities are called "restriction inequalities"

In this context, the surface measure σ on $\mathbb{S}^d$, used in defining the norm of $\mathcal{R}g$, is derived from Lebesgue measure adapted to the sphere's geometry, facilitating integration over $\mathbb{S}^d$ that reflects its curvature. The measure σ is essential for applying Lebesgue integration theory to spherical domains.

Lemma 2.

$$\max_{\omega \in \mathbb{S}^{d+1}} |\mathcal{R}g(\omega)| \leq \|g\|_{L_1(\mathbb{R}^{d+1})}$$

.

Proof. Let $g \in L_1(\mathbb{R}^{d+1})$ and consider $\omega \in \mathbb{S}^{d+1}$. By definition, we have

$$\mathcal{R}g(\omega) = \int_{\mathbb{R}^{d+1}} e^{-ix\cdot\omega} g(x)\, dx.$$

Taking the absolute value,

$$|\mathcal{R}g(\omega)| = \left| \int_{\mathbb{R}^{d+1}} e^{-ix\cdot\omega} g(x)\, dx \right|.$$

Using the triangle inequality for integrals, we obtain

$$|\mathcal{R}g(\omega)| \leq \int_{\mathbb{R}^{d+1}} |e^{-ix\cdot\omega}||g(x)|\, dx.$$

Since $|e^{-ix\cdot\omega}| = 1$ for all x, ω,

$$|\mathcal{R}g(\omega)| \leq \int_{\mathbb{R}^{d+1}} |g(x)|\, dx = \|g\|_{L_1(\mathbb{R}^{d+1})}.$$

Thus, for any $\omega \in \mathbb{S}^{d+1}$,

$$|\mathcal{R}g(\omega)| \leq \|g\|_{L_1(\mathbb{R}^{d+1})}.$$

Taking the maximum over all $\omega \in \mathbb{S}^{d+1}$,

$$\max_{\omega \in \mathbb{S}^{d+1}} |\mathcal{R}g(\omega)| \leq \|g\|_{L_1(\mathbb{R}^{d+1})}.$$

$\square$

For g in $L_2(\mathbb{R}^{d+1})$, a direct application of the formula for $\mathcal{R}g$ is ill-defined due to the potential divergence of the integral defining $\hat{g}(\xi)$ for some ξ. Instead, $\hat{g}$ is constructed via approximation, where for each $g \in L_2(\mathbb{R}^{d+1})$, a sequence $\{g_n\} \subset L_1 \cap L_2$ converges to g in L_2. The Fourier transform $\hat{g}$ is then the limit in L_2 of $\{\hat{g_n}\}$, extending the operator from $L_1 \cap L_2$ to all of L_2 by continuity.

Lemma 3. *Given any function $g \in L_2(\mathbb{R}^{d+1})$, the Fourier transform $\hat{g}$ exists in $L_2(\mathbb{R}^{d+1})$.* [2],[3]

Proof. For $g \in L_2(\mathbb{R}^{d+1})$, there exists a sequence $\{g_n\} \subset L_1 \cap L_2$ such that $\|g_n - g\|_{L_2} \to 0$ as $n \to \infty$. In other words, g_n approximates g in the L_2 sense. This is because in both L_1 and L_2, continuous functions are dense, meaning that you can approximate a given function in the space within any given margin of error.

Due to congvergence in L_2, we claim that $\{g_n\}$ is a Cauchy sequence in L_2, or for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have $\|g_n - g_m\|_{L_2} < \epsilon$.

Plancherel's theorem guarantees that $\|\hat{f}\|_{L_2} = \|f\|_{L_2}$ for all $f \in L_2(\mathbb{R}^{d+1})$. Therefore, $\{\hat{g_n}\}$ is also a Cauchy sequence in L_2 since

$$\|\hat{g_n} - \hat{g_m}\|_{L_2} = \|g_n - g_m\|_{L_2} < \epsilon$$

for $m, n > N$. Given that $L_2(\mathbb{R}^{d+1})$ is a complete metric space, the Cauchy sequence $\{\hat{g_n}\}$ converges to some limit in L_2. Let this limit be $\hat{g}$. By definition, $\hat{g}$ is the Fourier transform of g in $L_2(\mathbb{R}^{d+1})$.

□

Because "functions" in L_2 are really equivalent classes of functions almost everywhere, the restriction of the Fourier transform $\hat{g}$ to the sphere for functions in $L_2(\mathbb{R}^{d+1})$ is ill-defined. However, Stein revealed that the sphere's curvature allows for the possibility of meaningful restriction for certain function spaces beyond L_1, specifically for some $p > 1$. This justifies the **restriction conjecture** as follows:

Conjecture 4. *For $1 \leq p \leq \frac{2(d+1)}{d+2}$ and $q \leq \frac{d}{d+2}p'$*

$$\|\mathcal{R}g\|_{L_q(\mathbb{S}^d;d\sigma)} \leq C\|g\|_{L_p(\mathbb{R}^{d+1})}, \tag{1}$$

where p' denotes the Holder conjugate of p.

The conjecture is known for $1 \leq p \leq 2$. [reference needed] The adjoint of the restriction operator, denoted $\mathcal{R}^*$ or $\mathcal{E}$ for extension operator, relates to the restriction operator $\mathcal{R}$ via an operation similar to the inner product :

$$\langle \mathcal{R}g, f \rangle = \langle g, \mathcal{R}^* f \rangle$$

For functions $g \in L_p(\mathbb{R}^{d+1})$ and $f \in L^{p'}(\mathbb{S}^d)$, this relationship is expressed through integrals over $\mathbb{R}^{d+1}$ and $\mathbb{S}^d$, leading to the definition of

$$\mathcal{R}^* f(x) = \int_{\mathbb{S}^d} e^{ix\omega} f(\omega)\, d\sigma$$

Because $\mathcal{R}^*$ starts from a function on the manifold and extends it to a function on $\mathbb{R}^{d+1}$, it is often called the extension operator and denoted by $\mathcal{E}$.

Because of the close relationship between restriction and extension, the restriction conjecture has an equivalent form in terms of extension.

