Winter 12-12-2023

# First Order Approximation on the Basilica Julia Set

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December 15, 2023

**Abstract**

This project combines fractal geometry and analysis. We consider the basilica Julia set of the quadratic polynomial $P(z) = z^2 - 1$, with its successive graph approximations defined in terms of the external ray parametrization of the set. Following the model of Kigami and later Strichartz, we exploit these graph approximations to define derivatives of functions defined on the fractal, an endeavor complicated by asymmetric neighborhood behaviors at approximated vertex points across levels, and by the topology of these vertices. We hence differentiate even and odd levels of approximations of the Julia set and construct, accordingly, normal derivatives corresponding to each level category at the vertices, given their assigned ray names. We also discuss how a localized harmonic function serves as the tangent line, from which local linear approximation near vertices are obtained.

## 1 Introduction

Theory of analysis on fractals has been rapidly and extensively developed, especially for connected and finitely ramified fractal (may be disconnected by removing a finite number of points) sets. For example, with the pre-existing theory on this type of fractals analogous to the theory of analysis on manifolds, [1] gives a comprehensive account on the construction of a family of derivatives and Taylor approximations, following the methods of Kigami (see [2]). [3] extended the crux of Kigami's approach of building a Laplacian from energy and measure to the Julia set of polynomial $P(z) = z^2 - 1$, usually referred to as the basilica Julia set. The approach was also extended to an infinite family of quadratic Julia sets in [4]. We work only with the basilica Julia set for this project, constructing normal derivatives at vertices in graphs that approximate the basilica Julia set, and obtaining local first-order approximation at these points. As a specific type of the Julia set family $\mathcal{J}_c$ for $P(z) = z^2 + c$ with $c$ in the Mandelbrot set, the basilica Julia set is a nonlinear fractal both connected and finitely ramified and hence conforms nicely to the Kigami paradigm, which [4] gave a detailed account that we summarize here:

First, approximate the fractal by a sequence of graphs $\Gamma_m$, or a filtration, with vertices $V_m$ and edge relation $x \underset{m}{\sim} y$. In our study, there are self-loop at a vertex and multiple edges between two vertices. A nested sequence of vertex sets is required,

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \tag{1.1}$$

the union of which,

$$V_* = \bigcup_{m=0}^{\infty} V_m, \tag{1.2}$$

is a set dense in the fractal.

Then, on each graph $\Gamma_m$ construct the energy

$$\mathcal{E}_m(u, v) = \sum_{x \underset{m}{\sim} y} c_m(x, y)(u(x) - u(y))(v(x) - v(y)), \tag{1.3}$$

where $c_m(x, y)$ are positive. For a function $u$ defined on the vertex set $V_m$, its harmonic extension $\tilde{u}$ to $V_{m+1}$ is defined to be the extension that minimizes energy, such that:

$$\mathcal{E}_{m+1}(\tilde{u}, \tilde{u}) \le \mathcal{E}_{m+1}(u, u), \tag{1.4}$$

where the restriction of $\tilde{u}$ to $V_m$ is $u$: $\tilde{u}|_m = u$. The paradigm also prescribes the identity

$$\mathcal{E}_{m+1}(\tilde{u}, \tilde{u}) = \mathcal{E}_m(u, u). \tag{1.5}$$

Then, any function $u$ defined on $V_*$ will have increasing energy:

$$\mathcal{E}_0(u, u) \leq \mathcal{E}_1(u, u) \leq \cdots , \tag{1.6}$$

hence the definition

$$\mathcal{E}(u, u) = \lim_{m\to\infty} \mathcal{E}_m(u, u) \tag{1.7}$$

is reasonable. The domain $dom\, \mathcal{E}$ is defined to be functions with $\mathcal{E}(u, u) < \infty$, then

$$\mathcal{E}(u, v) = \lim_{m\to\infty} \mathcal{E}_m(u, v) < \infty \tag{1.8}$$

for $u, v \in dom\, \mathcal{E}$.

Now choose a measure $\mu$ on the fractal and define a Laplacian $\Delta_\mu$ by the weak formulation

$$-\mathcal{E}(u, v) = \int (\Delta_\mu u) v d\mu \quad \text{for } v \in dom\, \mathcal{E}. \tag{1.9}$$

The pointwise formula for $\Delta_\mu u$ on $V_*$ is the limit of graph Laplacians

$$\Delta_\mu u(x) = \lim_{m\to\infty} \frac{1}{\int \psi_x^{(m)} d\mu} \Delta_m u(x), \tag{1.10}$$

where the graph Laplacian at level $m$ at a vertex $x$ is defined as

$$\Delta_m u(x) = \sum_{x \underset{m}{\sim} y} c_m(x, y)(u(y) - u(x)), \tag{1.11}$$

and $\psi_x^{(m)}$ is the harmonic extension of the function $y \to \delta_{xy}$ on $V_m$. There are several choices for the measure to construct a Laplacian. In our study, we adopt the simplest measure introduced in [4], the equilibrium measure. We give more descriptions of this measure later after introducing the method we parametrize the Julia set.

The last thing is that we also want the energy $\mathcal{E}$ to be $P$-invariant in the context of finitely ramified self-similar fractals:

$$\mathcal{E}(u \circ P, v \circ P) = c\mathcal{E}(u, v), \tag{1.12}$$

or satisfying a weaker requirement

$$\mathcal{E}(u \circ P^{(k)}, v \circ P^{(k)}) = c_k \mathcal{E}(u, v). \tag{1.13}$$

The parameterization of $\mathcal{J}$ utilized in this project directly induces the sequence of discrete graph approximations $\Gamma_m$ for $\mathcal{J}$ we study. We use the method of external ray developed by Doaudy and Hubbard to parameterize the Julia set, which is, in short, a continuous mapping $\phi$ from the unit circle onto the Julia set. [4] also uses the method for their study and parametrizes the unit circle $\mathbb{S}$ by $x = \frac{\theta}{2\pi}$ in $[0, 1]$ for simplicity, which we adopt as well for this project. For the basilica Julia set $\mathcal{J}$ we study, the mapping $\phi : \mathbb{S} \to \mathcal{J}$ is continuous but two-to-one onto $\mathcal{J}$. Two points on the parametrized unit circle are mapped to one point of $\mathcal{J}$; in another wording, one point of $\mathcal{J}$ will have two external ray names or a corresponding identified external ray pair. Hence $\mathcal{J}$ is realized by an infinite set of identifications of points on the parametrized unit circle. Section 2 explicitly describes these identifications, providing illustrations for them at various graph approximation levels. The section also details how the dynamics of $P = z^2 - 1$ can be clearly tracked from the external ray parameterization, as $\phi$ assigns ray names to points of $\mathcal{J}$ with respect to the action of $P$ on $\mathcal{J}$. The external ray parametrization also naturally inspires the sequence of graph approximations $\Gamma_m$ on $\mathcal{J}$ used in [4], acquired by identifying each external ray pair in $\Gamma'_m$ into the single point in $\mathcal{J}$ which is the image of the pair under $\phi$. Regarding how to obtain vertices of each graph approximation, given any vertex set $V_m$ for the graph approximation of $\mathcal{J}$ at level $m$, $\Gamma_m$, vertices $V_{m+1}$ are defined to be the pre-image of $V_m$ under $P$.

We study definition of the P-invariant energy $\mathcal{E}$ in [4] and summarize its work in section 3. The discrete graph energies $E_m$ on graph approximations $\Gamma_m$ are constructed to be the same as energies $E'_m$ on $\Gamma'_m$, graph approximations parametrized by the method of external ray. However, while being $P$-invariant, $E_m$ does not satisfy the equality in (1.5): $\mathcal{E}_{m+1}(\tilde{u},\tilde{u}) = \mathcal{E}_m(u,u)$, where $\tilde{u}$ is the harmonic extension of $u$ to $V_{m+1}$. We only have $E_{m+k}(\tilde{u},\tilde{u})$ as a constant multiple, $2^{k-1}$, of $E_m(u,u)$, the value of $k$ depending on the Julia set and $k = 2$ for the basilica Julia set we work with. Hence the definition of energies $\mathcal{E}_m$ as a renormalization of $E_m$ is introduced, where

$$\mathcal{E}_{m+k}(\tilde{u},\tilde{u}) = \mathcal{E}_m(u,u) \tag{1.14}$$

is satisfied. Still, $k = 2$ for the basilica Julia set. It turns out that (1.5) does not hold for $k \geq 1$ and (1.14) is all we can have. Therefore, $k$ different energies,

$$\mathcal{E}^j(u,v) = \lim_{m\to\infty} \mathcal{E}_{j+km}(u,v),\ j = 0,1,\cdots,k-1$$

are obtained. [4] defined $\mathcal{E}$ by taking the average of these energies

$$\mathcal{E}(u,v) = \frac{1}{k}\sum_{j=0}^{k-1} \mathcal{E}^{(j)}(u,v)$$

and $\mathcal{E}$ is obtained as a $P$-invariant energy. Section 3 delineate the process of defining $\mathcal{E}$ with more details. We also discuss how to compute the harmonic extension of any function, based on which we claim that the only global harmonic functions on $\mathcal{J}$ are the constant. More importantly, we present a key feature of the harmonic extension which not only is crucial for a more thorough understanding of the implication of harmonically extending a function, but the feature also appears to contribute to the proof of zero graph Laplacian for a local harmonic function, the definition of which will be presented in section 6.

In section 4 we discuss the concept of a neighborhood, or a cell, and behaviors of neighbors for any vertex in $V_*$. With the parametrization method of external ray, any vertex in $V_*$ is born with either a single or two neighbors and will eventually gain four neighbors three graph approximation levels later than the level at which it is born. The process for a vertex to gain its first four neighbors is also worded as the formation of the first four-neighbor cell of the vertex, which we delineate in the section. A vertex in $V_*$ won't gain more than four neighbors in its cell due to the external ray parametrization. The number of neighbors for any vertex remains three approximation levels after the vertex's birth, while younger neighbors continue to substitute the older ones. We call this substitution as neighbors approaching closer to a vertex. It turns out that not all four neighbors would approach closer to a vertex whenever going up by one graph approximation level, from $\Gamma_m$ to $\Gamma_{m+1}$. Instead, neighbors approach alternately, and only going up by two consecutive levels would all neighbors of a vertex approach closer. We describe this situation thoroughly and explains its cause from the parametrization method of external ray. As a consequence of this asymmetric approaching behaviors of neighbors, we also define two categories of graph approximation levels, even and odd levels, given that two of the neighbors of any vertex always approach closer at even levels, the other two at odd levels.

Building on the comprehension of neighborhoods and approaching behaviors from the previous section, also motivated by the differentiation between even and odd levels of approximation, in section 5 we introduce the definitions of even and odd normal derivatives, $\partial^E$ and $\partial^O$, at any point in $V_*$ for a continuous function of $\mathcal{J}$. Given that any vertex in $V_*$ will continue to have four neighbors approaching from distinct directions, and that values of any function on these neighbors can be completely arbitrary, we obtain four derivatives at any vertex to represent the slope at each direction, for each category of approximation levels. We prove in this section the identity $dom\partial^E = dom\partial^O$, which means that the existence of either even or odd derivatives of a function at any $x \in V_*$ would imply the existence of the other. We also conjecture on the reason why our endeavor to prove a potential compatibility condition of the derivatives eventually failed.

In section 6, we exploit local harmonic functions restricted on an $m$-level cell of any vertex $x \in V_*$ and define them as the $m$-level tangent lines at $x$. By restricting on an $m$-level cell of $x$, in short, we mean that the local harmonic functions are defined on vertices in $V_* \setminus V_m$ that fall between four neighbors of $x$ at $\Gamma_m$ and $x$, together with these neighbors. Given that there are four derivatives at any $x \in V_*$, we define accordingly four local tangent lines at $x$ with respect to each derivative. We demonstrate how to perform a local linear approximation of any function $u$ restricted on an $m$-cell of $x$, based on the value $u(x)$ and the four derivatives at $x$ of the function $u$. We also show that the graph Laplacian $\Delta_m u(x)$ is zero for an $m$-level local tangent anywhere within its domain, except the four neighbors of $x$ in graph $\Gamma_m$, by exploiting the core feature of harmonic extensions of functions discussed before.

# 2 External Ray Parameterization and Induced Graph Sequence of $\mathcal{J}$

Studying functions defined on such an intricate setting as the basilica Julia set will be hard. Hence we would first like to develop, as said in the introduction, a sequence of graph approximations whose vertex sets converge to a set dense in $\mathcal{J}$, and study functions defined on each distinct graph approximation. Our graph approximations are naturally induced from the external ray parametrization of $\mathcal{J}$ utilized for this study. This section details the method of this parametrization and how we obtain the sequence of graph approximations, $\Gamma_m$, from the corresponding parametrizations $\Gamma'_m$.

The graph approximation of $\mathcal{J}$ starts from the attracting fixed point $z_0 = \frac{1-\sqrt{5}}{2}$, which lies in the basilica $\mathcal{J}$, of the quadratic polynomial $P = z^2 - 1$. Therefore, we begin with the vertex containing only the fixed point and let $V_0 = \{z_0\}$. The main point this paragraph aims to describe is the assignment of external ray names, or the points on the parametrized unit circle, to this fixed point $z_0 \in V_0$. Recall from the introduction that $\phi : \mathbb{S} \to \mathcal{J}$ is a continuous, but two-to-one mapping from the unit circle onto $\mathcal{J}$, because of which we define $\phi^{-1}(z_0) = \{\theta : \phi(\theta) = z_0\}$. Given that $P(z_0) = z_0$, we can write $\phi^{-1}(P(z_0)) = \phi^{-1}(z_0)$. Let $x_0 = \phi^{-1}(z_0)$. Namely, let $x_0$ denotes any point on the unit circle that is mapped to our fixed point $z_0$. As mentioned in the introduction, we parametrize the circle by $x = \frac{\theta}{2\pi}$ in $[0, 1]$, so $x \in [0, 1]$. It follows that $2x_0 \mod 1 = x_0$, if considering the polynomial $P$ operating on $x_0$ in terms of polar multiplication. Whence we take the vertex set of parametrized first level approximation to be points on the unit circle mapped to $z_0$, $V'_0 = \{x : x = \phi^{-1}(z_0)\}$, and assign them with external ray names within $[0, 1]$ satisfying $2x \mod 1 = x$. Namely, $V'_0 = \{\frac{1}{3}, \frac{2}{3}\}$. Given that $V_0$ contains only a single point, the two points in $V'_0$ are identified and adjoined by an edge in the induced graph $\Gamma'_0$. These two points divide the unit circle into 2 intervals, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, \frac{4}{3}(\text{mod } 1)]$, of length $\frac{1}{3}$ and $\frac{2}{3}$.