Conjecture 5. *For $q \geq \frac{d+2}{d}p'$ and $q > \frac{2(d+1)}{d}$*

$$\|\mathcal{E}f\|_{L_q(\mathbb{R}^{d+1})} \leq C\|f\|_{L_p(\mathbb{S}^d;d\sigma)}, \tag{2}$$

where p' denotes the Holder conjugate of p.

Lemma 6. *The restriction conjecture for $\mathcal{R} : L_p(\mathbb{R}^{d+1}) \to L_q(\mathbb{S}^d)$ is equivalent to the extension conjecture for the adjoint operator $\mathcal{R}^* : L^{q'}(\mathbb{S}^d) \to L^{p'}(\mathbb{R}^{d+1})$.*

The proof of such equivalence requires the following lemma shown in [3]

Lemma 7. *Suppose $1 \leq p, q \leq \infty$ are conjugate exponents. If $g \in L_q$, then*

$$\|g\|_{L_q} = \sup\left\{\left|\int fg\right| : \|f\|_{L_p} \leq 1\right\}$$

Proof. (Proof of 3.5) Suppose $\mathcal{R}$ is bounded from L_p to L_q, there exists a constant C such that for all $g \in L_p(\mathbb{R}^{d+1})$,

$$\|\mathcal{R}g\|_{L_q(\mathbb{S}^d)} \leq C\|g\|_{L_p(\mathbb{R}^{d+1})}.$$

By Hölder's inequality, it follows that for all $f \in L^{q'}(\mathbb{S}^d)$,

$$\left|\int_{\mathbb{S}^d} (\mathcal{R}g)f \; d\sigma\right| \leq \|\mathcal{R}g\|_{L_q(\mathbb{S}^d)}\|f\|_{L^{q'}(\mathbb{S}^d)} \leq C\|g\|_{L_p(\mathbb{R}^{d+1})}\|f\|_{L^{q'}(\mathbb{S}^d)}.$$

The inner product of $\mathcal{R}g$ with f is equal to the inner product of g with $\mathcal{R}^*f$, which means

$$\left|\int_{\mathbb{R}^{d+1}} g(\mathcal{R}^*f) \; dx\right| \leq C\|g\|_{L_p(\mathbb{R}^{d+1})}\|f\|_{L^{q'}(\mathbb{S}^d)}.$$

This implies that $\mathcal{R}^*f \in L^{p'}(\mathbb{R}^{d+1})$ and

$$\|\mathcal{R}^*f\|_{L^{p'}(\mathbb{R}^{d+1})} \leq C\|f\|_{L^{q'}(\mathbb{S}^d)},$$

showing that $\mathcal{R}^*$ is bounded from $L^{q'}(\mathbb{S}^d)$ to $L^{p'}(\mathbb{R}^{d+1})$.

The reverse implication follows by the same argument, starting with the boundedness of $\mathcal{R}^*$ and showing the boundedness of $\mathcal{R}$ by considering the duality pairing in the opposite order.

□

0.5 Sharp Restriction/Extension Inequalities

With the two forms of Restriction Conjecture in mind, we ask the immediate follow-up question of the function which potentially satisfies the equality (which originates from an extreme case from the original inequality). The equivalence of the two forms of the supremum in the extension conjecture simplifies our analytical approach. By transitioning to a discussion on the supremum, we aim to explore the upper bounds of the operator's amplification effect more comprehensively. This exploration is vital for identifying sharp constants and understanding the conditions that lead to the existence of extremizers, which are functions that achieve or closely approach these upper bounds.

Define

$$S_{p\to q} := \sup_{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}<\infty} \frac{\|\mathcal{E}f\|_{L_q\mathbb{R}^{d+1}}}{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}}$$

The supremum encapsulates the core challenge of the extension conjecture: to quantify the maximal effect of the extension operator on the L_p norm of functions from $\mathbb{S}^d$ to $\mathbb{R}^{d+1}$. By considering the supremum, we can directly measure the maximal amplification effect the operator has on the L_p norm of functions when transferred from the sphere to Euclidean space. In fact, Flock and Stovall[4] approached this question by considering sequences of functions for which the ratio of the extension norm to the norm of the function converge to $S_{p\to q}$. Such sequences are called *extremizing sequences* or *extremal sequences*.

Flock and Stovall[4] proved a dichotomy of extremizing sequences g_n, where either

1. $g_n \to g$. Here, an extremizer g exists.
2. g_n can be written as $F_n + t\tilde{F_n}$, where F_n and $\tilde{F_n}$ are functions converging on the North and South pole.

Our studies focus on the latter case. Suppose we had the isoperimetric inequality restricted to the set of regular polygons. Then as the number of edges increase, the shape approximates a circle, hence we would get something very close to the original isoperimetric inequality, but it can never be sharp. However, the best constant will not change just because no equality can be reached. Consider

$$S_{p\to q} := \sup_{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}<\infty} \frac{\|\mathcal{E}f\|_{L_q\mathbb{R}^{d+1}}}{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}} = \sup_{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}=1} \|\mathcal{E}f\|_{L_q(\mathbb{R}^{d+1})}.$$

To show the equivalence of the two forms, let's start with any function f such that $\|f\|_{L_p(\mathbb{S}^d;d\sigma)} < \infty$. Define $g = \frac{f}{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}}$, ensuring that $\|g\|_{L_p(\mathbb{S}^d;d\sigma)} = 1$. The extension operator applied to g gives:

$$\|\mathcal{E}g\|_{L_q(\mathbb{R}^{d+1})} = \frac{\|\mathcal{E}f\|_{L_q(\mathbb{R}^{d+1})}}{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}}.$$

Hence, the supremum over all f with finite L_p norm is equivalent to the supremum over all normalized f, proving that the two formulations of $S_{p\to q}$ are indeed equal.