We then define $V_1 = \{z_1 : P(z_1) = z_0\}, z_0 \in V_0$. In other words, we consider the whole pre-image of $z_0$ to be contained in $V_1$, including $z_0$ itself given it is a fixed point of $P$. The definition of $V'_1$ follows as $V'_1 = \{x_1 : 2x_1(\text{mod } 1) = x_0\}, x_0 \in V'_0$, under the same intuition of applying $P$ on $V'_1$ in a polar sense. Points in $V'_1$, therefore, are either of the form $\frac{1}{2}x_0$ or $\frac{1}{2}x_0 + \frac{1}{2}$, $x_0$ varying over $V'_0$; explicitly, $V'_1 = \{\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}\}$, where $\frac{1}{3}, \frac{2}{3}$ and $\frac{1}{6}, \frac{5}{6}$ are identified pairs of $z_0$ and its pre-image other than itself. In other words, $\phi(\{\frac{1}{3}, \frac{2}{3}\}) = z_0$, $\phi(\{\frac{1}{6}, \frac{5}{6}\}) = V_1 \setminus V_0$. Note that the newly added points in $V'_1$, $\frac{1}{6}, \frac{5}{6}$, all lie in the long interval from $\Gamma'_0$, $[\frac{2}{3}, \frac{1}{3}]$, and that the four points in $V'_1$ together divide the circle into two cycles of intervals of length $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ and $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, half length of the other. Also, intervals with different lengths are laid alternately on the unit circle: A long interval with length $\frac{1}{3}$ has to be only adjacent to short ones with half the length, and vice versa. Given all these observations, we can describe the passage from vertex set $V'_0$ to $V'_1$ as subdividing the long interval $[\frac{2}{3}, \frac{1}{3}]$ in $\Gamma'_0$ of length $\frac{2}{3}$ into three sub-intervals, by laying two new identified points in $V'_1 \setminus V'_0$ within $[\frac{2}{3}, \frac{1}{3}]$. The longer sub-interval $[\frac{5}{6}, \frac{1}{6}]$ is between the identified pair and has length $\frac{1}{3}$, exactly the length of the short interval in $\Gamma'_0$, the parametrized graph approximation from the previous level. This sub-interval, together with the short interval $[\frac{1}{3}, \frac{2}{3}]$ from $\Gamma'_0$, becomes a long interval for $\Gamma'_1$, the external ray parametrization for approximated $\mathcal{J}$ at the current approximation level. The other two shorter sub-intervals, $[\frac{1}{6}, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{5}{6}]$, have the same length $\frac{1}{2} \cdot \frac{1}{3}$ and are considered new short intervals for $\Gamma'_1$. Note that the interval between any point in $V'_1 \setminus V'_0$ and a point in $V'_0$ is short. As shown in Figure 2.1, $\Gamma'_1$ has two pairs of identified points and two edges adjoining each pair. It follows that the induced graph approximation $\Gamma_1$ will have two vertices, two edges adjoining its vertices, and two self-edges at each point. Demonstration of $\Gamma_1$ will be later shown in Figure 2.3.

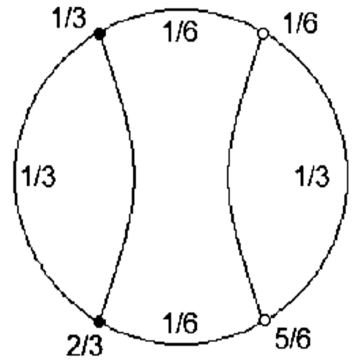



Figure 2.1. $\Gamma'_1$: The external ray parametrization for $\mathcal{J}$ at graph approximation level $m = 1$. Vertices in $V'_0$ are in solid dots and in $V'_1 \setminus V'_0$ are open dots. The length of intervals are marked.

The definition of the vertex set $V_m$ is similar as that of $V_1$, where we consider to be the whole pre-image of $V_{m-1}$: $V_m = \{z_m : P(z_m) = z_{m-1}\}$, $z_{m-1} \in V_{m-1}$. Given that $P = z^2 - 1$ is a quadratic polynomial, the pre-image of $P^{-1}(z)$, $z \in V_*$ always consists of two new distinct points, except the fixed points $z_0$ where $P^{-1}(z_0)$ also contains itself. We first explain the cardinality of $V_m$ and $V'_m$. Let $x \in V_m \setminus V_{m-1}$ be any vertex, and the two ray names of $x$ in $V'_m \setminus V'_{m-1}$. Consider the passage from $V_m$ to $V_{m+1}$ regarding only $x$: The pre-image $P^{-1}(x) \in V_{m+1} \setminus V_m$ will contain two points distinct from $x$, implying that four new points from the parametrized unit circle will be newly added to $V'_{m+1}$ since each vertex in the pre-image of $x$, $P^{-1}(x) \in V_{m+1}$, will be assigned two external ray names in $[0, 1]$. In other words, whenever a point $x$ was born at level $m$ which means that one new point is added to $V_m$ and two to $V'_m$, two other points in $P^{-1}(x)$ will be added to $V_{m+1}$, and four in $V'_{m+1}$. Therefore, the cardinality of $V'_{m+1} \setminus V'_m$ is twice that of $V'_m \setminus V'_{m-1}$. For example, Figure 2.2 shows the transition from $V'_2$ to $V'_3$. There are four open dots or two identified pairs in $\Gamma'_2$, eight open dots in $\Gamma'_3$. It can be naturally deduced that $|V'_m \setminus V_{m-1}| = 2^m$ and that $|V'_m| = 2^{m+1}$. Points in $V'_m$ are identified into $2^m$ vertices in $V_m$.

Now we explain the passage from $V'_m$ to $V'_{m+1}$. The $2^m$ points in $V'_m$ subdivide the unit circle into $2^m$ cycles of intervals, each cycle consisting of two intervals with length $\frac{1}{2^m}\frac{1}{3}$ and $\frac{1}{2^m}\frac{2}{3}$, and intervals are laid with alternate lengths on the circle. Then, within each long interval of length $\frac{1}{2^m}\frac{2}{3}$ in $\Gamma'_m$ we insert two new points that will be identified in $V_{m+1}$, subdividing the long interval into three sub-intervals. The longer sub-interval between the newly added identified pair has length $\frac{1}{2^{m+1}}\frac{2}{3}$, the same as that of any short interval in $\Gamma'_m$. The other two shorter sub-intervals both have length $\frac{1}{2^{m+1}}\frac{1}{3}$, half the length of interval between the newly laid identified pair. In summary, short intervals from $\Gamma'_m$ remain undivided, becoming long ones in $\Gamma'_{m+1}$ and having the same length as the longer sub-intervals. Shorter sub-intervals are considered the short intervals for $\Gamma'_{m+1}$. Similarly as the situation for $\Gamma'_1$, notice that the interval between any points in $V'_{m+1} \setminus V'_m$ and its neighbor not in the same identified pair is short in $\Gamma'_{m+1}$. For instance, take the identified pair $\frac{5}{12}, \frac{7}{12} \in V'_2 \setminus V'_1$. The interval between the pair themselves is long in $\Gamma'_2$ and has length $\frac{1}{6}$, but the intervals between them and their another neighbors not in the pair are short: $[\frac{1}{3}, \frac{5}{12}]$ and $[\frac{7}{12}, \frac{2}{3}]$. Lastly, external ray names for newly added points can be derived by simple additions regarding names of their neighbors from the previous level and the length of intervals.

We would like to mention one more observation on this general passage. As said in the last paragraph, the interval between any two newly added identified points in $V'_{m+1} \setminus V'_m$ will always have length $\frac{1}{2^{m+1}}\frac{2}{3}$, which is considered a long interval in $\Gamma'_{m+1}$. Therefore, within the interval between each identified pair newly born in $\Gamma'_{m+1}$, another identified pair, the interval between which will have length $\frac{1}{2^{m+2}}\frac{2}{3}$, will be immediately inserted during the next transition to $V'_{m+2}$. In other words, within the interval between any identified pair newly born in the current level of parametrization, a new identified pair will be immediately laid in the next level. Figure 2.2 also demonstrates this fact: While $\frac{5}{12}, \frac{7}{12} \in V'_2 \setminus V'_1$, the interval $[\frac{5}{12}, \frac{7}{12}]$ is sub-divided into three intervals $[\frac{5}{12}, \frac{11}{24}], [\frac{11}{24}, \frac{13}{24}]$, and $[\frac{13}{24}, \frac{7}{12}]$ in $\Gamma'_3$. The only special case here is the fixed point $z_0$ of $P$. Having two intervals between its identified pair in $\Gamma'_0$, one identified pair with ray names $\frac{1}{6}, \frac{5}{6}$ is inserted in $\Gamma'_1$ in the interval $[\frac{2}{3}, \frac{1}{3}]$, the other one named $\frac{5}{12}, \frac{7}{12}$ being inserted in $\Gamma'_2$ in the interval $[\frac{1}{3}, \frac{2}{3}]$.

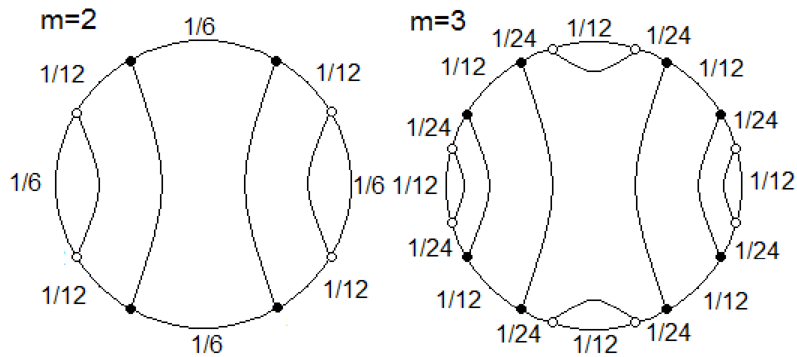



Figure 2.2. The external ray parametrization for the basilica, when $m = 2$ and 3. Vertices in $V'_{m-1}$ are in solid dots and in $V'_m \setminus V'_{m-1}$ are open dots. The length of intervals are marked.

This last paragraph of the section explains, in general, the two-to-one mapping $\phi$ from the basilica Julia set to the unit circle, with regard to points in $V_m$. As previously said, the single point $z_0$ in $V_0$, being the fixed point of $P$ that is a junction point, is mapped to the identified pair $\{\frac{1}{3}, \frac{2}{3}\}$ on the circle. Points in $V_1$ are the two pre-images of $z_0$ including itself. Inductively, points in $V_{m+1}$ are the pre-images of points in $V_m$ under $P$, where it follows that $V'_m = \{x_m : 2x_m (\text{mod } 1) = x_{m-1}\}$, $x_{m-1}$ varying throughout $V'_m$. We then identify each pair of points newly laid in each long interval. It's worth noticing that the external ray names of points in $V'_m$ with regard to the unit circle respect the actual topology of corresponding points in $V_m$, which makes the correspondence between an identified ray pair and their relative single vertex in $V_m$ clearly identified. We've also discussed how intervals in $\Gamma'_1$ are mapped to edges in $\Gamma_1$: Two short intervals with length $\frac{1}{6}$ are mapped to edges connecting vertices in $V_1$, and the two long intervals of length $\frac{1}{3}$, between the identified pairs, are each mapped to one self-edge at each point in $V_1$. These self-edges are illustrated in Figure 2.3 as loops joining the central circle at either end. Similarly, all intervals between identified ray pairs are mapped to loops at relative vertices, while those between distinct points being mapped to different portion of the central circle or pre-existing loops.

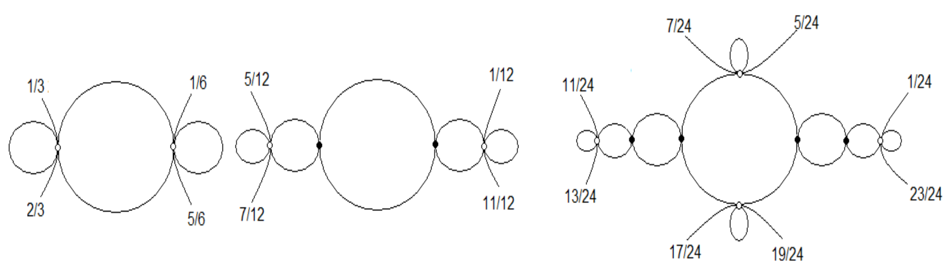



Figure 2.3. The graph approximation of the basilica Julia set when $m = 1, 2$, and 3.

# 3 Energy, Harmonic Extensions, and Harmonic Functions

Energy, as one of the fundamental notions of this paper, provides useful insights into harmonic functions and their first derivatives, which later will be shown to shed light on our definition of normal derivatives at $V_*$ of continuous functions of $\mathcal{J}$ in section 5, one of the two end goals of this project. Definitions of discrete graph energy $E_m$ and the energy $\mathcal{E}$ in accordance with the Kigami paradigm which takes functions on$\mathcal{J}$ used in this paper are adopted from [4], the results and work of which this section aims for summarizing.

The weak formulation (1.9) defining the Laplacian with a chosen measure $\Delta_\mu$ immediately necessitates a definition for the energy $\mathcal{E}$, which takes functions whose domain is on $\mathcal{J}$. According to the Kigami paradigm, $\mathcal{E}$ should be $P$-invariant and also realize the identity (1.5). The first step is to define discrete energies $E_m$ on graph approximations $\Gamma_m$, by considering the energy $E'_m$ on the parametrized graphs $\Gamma'_m$ defined by

$$E'_m(u,v) = \sum_{x' \underset{m}{\sim} y'} \frac{1}{|x'-y'|}\big(u(x')-u(y')\big)\big(v(x')-v(y')\big) \qquad (x',y' \in V'_m), \tag{3.1}$$

where $|x'-y'|$ equals to the length of the interval connecting the consecutive vertices $x', y'$. With this definition, it is reasonable to define

$$E_m(u,v) = E'_m(u,v), \tag{3.2}$$

because functions on $V_m$ can also be regarded as on $V'_m$, by assigning the same value to the corresponding external ray pair mapped to a vertex in $V_m$. For any graph $\Gamma_m$, some of its edges may be with repeating neighbors and some are self-edges: For example, as shown in Figure 2.3, in $\Gamma_1$ the fixed point with external ray names $\frac{1}{3}, \frac{2}{3}$ has a self-loop, and there are two edges between the fixed point and its another pre-image, which has corresponding external ray pair $\frac{1}{6}, \frac{5}{6}$. To accommodate these features, $E_m(u,v)$ is in the form

$$E_m(u,v) = \sum_{x \underset{m}{\sim} y} c_m(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big) \qquad (x,y \in V_m), \tag{3.3}$$

where $c_m(x,y)$ is the sum of $\frac{1}{|x'-y'|}$ for all adjacent vertex pairs $(x',y') \in V'_m$ mapped to $(x,y)$.