The reformulation sets the stage for the conjecture to propose specific bounds and conditions under which the supremum is finite or achieves a particular value. For the case where $g_n \to g$, we hypothesize that

Conjecture 8 (Conjecture for the sphere)**.** *For $q \geq \frac{d+2}{d}p'$ and $q > \frac{2(d+1)}{d}$*

$$S_{p\to q} < \infty$$

and further when $q = \frac{d+2}{d}p'$, for $f := 1$

$$S_{p\to q} = \frac{\|\mathcal{E}f\|_{L_q\mathbb{R}^{d+1}}}{\|f\|_{L_p(\mathbb{S}^d;d\sigma)}}.$$

The discussion of how the condition of antipodally concentrating profile is derived with rigorous mathematics discussion, intriguing and expansive as it could be, is beyond the scope of this paper. A brief summary is that for a family of symmetrically concentrating profiles known to be likely candidates when the inequality becomes an equality,

$$\lim_{n\to\infty} \frac{\|\mathcal{E}g_n\|_{L_q(\mathbb{R}^{d+1})}}{\|g_n\|_{L_p(\mathbb{S}^d;d\sigma)}} \propto \alpha_{p,q}.$$

.

For $1 \leq p < q = \frac{d+2}{d}p'$, we define

$$\alpha_{p,q} := \max_{t\in[0,1]} \frac{\|1 + te^{i\theta}\|_{L_q([0,2\pi],d\theta/2\pi)}}{(1+t^p)^{1/p}}. \tag{3}$$

The parameter t serves as the ratio between the norms of the extensions of two antipodally concentrating profiles.

We are interested in the value of t at which extremum can be obtained by the function of the right hand side, as well as the actual magnitude of the extremum itself, although perhaps to a lesser extent. Specifically, given numerical evidence that $\max\{\alpha_{p,q}\}$ occurs at either 0 or 1. we consider the following questions:

1. Can we provide a mathematically sound proof of $\alpha_{p,q}$ being maximized at either 0 or 1?

2. Can we give a cutoff in (p, q) for when the maximum values moves from $t = 0$ to $t = 1$?

Before the discussion of the endpoint conjectures, we must discuss several preliminary facts about the quantity.

We define

$$
\begin{aligned}
I(t,p,q) &:= \frac{\|1+te^{i\theta}\|_{L_q([0,2\pi],d\theta/2\pi)}}{(1+t^p)^{1/p}} \\
&= \frac{1}{(1+t^p)^{1/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} |1+t\cos(\theta)+ti\sin(\theta)|^q \ d\theta \right)^{1/q} \\
&= \frac{1}{(1+t^p)^{1/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(t^2\cos^2(\theta)+2t\cos(\theta)+\sin^2(\theta)\right)^{q/2} \ d\theta \right)^{1/q} \\
&= \frac{1}{(1+t^p)^{1/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(t^2+2t\cos(\theta)+1\right)^{q/2} \ d\theta \right)^{1/q}.
\end{aligned}
$$

The range of $t \in [0,1]$ has led us to speculate in a variety of forms. For instance, let $\phi \in [0, \frac{\pi}{2}]$, we attempted the parameterization of $t = \sin(\phi)$, where

$$
I(\phi,p,q) := \frac{1}{(1+\sin^p(\phi))^{1/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\sin^2(\phi)+2\sin(\phi)\cos(\theta)+sin^2(\theta)\right)^{q/2} \ d\theta \right)^{1/q}.
$$

However, this form of parameterization did not simplify the already complicated function, and impeded the attempt of finding the derivative of the already complicated function.

Studies on the Ratio Parameter

0.6 Evolving Conjectures on $\alpha_{p,q}$

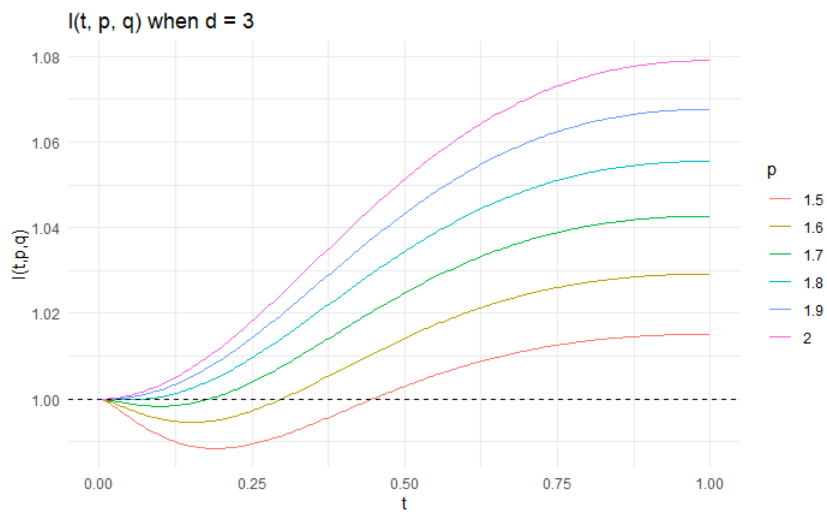

Figure 1 Overall Trend for $I(t, p, q)$

The function $I(t, p, q)$, as depicted in Figure 2 with a multitude of choices for p, presents a complex behavior where the location of its maximum shifts based on the parameter p. For certain values of p, the maximum of $I(t, p, q)$ is achieved when $t = 0$, while for other values, the maximum is at $t = 1$. This suggests the existence of threshold values of p at which the behavior of the function transitions. Initial studies on visualizations like Figure 1 on the quantity has enabled us to propose the following conjectures:

Conjecture 1. *There exists a critical value p_c such that for all $p < p_c$, the maximum of $I(t, p, q)$ is achieved at $t = 0$, and for all $p > p_c$, the maximum is achieved at $t = 1$.*

Conjecture 2. *For $p_1 > p_2$, the function $I(t, p_1, q_1)$ and is always bigger than the function $I(t, p_2, q_2)$ at any given $t \in (0, 1)$, where the pairs (p_1, q_1) and (p_2, q_2) are defined by $q_i = \frac{d+2}{d} p_i'$.*

While this was verified by several of our visualizations, this encountered what we would call "anomalies for small p", where we see on Figure 2 where p is around 1.2 and t around 0.25. This shows both the strength and limitation of numerical observations: while it can rule out incorrect hypothesis with relative efficiency, it does not provide information as much as a concrete proof.