As stated while introducing the Kigami paradigm, the harmonic extension of a function defined on $V_m$ to $V_{m+1}$ minimizes energy. With the formula for energy $E_m$, we should now be able to explicitly compute the harmonic extension. Suppose $u$ is defined on $V_m$ and its harmonic extension $\tilde{u}$ to $V_{m+1}$, and $x \in V_{m+1} \setminus V_m$ any new point consisting of two identified points between points $y', z'$ on $\Gamma'_m$. Points $y', z'$ may or may not be an identified pair depending on structures of neighborhood for $x$, which will be fully explained in section 4. Regardless of the topology of $x$, however, energy $E_{m+1}(\tilde{u},\tilde{u})$ will be minimized by setting $\tilde{x} = \frac{1}{2}\big(u(y')+u(z')\big)$ with full details explained in [4]. But the intuition of why this way of extending minimizes $E_{m+1}$ follows rather directly from (3.3): Let $u'$ be an arbitrary extension of $u$ to approximation level $m+1$. We would only have to consider the components of $E_{m+1}(u',u')$ induced by any new point $x \in V_{m+1} \setminus V_m$; namely,

$$\sum_{x \underset{m+1}{\sim} y} c_{m+1}(x,y)\big(u(x)-u(y)\big)\big(u(x)-u(y)\big) \qquad (x \in V_{m+1}). \tag{3.4}$$

With given conductance $c_{m+1}(x,y)$ whose value is determined solely by how the ray parametrization works, (3.4) is clearly minimized by taking the value at any new point to be the average of its neighbors.

It's worth noticing that the energy $E_m$ such defined on graph $\Gamma_m$ is not $P$-invariant; namely, $E_{m+1}(\tilde{u},\tilde{u})$ is not a constant multiple of $E_m(u,u)$, a consequence caused by how we parametrized $\mathcal{J}$ utilizing the method of external rays: Any new point in $V_{m+1} \setminus V_m$ consisting of two identified points will only be inserted in the long intervals of length $\frac{1}{2^m} \cdot \frac{2}{3}$ between points in $V'_m$, where the contribution associated with these intervals to $E_{m+1}$ are multiplied by a constant as to $E_m$, and to make the contribution of graph energy associated with every interval in $\Gamma'_m$ multiplied by the same constant, namely, to insert two new identified points which actually contribute to the graph energy within all intervals, we will have to go up two consecutive graph approximation levels. This asymmetry of neighbor behaviors in graph approximations will later be explained in section 4. [4], accordingly, gives the equality

$$E_{m+k}(\tilde{u},\tilde{u}) = 2^{k-1}E_m(u,u), \tag{3.5}$$

and details the proof. Specifically, $k=2$ for $\mathcal{J}$ in (3.5) and all following equations or inequalities.

The equality shown in (3.5) necessitates some manipulations of the definition of $E_m$ in (3.3), as $\lim_{m\to\infty} E_m(u,u) \to \infty$ for any function $u$ with a non-zero graph energy, hence providing almost no useful information on the function itself. Moreover, the Kigami paradigm prescribes its energy $\mathcal{E}$ to be such constructed that (1.5) is satisfied. Now, to first achieve identity of energies between a function defined on approximation level $m$ and its harmonic extension on $m+k$, the definition

$$\mathcal{E}_m(u,v) = 2^{(\frac{1-k}{2})m}E_m(u,v) \tag{3.6}$$

is proposed, in order to replace (3.5) with

$$\mathcal{E}_{m+k}(\tilde{u},\tilde{u}) = \mathcal{E}_m(u,u), \tag{3.7}$$

so that with this identity, for each $j = 0, 1$ we have,

$$\mathcal{E}_{j+km}(u,u) \leq \mathcal{E}_{j+k(m+1)}(u,u), \tag{3.8}$$

for $u$ defined on $V_k$. Whence, it is reasonable to define from the inequality,

$$\mathcal{E}^j(u,v) = \lim_{m\to\infty} \mathcal{E}_{j+km}(u,v). \tag{3.9}$$

The last step is constructing the energy $\mathcal{E}$ that is $P$-invariant. [4] proves that, if define

$$\mathcal{E}(u,v) = \frac{1}{k}\sum_{j=0}^{k-1} \mathcal{E}^{(j)}(u,v), \tag{3.10}$$

then the $P$-invariance condition $\mathcal{E}(u \circ P, v \circ P) = 2^{1+\frac{1}{k}}\mathcal{E}(u,v)$ is satisfied for $\mathcal{E}$. An energy in a continuous sense, also in accordance with the Kigami paradigm, is thus acquired.

Now we are finally able to see what information about a harmonic function and its first derivatives our continuous energy $\mathcal{E}$ provides. As shown before, the harmonic extension of a function defined on $V_m$ to $V_{m+1}$ by assigning any new point in $V_{m+1} \setminus V_m$ the average of its neighbor(s). Due to the method of external ray parametrization, a vertex in $V_*$ can only be born with either one or two neighbors and eventually will have four neighbors throughout the sequence of graph approximations, a core neighborhood behavior that will be fully explained in the next section. For now, return to our function $u$ defined on $V_m$ and its harmonic extension to $V_{m+1}$, $\tilde{u}$. We are interested in whether a point $x \in V_{m+1} \setminus V_m$, where $\tilde{u}(x)$ is assigned to be the average of either one or two neighbors the point $x$ is born with, remains to be the average of its neighbors in all the following graph approximations $\Gamma_{m+2}, \Gamma_{m+3}, \cdots$, if we keep harmonically extending $u$. Namely, we would like to know if $\tilde{u}(x)$ remains to be the average of neighbors of $x$ regardless of the number of neighbors and the graph approximation levels. We find out and prove that this is exactly the case.

**Theorem 3.1.** *Let $u$ be a function defined on the vertex set $V_m$ and denote the function's harmonic extension to $V_{m+1}$ by $\tilde{u}_1$. Let $x \in V_{m+1} \setminus V_m$. By definition of harmonic extension, $\tilde{u}(x)$ is assigned the value as the average of its neighbor(s). Denote the harmonic extension of $u$ to $V_{m+n}$ as $\tilde{u}_n$, to $V_*$ as $\tilde{u}_*$, with $\tilde{u}_*(x) = \tilde{u}_n(x) = \tilde{u}_{n-1}(x) = \cdots = \tilde{u}_1(x)$. Then, the value at $x$ remains to be the average of its neighbors along the sequence of harmonic extensions, $(\tilde{u}_n)$.*

*Proof.* There are two cases to discuss here: whether $x$ is born with only one or two neighbors. The first case is trivial: $\tilde{u}(x)$ will be the same as its first neighbor, and any point born in later levels in the cell of $x$ is also assigned the same value. Regardless of the level of graph approximation, the neighborhood of $x$ is always constant.