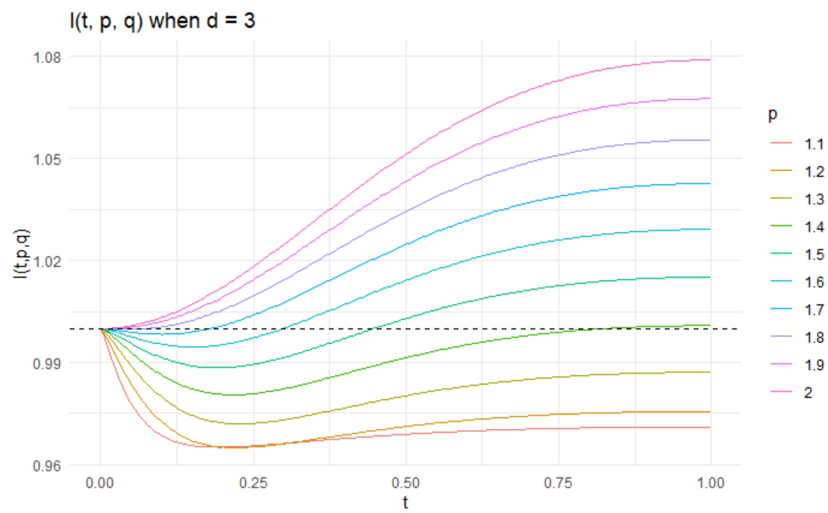

Figure 2 While the endpoints seem to be monotonous with respect to p, this cannot be generalized to the entirety of the function.

To further investigate small values of p, we make a third visualization on $p \in [1.05, 1.15]$. The result shows that the monotonous increase of $I(1, p, q)$ with respect to p no longer holds here.

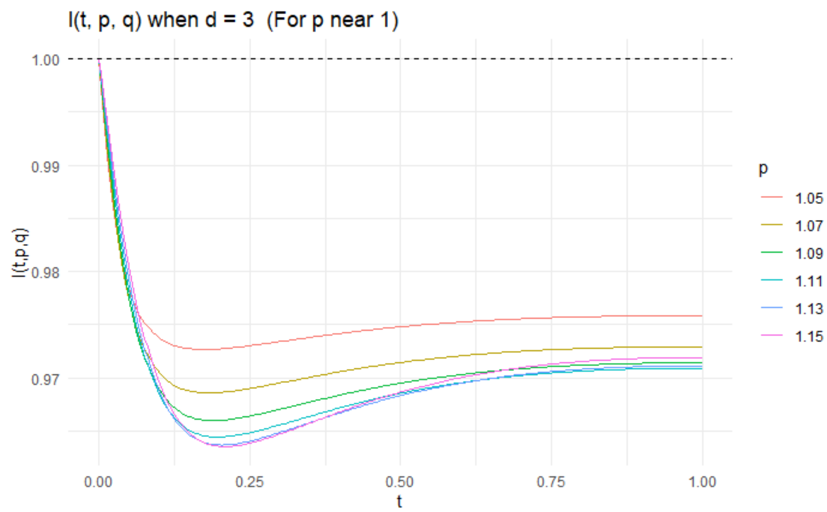

Figure 3 Comparisons of $p \in [1.05, 1.15]$ reveal further complications on $I(t, p, q)$

The continuous attempt (and failure) to find a symbolic, mathematically rigorous way to indicate the property of $I(t, p, q)$ serve as a recurrent theme in our paper. And when such efforts fail, we resort to numerical observations.

Initially, this works well for cases where p is large. We considered generalizing the statement of monotonicity to **any** $t \in (0, 1)$. However, as the value of p decreases, we discovered that the case is not necessarily true for small p.

Our first conjecture posits that such threshold values exist, and understanding their nature is crucial for the full characterization of the function.

Consider the function $I(t, p, q)$ for $d = 3$. We observe that as p varies, the function's maximum shifts from $t = 0$ to $t = 1$. The general behavior of the functions depicted in the graph can be described in several distinct phases based on the value of the parameter p.

We observe 3 cases for the behavior of the function $I(t, p, q)$:

1. Case: For p that is closer to 2, the function increases monotonously on $(0, 1)$.

2. Case: For smaller p that is larger than some cutoff, which will be discussed later, the function decreases first, and then increases to a value above 1.

3. Case: For p even smaller than the cutoff, the function decreases first, but then increases to a value that is below 1.

For lower values of p, the functions appear to be decreasing rapldiy near $t = 0$ and then increase in a flat manner as t approaches 1, but never growing back to 1 again. This indicates that the function's sensitivity to changes in t is minimal at the lower end of the t range. As p increases, a point of inflection emerges, beyond which the function's growth rate increases significantly. This point of inflection moves closer to $t = 0$ as p further increases.

At intermediate values of p, the function's curvature changes, suggesting a complicated relationship between t and the function's value. We have managed to evaluate the values of the function symbolically at 1, but failed to generalize it to any arbitrary value in $t \in (0, 1)$

As p approaches and exceeds a certain threshold value, the functions demonstrate a marked increase at lower values of t, signifying a shift in the behavior of the function. This shift is indicative of a phase transition-like phenomenon where the function's sensitivity to t is now greatest at the lower end of the t range, which is a stark contrast to the behavior observed for lower values of p.

The evaluation of $I(1, p, q)$ enables us to look into the cutoff of p where its behavior change. While we have observed anomalies for small p, we believe that there exist for any d, there exists a critical value p_c such that for the pair (p_c, q_c) defined by d, $I(t, p_c, q_c)$ either strictly increases or decreases before bouncing back to 1. (and we would expect to observe the maximum at $t = 1$) and for any $p < p_c$ the monotonicity no longer exists, but we still could see a maximum at either $t = 0$ (when p is near 1) or $t = 1$ (when p gets larger). In other words, this p_c separates **case 1,2** from **case 3**.