The second case is more complicated. Suppose $x$ is born at $V_{m+1}$ with two neighbors, and is assigned to be the average of them. Let the values of these two neighbors of $x$ be $a$ and $b$. At level $m+2$ $x$ gains another new neighbor, which is born on the self-loop of $x$ hence assigned the same value as $\tilde{u}(x)$. The two neighbors $x$ is born with approach closer at level $m+3$. By the rule of harmonic extension, the two new neighbors are assigned values $\frac{a+x}{2}$ and $\frac{b+x}{2}$. Note that $x$ is still the average of the sum of $\frac{a+x}{2}$ and $\frac{b+x}{2}$. At level $m+4$ where $x$ gains its all four neighbors and all following levels, two of the neighbors of $x$ will always have the same value as $\tilde{u}(x)$, and the other two always sum twice as $\tilde{u}(x)$. Hence, under harmonic extension, $\tilde{u}(x)$ is always the average of its neighbor(s), regardless the level of graph approximation and number of its neighbors. □

Generally, a harmonic function $u$ is defined to be a twice continuously differentiable function with zero Laplacian, meaning that $\Delta u = 0$. In our situation, a global harmonic function defined on $V_*$ can be realized as harmonically extending a function defined on $V_0$ to $V_*$; in other words, a function is harmonic if being harmonically extended for all graph approximation levels starting at $m = 0$. With this definition, it follows from and rule of harmonic extension and Theorem 3.1 that a harmonic function on $V_*$ should only have values of points as the average of their neighbors throughout the sequence graph approximations $\Gamma_m$ of $\mathcal{J}$. As a result, the only global harmonic function on $V_*$ is simply constant. Consider the following counter-example which illustrates more clearly why this is the case: Suppose that we didn't harmonically extend a function from the very first passage from $m = 0$ to $m = 1$, but rather extended it starting from $m = 1$. Say at $m = 1$ we assign our fixed point named $\frac{1}{3}, \frac{2}{3}$ some arbitrary value $a$, and its another pre-image $\frac{1}{6}, \frac{5}{6}$ some different value $b$. It can be immediately seen that the function will not be harmonic even if being harmonically extended in all later passages. During the passage from $m = 1$ to $m = 2$, we assign the new point born $\frac{5}{12}, \frac{7}{12}$ adjacent to $\frac{1}{3}, \frac{2}{3}$ the same value $a$, the other point adjacent to $\frac{1}{6}, \frac{5}{6}$ the value $b$ similarly. The issue is obvious: $a \neq b \iff a, b \neq \frac{a+b}{2}$. Neither the value of $\frac{1}{3}, \frac{2}{3}$ nor of $\frac{1}{6}, \frac{5}{6}$ is the average of their two neighbors, whence the function fails to be globally harmonic starting at this level.

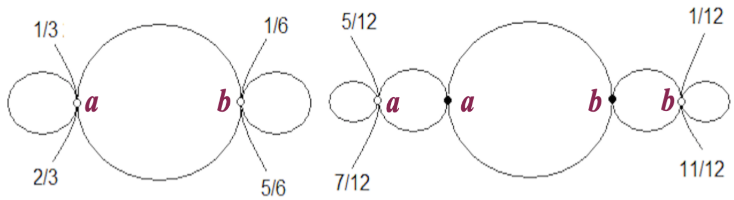



Figure 3.. A simple counter-example. The function would fail to be globally harmonic if not harmonically extended starting from the beginning passage from $m = 0$ to $m = 1$.

The fact that global harmonic functions on $V_*$ with the ray parameterization have to be constant implies zero energy $\mathcal{E}$. By the weak formulation (1.9), that they must also have zero laplacian $\Delta_\mu$ given their zero energy $\mathcal{E}$, which is in accordance to the general definition of a harmonic function. They also have zero first derivatives given that they are constant. These implications appear to be veritably helpful in our attempts on the definition of normal derivatives of continuous functions of $\mathcal{J}$, later described in section 5. They also bring forth the discussion of the untilization of local, piecewise harmonic functions as the potential candidate for first-order tangent in section 6, given that global harmonic functions, being simply constant, providing not much helpful information on a linear approximation of other functions.

# 4 Neighborhoods

In order to construct a derivative for functions defined on our approximation of $\mathcal{J}$ we first need to examine how the neighborhood of any vertex in $V_*$ behaves, at its born time and in following levels of approximations. We observed and will demonstrate in section 4.1 that any vertex in $V_m$ was born with either one or two neighbors and will eventually have four neighbors approaching asymmetrically the vertex, starting from graph approximation $\Gamma_{m+3}$. The junction point, also the fixed point $z_0$ of $P$ in our case, is the sole exception born with no neighbor and starting to have four neighbors in $\Gamma_4$, four levels after $V_0$. We also write out the ray names of neighbor(s) of any vertex in the approximation level it is born and in any later level, until and including the level where it has four neighbors, with respect to the vertex's two ray names. Due to the method of external ray, of the first four neighbors of any point in $V_*$, one pair of neighbors will be closer to $x$ than another pair. We then explain in section 4.2 how the four neighbors of any vertex approaches arbitrarily close but alternately to the vertex, the asymmetric behaviors of which result from our very choice of external ray parametrization of $\mathcal{J}$. In plain English, starting at the level where a point gaining its all four neighbors, the passage from the current level $m$ to the next level $m + 1$ only makes two of the neighbors approach closer to $x$, while the passage from $m + 1$ to $m + 2$ cause the other two neighbors of $x$ that stayed during the previous passage to approach closer. This asymmetry leads to our differentiation between even and odd levels of approximations in terms of each individual vertex: Given any vertex born at level $m_0$ and gaining its first four neighbors at level $m_0 + 3$, we classify all levels that can be written in the form $m_0 + 3 + 2n, n \in \mathbb{N}$ as even, others as odd in the form $m_0 + 3 + (2n + 1)$. We make this classification because within each category of levels, the passage from a current graph approximation, say $\Gamma_{m_0+3+2n}$, to the next one along the sequence, $\Gamma_{m_0+3+2(n+1)}$, would result in all four neighbors of a vertex approaching closer to it, since we actually go up by two consecutive approximation levels. It's also the case that at any level, a pair of neighbors would be closer to $x$, while another pair further. At the end of the section we give a general formula for explicit ray names of all four neighbors of any vertex, for all approximation levels starting from where the vertex first gains its four neighbors.

## 4.1 Structures of Neighborhoods in Earlier Levels

Suppose we are currently at approximation level $m > 0$, with the approximated external ray parametrization $\Gamma'_m$. As described in section 2, within the interval between each identified ray pair of $\Gamma'_m$, a new identified pair will be laid during the passage from $V'_m$ to $V'_{m+1}$. Each of these inserted pairs is identified into a single vertex in $\Gamma_{m+1}$ with only one neighbor. A simple example can be the pre-image of $z_0$ other than itself whose ray names are $\frac{1}{6}, \frac{5}{6}$. All positive reals in the Julia set are also straightforward examples of points parametrized to born with only one neighbor. We would like to make a further classification of vertices in $V_*$ that were born with one neighbor before our discussion of neighborhood behaviors. We differentiate the positive reals from and all others only due to slightly different external ray naming patterns for the two categories of vertices; that said, both the process of the formation of a stable neighborhood with four neighbors and the approaching behaviors of neighbors are exactly the same, for approximated positive reals and for all other vertices in $V_*$ born with one neighbor. The only difference we will address here is the external ray naming pattern, and hence the general formula for ray names of neighbors for these two types of vertices. We also preserve this classification in later definition of normal derivatives of continuous functions of $\mathcal{J}$ in section 5 for accurate reference to neighbors of vertices of different types.