We looked into this by defining a hypothesized critical point $p_\epsilon(d)$ where given a certain integer d that represents dimension and a small real number $\epsilon > 0$, we have $I(t, p_\epsilon(d), q_\epsilon(d))$ attains its maximum value at 0, whereas $I(t, p_\epsilon(d) + \epsilon, q_{2\epsilon(d)})$ attains its maximum value at 0. We use the binary search algorithm enabled by the conjecture that the value $I(1, p, q)$ is monotonous with respect to p. Previously, we have tried symbolic means to solve $\beta_{p,q} = 1$, but complications in computation have prevented us from obtaining any meaningful results. Here, the pair $(p_\epsilon(d), q_\epsilon(d))$ and $(p_\epsilon(d) + \epsilon, q_{2\epsilon(d)})$ are both (p, q) pairs defined by d. Importantly, this definition of cutoff asserts that if p is bigger than $p_\epsilon(d)$, the function has to attain its maximum at 1, a hypothesis substantiated by visualization and numerical analysis but lacks actual proof.

Here is the visualization of the cutoff between where the maximum would be attained with respect to dimensionality. The threshold starts at around 1.6 and slowly approaches 2. We draw this figure with $\log(d)$ for a compact view.

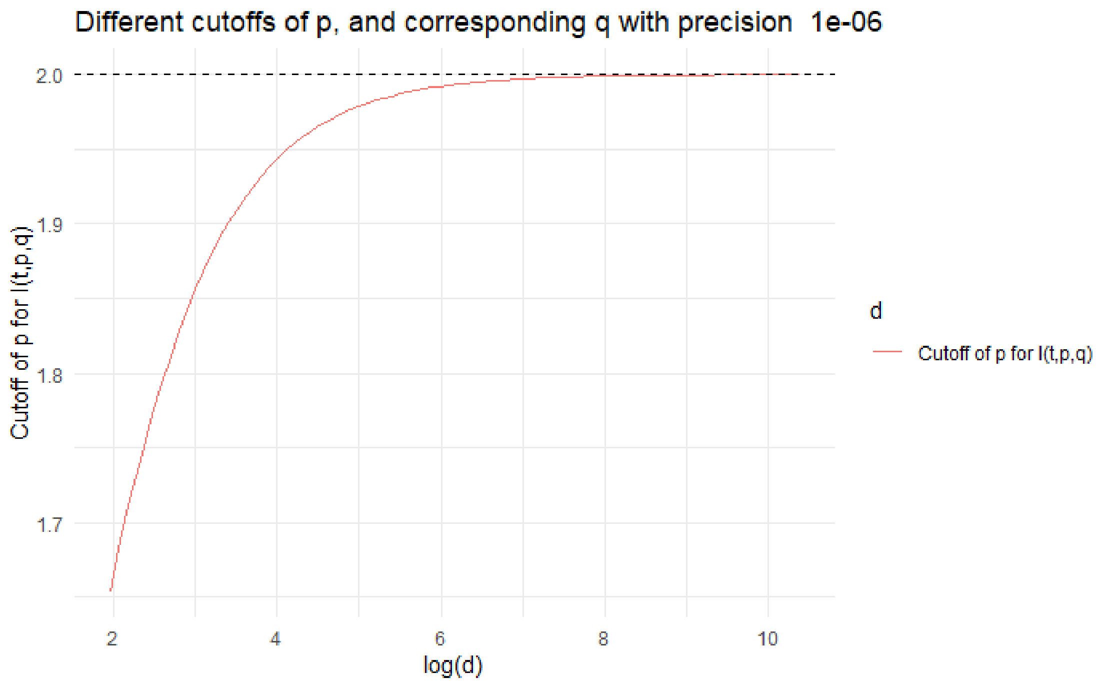

Figure 4 The cutoff of p, which separates **case 1,2** from **case 3**.

In addition, here is a list of cutoff values for different values of d. This table indicates how rapidly the cutoff $p_\epsilon(d)$ approaches 2 from below:

d	$p_\epsilon(d)$
16	1.82498918
40	1.92460603
99	1.96858460
245	1.98714373
602	1.99474074
1480	1.99785666
3641	1.99912813
8955	1.99964564

Table 1 Selected Values of $p_\epsilon(d)$

0.7 Special values of $\alpha_{p,q}$

This sections deals with $I(t,p,q)$ when the variable t and parameters (p,q) are special. Recall that

$$I(t,p,q) := \frac{\|1+te^{i\theta}\|_{L_q([0,2\pi],d\theta/2\pi)}}{(1+t^p)^{1/p}} = \frac{1}{(1+t^p)^{1/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} |1+t\cos(\theta)+ti\sin(\theta)|^q \ d\theta \right)^{1/q}$$

and thus when $t=0$, $I(t,p,q)=1$.

The case $t=1$ is relatively complicated to $t=0$, as we consider the fraction part, $\frac{1}{2^{1/p}}$, and the integral part,

$$\frac{1}{2\pi} \int_0^{2\pi} (2+2\cos(\theta))^{q/2} \ d\theta$$

Note that

$$1+\cos(\theta) = 1+\cos(2\cdot\theta/2) = (1+2\cos^2(\theta/2)-1) = 2\cos^2(\theta/2)$$

Thus

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} (2+2\cos(\theta))^{q/2}\ d\theta \\
&= \frac{2^{q/2}}{2\pi}\int_0^{2\pi} (1+\cos(\theta))^{q/2}\ d\theta \\
&= \frac{2^{q/2}}{2\pi}\int_0^{2\pi} \left(2\cos^2(\theta/2)\right)^{q/2}\ d\theta \\
&= \frac{2^{q}}{2\pi}\int_0^{2\pi} \cos^q(\theta/2)\ d\theta \\
&= \frac{2^{q}}{\pi}\int_0^{2\pi} \cos^q(\theta/2)\ d\theta/2 \\
&= \frac{2^{q}}{\pi}\cdot\left[2\int_0^{\pi/2} \cos^q(\varphi)\ d\varphi\right]
\end{aligned}$$

where the last line holds due to periodicity.
Now recall the general trigonometric form of the **Beta Function** $\mathcal{B}(a,b)$

$$\mathcal{B}\left(\frac{a+1}{2},\frac{b+1}{2}\right) = 2\int_0^{\pi/2} \sin^a x \cos^b x\ dx$$

Recall that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Substitute $a=0$ and $b=q$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} (2+2\cos(\theta))^{q/2}\ d\theta = \frac{2^q}{\pi}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} = 2^q\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}$$

And consequently,

$$\begin{aligned}
I(1,p,q) &= \frac{1}{2^{1/p}}\cdot\left(2^q\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \\
&= \frac{1}{2^{1/p}}\cdot 2\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \\
&= 2^{1-1/p}\cdot\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{p'}}\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}}
\end{aligned}$$

We denote this value as $\beta_{p,q}$.

Recall the Beta function's definition in terms of the Gamma function:

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

By setting $x = \frac{q}{2}$ and $y = \frac{q}{2} + 1$, we can align the denominator of the expression with the Beta function's form:

$$\mathcal{B}\left(\frac{q}{2}, \frac{q}{2}+1\right) = \frac{\Gamma(\frac{q}{2})\Gamma(\frac{q}{2}+1)}{\Gamma(q+1)}.$$

And consequently,

$$I(1,p,q) = 2^{\frac{1}{p'}} \left(\frac{2^{1-q}}{q\pi\mathcal{B}(\frac{q}{2}, \frac{q}{2}+1)}\right)^{\frac{1}{q}}$$

Note that the Beta function can also be defined by the **integral form**:

$$\mathcal{B}(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}\,dz.$$

Substituting $x = \frac{q}{2}$ and $y = \frac{q}{2} + 1$, we get:

$$\mathcal{B}\left(\frac{q}{2}, \frac{q}{2}+1\right) = \int_0^1 z^{\frac{q}{2}-1}(1-z)^{\frac{q}{2}}\,dz.$$

We note that the integrand is always positive and is less than or equal to 1 for $t \in [0,1]$ because $t^{\frac{q}{2}-1} \leq 1$ for $t \in [0,1]$ and $(1-t)^{\frac{q}{2}} \leq 1$ for $t \in [0,1]$. Therefore, the entire integrand is less than or equal to 1 for all t in the domain of integration. Given this, a conservative bound in this context is:

$$\mathcal{B}\left(\frac{q}{2}, \frac{q}{2}+1\right) \leq 1.$$

And thus

$$I(1,p,q) = 2^{\frac{1}{p'}} \left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \geq 2^{\frac{1}{p'}} \left(\frac{2^{1-q}}{q\pi}\right)^{\frac{1}{q}}$$

Here is the trend of $\beta_{p,q}$ with respect to p when $d = 3$.

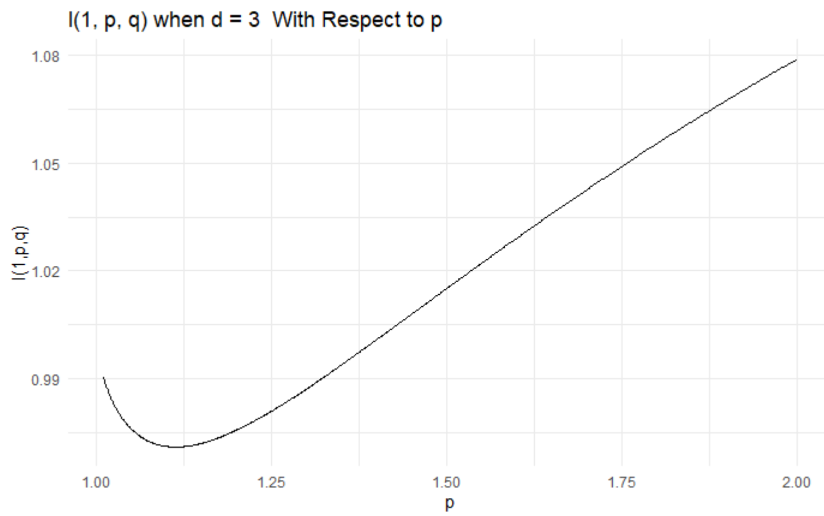

Figure 5 $\beta_{p,q}$ first decreases from $p = 1$ to $p = 1.1$, and monotonous increases from $p = 1.1$ to $p = 2$

Note that due to the **Legendre duplication formula**, we have:

$$\Gamma\left(j + \frac{1}{2}\right) = \frac{2^{1-2j}\Gamma(2j)}{\sqrt{\pi}\Gamma(j)}$$

Thus

$$\begin{aligned} I(1,p,q) &= 2^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})} \Gamma\left(\frac{q+1}{2}\right) \right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})} \frac{2^{1-q}\Gamma(q)}{\sqrt{\pi}\Gamma(q/2)} \right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{p'}} \left(\frac{2^{1-q}}{\pi} \frac{\Gamma(q)}{\Gamma(\frac{q+2}{2})\Gamma(q/2)} \right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{p'}} \left(\frac{2^{1-q}}{q\pi} \frac{\Gamma(q+1)}{\Gamma(\frac{q+2}{2})\Gamma(q/2)} \right)^{\frac{1}{q}} \end{aligned}$$

Where the last equality is due to $\Gamma(q+1) = q\Gamma(q)$.

With a similar process, we can work out $I(t,p,q)$ when t and q are special. One case we have looked at is when $q \in \mathbb{N}^+$. This is motivated by numerical observations for such choices of q.