Note that the positive reals are parametrized in accordance with their Cartesian coordinates: They appear only on the right end of the unit circle. It follows that the two parametrized points of each positive real are symmetric to each other horizontally, the intervals between them crossing the starting and end point of the circle named $\{0, 1\}$. The identified ray pair $\{\frac{1}{6}, \frac{5}{6}\}$, for example, contains the two values of the parameter mapped to the positive real $\frac{\sqrt{5}-1}{2} \in V_*$, and the interval between them, denoted by $[\frac{5}{6}, \frac{1}{6}]$, traverses the starting and end point of the circle. For a more generalized situation on how unique features of the positive reals distinguish themselves from others, choose any positive real $x \in V_{m_0} \setminus V_{m_0-1}$, $x \in \mathbb{R}^+$ of the approximated basilica Julia set, with its two ray names $x_1, x_2 \in V'_{m_0}$, $x_1 < x_2$. The interval between $x_1, x_2$ is denoted by $[x_2, x_1]$ since it traverses the point $0, 1$. Given what section 2 describes, $[x_2, x_1]$ subdivides a long interval with length $\frac{1}{2^{m_0-1}}\frac{2}{3}$ in $\Gamma'_{m_0-1}$ into three subintervals including itself with length $\frac{1}{2^{m_0}}\frac{2}{3}$, while the other two shorter sub-intervals have the same length $\frac{1}{2^{m_0}}\frac{1}{3}$. We can then deduce that the single neighbor of $x$ has the ray pair named $\{x_1 + \frac{1}{2^{m_0}}\frac{1}{3}, x_2 - \frac{1}{2^{m_0}}\frac{1}{3}\}$, and the interval between them is denoted as $[x_2 - \frac{1}{2^{m_0}}\frac{1}{3}, x_1 + \frac{1}{2^{m_0}}\frac{1}{3}]$. However, the external ray naming pattern for other vertices and their neighbors is different. Choose a vertex that is not a positive real, $y \in V_{m_0} \setminus V_{m_0-1}$, $y \notin \mathbb{R}^+$ and its two ray names $y_1, y_2 \in V'_{m_0} \setminus V'_{m_0-1}$, where $y_1 < y_2$. Note that $[y_1, y_2]$ denotes the interval between the vertex's identified ray pair. Since $x, y$ are both born in $V_{m_0}$, the length of the interval $[y_1, y_2]$ is the same as $[x_2, x_1]$. It follows that lengths of the two long intervals in $\Gamma'_{m_0-1}$ these two intervals are dividing, including length of the shorter sub-intervals thus formed in each case, are also the same. Therefore, the ray pair of the neighbor of $y$ is written as $y_1 - \frac{1}{2^{m_0}}\frac{1}{3}, y_2 + \frac{1}{2^{m_0}}\frac{1}{3}$, and the interval between this ray pair is written as $[y_1 - \frac{1}{2^{m_0}}\frac{1}{3}, y_2 + \frac{1}{2^{m_0}}\frac{1}{3}]$. Note a change of signs and exchanging position of ray names compared to the situation while assigning names for the neighbor of $x$: Recall that the neighbor $x$ is born with has the corresponding ray pair $\{x_1 + \frac{1}{2^{m_0}}\frac{1}{3}, x_2 - \frac{1}{2^{m_0}}\frac{1}{3}\}$, and the interval between these two ray points is denoted by $[x_2 - \frac{1}{2^{m_0}}\frac{1}{3}, x_1 + \frac{1}{2^{m_0}}\frac{1}{3}]$. Figure 4.1.1 demonstrates this difference in ray naming pattern for vertices $x, y$ in question.

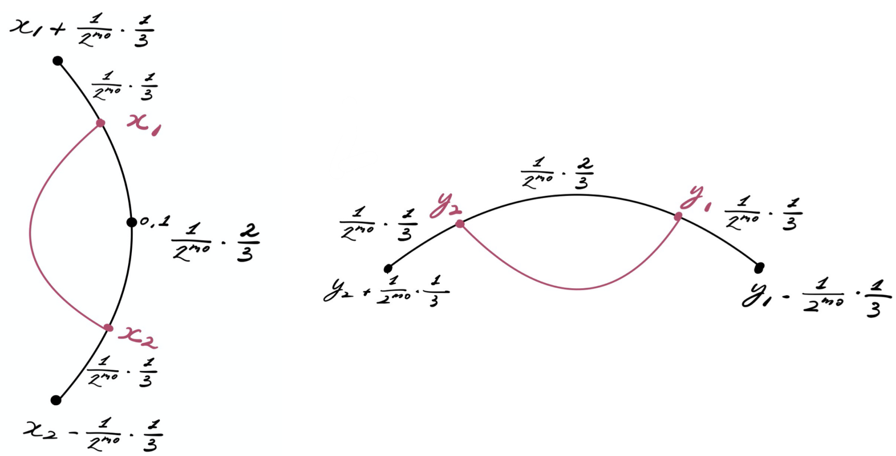


Figure 4.1.1. Demonstration of the neighbor for $x, y$ when they were born, in terms of the external ray parametrization. External ray names for each identified pair are labeled.

As stated before, all behaviors are the same for neighbors of positive reals and others born with one neighbor despite their external ray names. We will not belabor on but only give separately the ray names of neighbors of the positive reals, therefore, and instead focus on the difference between the formation of a stable four-neighbor cell of vertices born with different number of neighbors.

In previous discussion of this section, we have already written out two ray names of the neighbor with which a vertex is born. Now we continue describing the formation of a four-neighbor cell of vertices born with only one neighbor. Still, let $x \in V_{m_0} \setminus V_{m_0-1}$ with its two ray names $x_1, x_2 \in V'_{m_0}$, $x_1 < x_2$. And suppose without loss of generality $x \notin \mathbb{R}^+$ so that we write the interval between $x_1, x_2$ as $[x_1, x_2]$, the length of which denoted by $2l$ for the sake of simplicity, where $2l = \frac{1}{2^{m_0}}\frac{2}{3}$. As explained before, the two shorter intervals adjacent to $[x_1, x_2]$ have length $l$, and the two ray names of the neighbor $x$ is born with are $x_1 - l, x_2 + l$. Now, during the passage from $V_{m_0}$ to $V_{m_0+1}$, a new identified pair is laid between $x_1, x_2$ and subdividing $[x_1, x_2]$ into three sub-intervals, the longer sub-interval between the inserted identified pair with length $l$, and two shorter intervals adjacent to it with the same length $\frac{l}{2}$. This new external ray pair, written out as $\{x_1 + \frac{l}{2}, x_2 - \frac{l}{2}\}$, is identified into the second neighbor $x$ has. Note that we yet insert any identified pair in the intervals $[x_1 - l, x_1]$ and $[x_2, x_2 + l]$ at this point, given that they are not considered long intervals for $\Gamma'_{m_0}$. Hence the neighbor with which $x$ is born named $x_1 - l, x_2 + l$ remains from the previous approximation level as a neighbor of $x$.