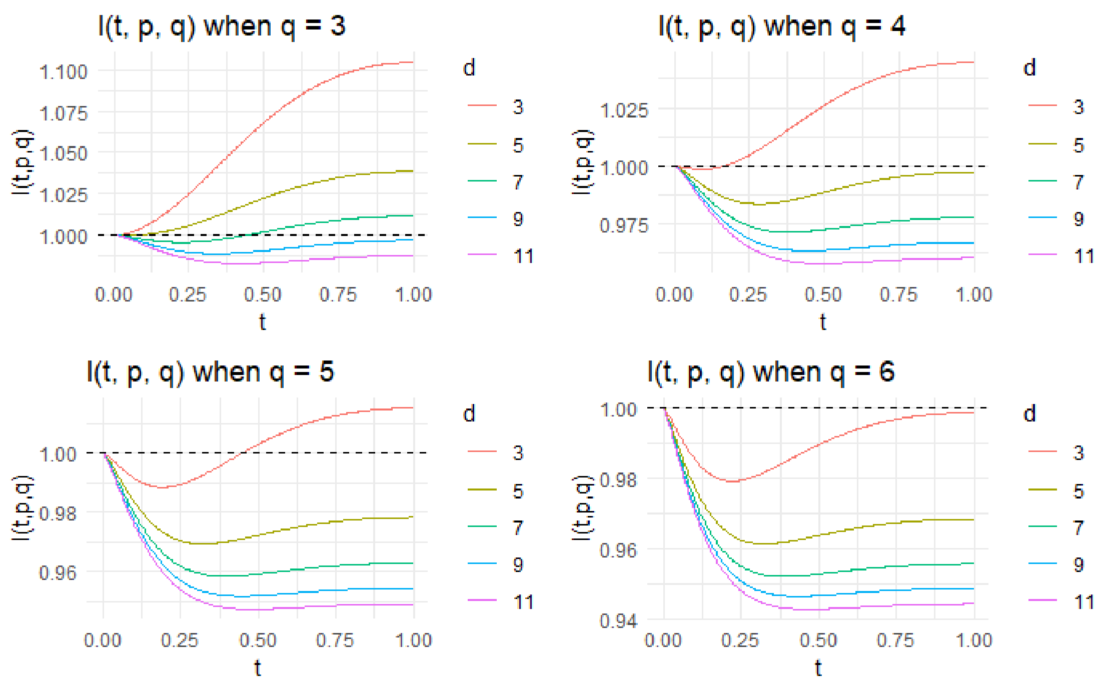

Figure 6 For values of $q > 2$ and d, $I(t,p,q) < 1$ for $t \in (0,1)$. We hypothesized that this is always true for large q but have not succeeded in proving it.

We can see that the trend of functions characterized by q is similar from previous discussions relevant to p. As q increases, $I(t,p,q)$ decreases (when we compare curves of the same color across different figures). Recall that we characterize the lower bound of $I(1,p,q) \geq 2^{\frac{1}{p'}} \left(\frac{2^{1-q}}{q\pi}\right)^{\frac{1}{q}}$. While this is a lower bound, it also served as a close approximation of $I(1,p,q)$.

We can take it one step further by making q even natural numbers, or $q = 2k, k \in \mathbb{N}^+, k \geq 2$. While this case has been often considered by previous literature due to simplicity of computation, our interest is due to the application of the binomial expansion, namely

$I(t,p,q) = \dfrac{1}{(1+t^p)^{1/p}} \left(\int_0^{2\pi} \left(t^2 + 2t\cos(\theta) + 1\right)^k \, d\dfrac{\theta}{2\pi} \right)^{\frac{1}{2k}}$, where the numerator can be written as

$$\begin{aligned}
&\int_0^{2\pi} \left(t^2 + 2t\cos(\theta) + 1\right)^k \, d\frac{\theta}{2\pi} \\
&= \int_0^{2\pi} \left[t^2 - 2t + 1 + 2t + 2t\cos(\theta)\right]^k \, d\frac{\theta}{2\pi} \\
&= \int_0^{2\pi} \left[(t-1)^2 + 2t(1+\cos(\theta))\right]^k \, d\frac{\theta}{2\pi} \\
&= \int_0^{2\pi} \left\{ \sum_{j=0}^{k} \binom{k}{j} (1-t)^{2k-2j} [2t(1+\cos(\theta)]^j \right\} \, d\frac{\theta}{2\pi} \\
&= \sum_{j=0}^{k} \binom{k}{j} \left\{ (1-t)^{2k-2j} (t)^j \int_0^{2\pi} \left\{ [(2 + 2\cos(\theta)]^j \right\} \, d\frac{\theta}{2\pi} \right\}
\end{aligned}$$

We take the step of

$$\left(t^2 + 2t\cos(\theta) + 1\right)^k = [(t-1)^2 + 2t(1+\cos(\theta))]^k$$

because it enables us to integrate $(1+\cos(\theta))^j$ after binomial expansion, which we can do thanks to the trigonometric form of Beta Function.

Note that when $j = 0$, the amount

$$\left\{ (1-t)^{2k-2j} (t)^j \int_0^{2\pi} \left\{ [(2 + 2\cos(\theta)]^j \right\} \, d\frac{\theta}{2\pi} \right\}$$

is $(1-t)^{2k-2j}$. We define this expansion on $t \in (0,1)$ and avoid the flaw caused by 0^0.

Here is a table of special values of $I(t,p,q)$ when $q = 2*2$, where the (p,q) pair is defined by $d = 3$.

0.8 Effort for Differentiation and Estimation of $I(t,p,q)$

We originally wanted to find the value of t such that the function $I(t,p,q)$ is maximized. A theoretical approach is to differentiate, which is the purpose of polynomial expansion.

t	Direct Integration	Binomial Expansion
0.1	0.95601455	0.95601455
0.2	0.92149780	0.92149780
0.3	0.89919695	0.89919695
0.4	0.88567976	0.88567976
0.5	0.87774286	0.87774286

Table 2 Comparison of Integration Methods

Recall that

$$\frac{1}{2\pi}\int_0^{2\pi} (2+2\cos(\theta))^{q/2}\ d\theta = \frac{2^q}{\pi}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}$$

,

Because $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, this can be further simplified to

$$I(t,p,q) = 2^q \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{q+2}{2})}$$

We can further simplify the expression of

$$I(t,p,q) = \frac{1}{(1+t^p)^{1/p}}\left(\sum_{j=0}^{k}\binom{k}{j}\left\{(1-t)^{2k-2j}(t)^j\int_0^{2\pi}\left\{[(2+2\cos(\theta)]^j\right\}\ d\frac{\theta}{2\pi}\right\}\right)^{\frac{1}{2k}}$$

when we consider $q = 2k$.

We attempted to separate the function by a product, denoting

$$f(t) = \frac{1}{(1+t^p)^{1/p}}$$

and $g(t)$ being the expanded polynomial for the case $q = 2k$ and $k \in N^+, k \geq 2$. We impose this restraint because previous studies indicated that it could lead to interesting results. To compute $\frac{d}{dt}I(t,p,q)$, we apply the product rule, which yields

$$\frac{d}{dt}I(t,p,q) = f'(t)g(t) + f(t)g'(t)$$

. The derivative of $f(t)$,

$$f'(t) = -\frac{pt^{p-1}}{(1+t^p)^{\frac{1}{p}+1}}$$

, involves standard calculus. However, the derivative of $g(t)$, $g'(t)$, necessitates differentiating under the integral sign and then taking the $2k$-th root's derivative.

A second point of interest lies in the initial observation of the funtion. Numerical evidence seems to suggest that the monotonicity of the funtion near 0 is indicative of its overall behavior:while we hypothesized that if the function is monotonous increasing at $t = 0$, it will be monotonous increasing on $t \in (0, 1)$ as well. To approach this, we attempted using Taylor Series and Generalized Binomial Expansion Theorem, only to be inhibited again by complications of differentiation.

We are interested in studying the Taylor Polynomials of $\alpha_{p,q}$ near $t = 0$. Recall that

$$I(t, p, q)^q = \frac{1}{(1 + t^p)^{q/p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left(t^2 + 2t\cos(\theta) + 1\right)^{q/2} \, d\theta \right)$$

For the denominator function $g(t) = (1 + t^p)^{q/p}$,

$$\begin{aligned}
g(0) &= (1 + 0^p)^{1/p} = 1 \\
g'(t) &= \frac{d}{dt}(1 + t^p)^{1/p} = t^{p-1}(1 + t^p)^{1/p-1} \\
g'(0) &= 0^{p-1}(1 + 0^p)^{1/p-1} = 0 \\
g''(t) &= \frac{d}{dt}\left[t^{p-1}(1 + t^p)^{1/p-1}\right] = (p-1)t^{p-2}(1 + t^p)^{1/p-1} + t^{2p-2}(1 + t^p)^{1/p-2} \\
g''(0) &= NA(\text{due to } \lim_{t \to 0}(p-1)t^{p-2} \text{ does not converge}).
\end{aligned}$$

For the numerator this is more complicated. Using the Leibniz rule, we have:

$$f(0) = 1$$
$$f'(t) = \frac{d}{dt}\frac{1}{2\pi}\int_0^{2\pi}\left(t^2 + 2t\cos(\theta) + 1\right)^{q/2} d\theta$$
$$= \frac{q}{2\pi}\int_0^{2\pi}\left(t^2 + 2t\cos(\theta) + 1\right)^{q/2-1} \cdot (t + \cos(\theta))d\theta$$
$$f'(0) = \frac{q}{2\pi}\int_0^{2\pi}\cos(\theta)d\theta = 0$$

$f''(t)$ is too complicated. Reducing the terms vanishing at $t = 0$ yields

$$f''(0) = \frac{q}{2\pi}\int_0^{2\pi}\left[1 + (q/2-1)\cdot 2\cos^2(\theta)\right] d\theta = q + \frac{q/2-1}{2\pi}\int_0^{2\pi} 2\cos^2(\theta)\, d\theta = \frac{3q-2}{2}$$

The last equality uses the fact that $2\cos^2(\theta) = 1 + \cos(2\theta)$

The generalized binomial theorem states for any real number α and $|x| < 1$, the expansion is:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$$

Here, $\alpha = -q/p$ and $x = t^p$. Let's write out the first three terms of the expansion:

0. Zeroth Term: - The constant term is always 1.

1. First Term: - The coefficient of the first term x (which in our case is t^p) is α, so the first term is $-\frac{q}{p}t^p$.

2. Second Term: - The coefficient of the second term x^2 (which in our case is $(t^p)^2 = t^{2p}$) is $\frac{\alpha(\alpha-1)}{2!}$. Substituting $\alpha = -q/p$ gives us the second term as $\frac{(-q/p)(-q/p-1)}{2}t^{2p}$.

3. Third Term:

$$\frac{(-q/p)(-q/p-1)(-q/p-2)}{3!}t^{3p}$$

This term arises from the $\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3$ part of the expansion, representing the third term involving t.

Combining these, the first three terms of the expansion of $g(t) = (1 + t^p)^{-q/p}$ are:

$$g(t) \approx 1 - \frac{q}{p}t^p + \frac{(-q/p)(-q/p-1)}{2}t^{2p} + \frac{(-q/p)(-q/p-1)(-q/p-2)}{3!}t^{3p}$$

We study the sign of $f'g - fg'$, which yields:

$$\frac{1}{(p^3 * t^3)}(p * q * (t^2 * (3q-2) + 4) * (2 * p^2 * t^{p+2} - 2 * p * t^{2p+2} * (p+q) + t^{3p+2} * (p+q) * (2 * p + q))/8$$
$$+t^4 * (3q-2) * (6 * p^3 - 6 * p^2 * q * t^p + 3 * p * q * t^{2p} * (p+q) - q * t^{3p} * (p+q) * (2 * p + q))/12)$$

But the complications prevent us from obtaining any meaningful results.

Future Work

The studies on $\alpha_{p,q}$, with its functional form $I(t, p, q)$ has enabled us to understand its derivation, and we can use it to derive some special values of t. However, trend of the function part of $\alpha_{p,q}$ still requires differentiation, which is prohibited by the complicated form of the function. Our discussions have reaffirmed the intricate nature of the restriction inequalities and their dependency on the dimensional settings and the values of p and q. The theoretical and numerical explorations have suggested that the conjectures on the monotonicity for $I(p, q)$ holds under certain conditions, but not for every t. While the hypothesized value for the cutoff of t where the maximum is attained at 1 instead of 0 is when p is significantly larger than when it would exhibit anomalies, there is no proof of monotonocity. This study really highlights the strength and drawback of numerical studies: while it provides us with immediate visualizations enabling interesting hypotheses, we lack concrete methods to prove them.

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