The passage from $V_{m_0+1}$ to $V_{m_0+2}$, while retaining the new-born neighbor in $V_{m_0+1}$ named $x_1 + \frac{l}{2}, x_2 - \frac{l}{2}$ still in the cell of $x$, substitutes the original neighbor in $V_{m_0}$ named $x_1 - l, x_2 + l$ with two neighbors newly appearing in $V_{m_0+1}$, so that $x$ will have three neighbors once the passage is completed. Given that intervals for the external ray parametrization at previous approximation level, $m_0+1$, have length either $l$ or $\frac{l}{2}$, with regard to the way ray parametrization works, one new identified pair will be inserted within each long interval of length $l$ in $\Gamma'_{m_0+1}$, again subdividing the long interval into three sub-intervals, the one longer sub-interval between any new identified pair of length $\frac{l}{2}$, which becomes the new long interval for $\Gamma'_{m_0+2}$, and two shorter sub-intervals adjacent to the longer sub-interval with half of its length, $\frac{l}{4}$. Under our context, with a concentration on the cell of $x$, the previous neighbor born in $V_{m_0+1}$ which has the ray pair $\{x_1 + \frac{l}{2}, x_2 - \frac{l}{2}\}$ stays as a neighbor for $x$ because no identified pair is laid within intervals $[x_1, x_1 + \frac{l}{2}]$ and $[x_2 - \frac{l}{2}, x_2]$: With the length $\frac{l}{2}$, they are not considered long intervals for $\Gamma'_{m_0+1}$, where intervals have length $l$ or $\frac{l}{2}$. However, note that the vertex with ray names $x_1 - l, x_2 + l$ is no longer in the cell of $x$ during this current transition to $V_{m_0+2}$. Within each of the intervals $[x_1 - l, x_1]$ and $[x_2, x_2 + l]$ with length $l$ one new identified pair is laid, subdividing each of them into three sub-intervals of length $\frac{l}{2}$, $\frac{l}{4}$, and $\frac{l}{4}$, the longer sub-interval again being the one between any new identified pair. The two vertices these two new pairs are identified into are the new neighbors appearing in the cell of $x$, at $V_{m_0+2}$. We can also explicitly write out their ray names, with respect to $x_1, x_2$: The identified pair laid within the interval $[x_1 - l, x_1]$ has ray names $x_1 - \frac{l}{4} - \frac{l}{2}, x_1 - \frac{l}{4}$, the other pair within $[x_2, x_2 + l]$ having names $x_2 + \frac{l}{4}, x_2 + \frac{l}{4} + \frac{l}{2}$. Although there is another identified pair inserted within the interval $[x_1 + \frac{l}{2}, x_2 - \frac{l}{2}]$ during our current transition, it fails to become a neighbor given its non-adjacency to $x$. We would only like to consider the cell of $x$, namely points in $V'_{m_0+2}$ adjacent to either of $x_1, x_2$, and the intervals between these points and either $x_1$ or $x_2$.

We've seen that there are already three neighbors of $x$ in its cell once transitioned into $V_{m_0+2}$. The passage from $V_{m_0+2}$ to $V_{m_0+3}$, which we are about to describe in this paragraph, completes eventually the formation of the first four-neighbor cell of $x$, also a prototype of later cells all consisting of four neighbors. Similar as behaviors in the previous passage, the passage to $V_{m_0+3}$ retains in the cell of $x$ the two new-born neighbors of $x$ in $V_{m_0+2}$, named, in terms of ray parametrization, $x_1 - \frac{l}{4} - \frac{l}{2}, x_1 - \frac{l}{4}$ and $x_2 + \frac{l}{4}, x_2 + \frac{l}{4} + \frac{l}{2}$. These two vertices continue to be neighbors of $x$ because intervals $[x_1 - \frac{l}{4}, x_1]$ and $[x_2, x_2 + \frac{l}{4}]$ with length $\frac{l}{4}$ are not considered long for the parametrized graph approximation $\Gamma'_{m_0+2}$, hence no new identified pair is inserted between them during the current transition. The other neighbor of $x$ in $\Gamma'_{m_0+2}$, nevertheless, is no longer in the cell of $x$ once transitioning into $V_{m_0+3}$. Recall from our discussion about the passage from $V_{m_0+1}$ to $V_{m_0+2}$ in the last paragraph that this neighbor has the corresponding ray pair $\{x_1 + \frac{l}{2}, x_2 - \frac{l}{2}\}$. The intervals $[x_1, x_1 + \frac{l}{2}]$, $[x_2 - \frac{l}{2}, x_2]$, with length $\frac{l}{2}$, are long intervals for $\Gamma'_{m_0+2}$ and are thus each inserted one new identified pair within, subdividing them each into three sub-intervals, as always. The interval between any new pair is with length $\frac{l}{4}$, half the length of the long interval from $\Gamma'_{m_0+2}$ it is subdividing, and the remaining two shorter intervals have length $\frac{l}{8}$, half the length as that of the interval between a newly inserted identified pair at the current level. Therefore, the two vertices which these two new pairs are identified into become neighbors newly appearing in the cell of $x$ at $m_0+3$. We give the external ray names of these two new-born neighbors, as before: The identified pair lying within $[x_1, x_1+\frac{l}{2}]$ has names $x_1+\frac{l}{8}, x_1+\frac{l}{8}+\frac{l}{4}$, and the other pair within $[x_2-\frac{l}{2}, x_2]$ is named $x_2-\frac{l}{8}-\frac{l}{4}, x_2-\frac{l}{8}$. At this point we've finally witnessed the whole process of the formation of the first four-neighbor cell of the vertex $x \in V_{m_0} \setminus V_{m_0-1}$, $x \notin \mathbb{R}^+$, which is also shown in Figure 4.1.2 and Figure 4.1.3, in external ray parametrization and in corresponding actual graph approximations of the Julia set $\mathcal{J}$.

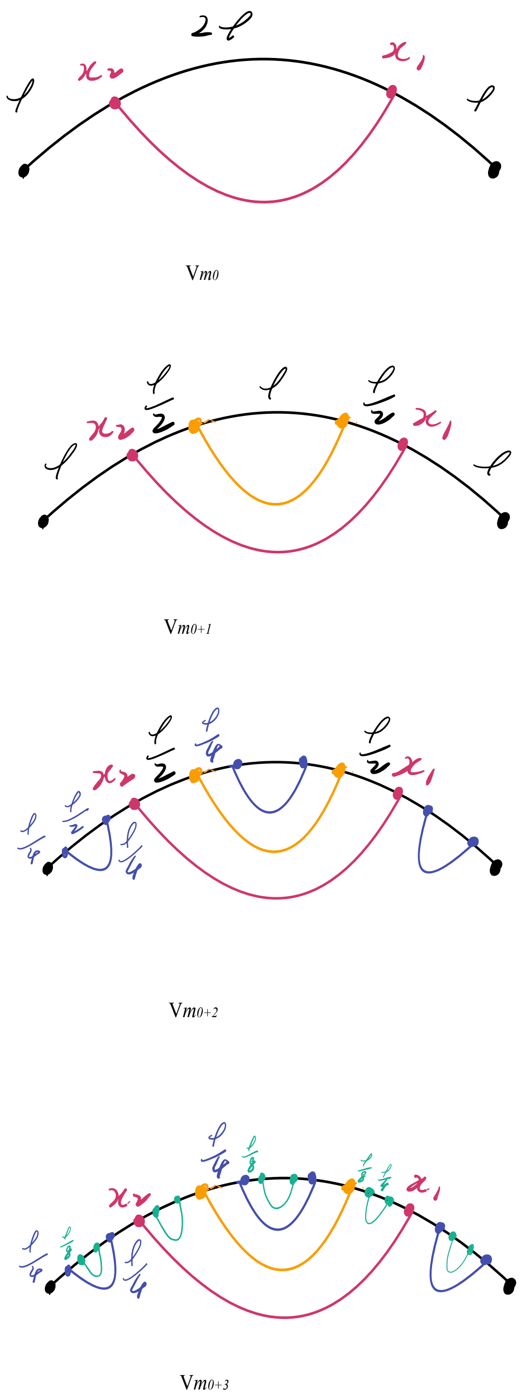



Figure 4.1.2. Demonstration of the formation of a four-neighbor cell for $x \in V_{m_0} \setminus V_{m_0-1}$ throughout the passages from $V_{m_0}$ to $V_{m_0+3}$, in terms of the external ray parametrization, where $x$ is born with one neighbor.

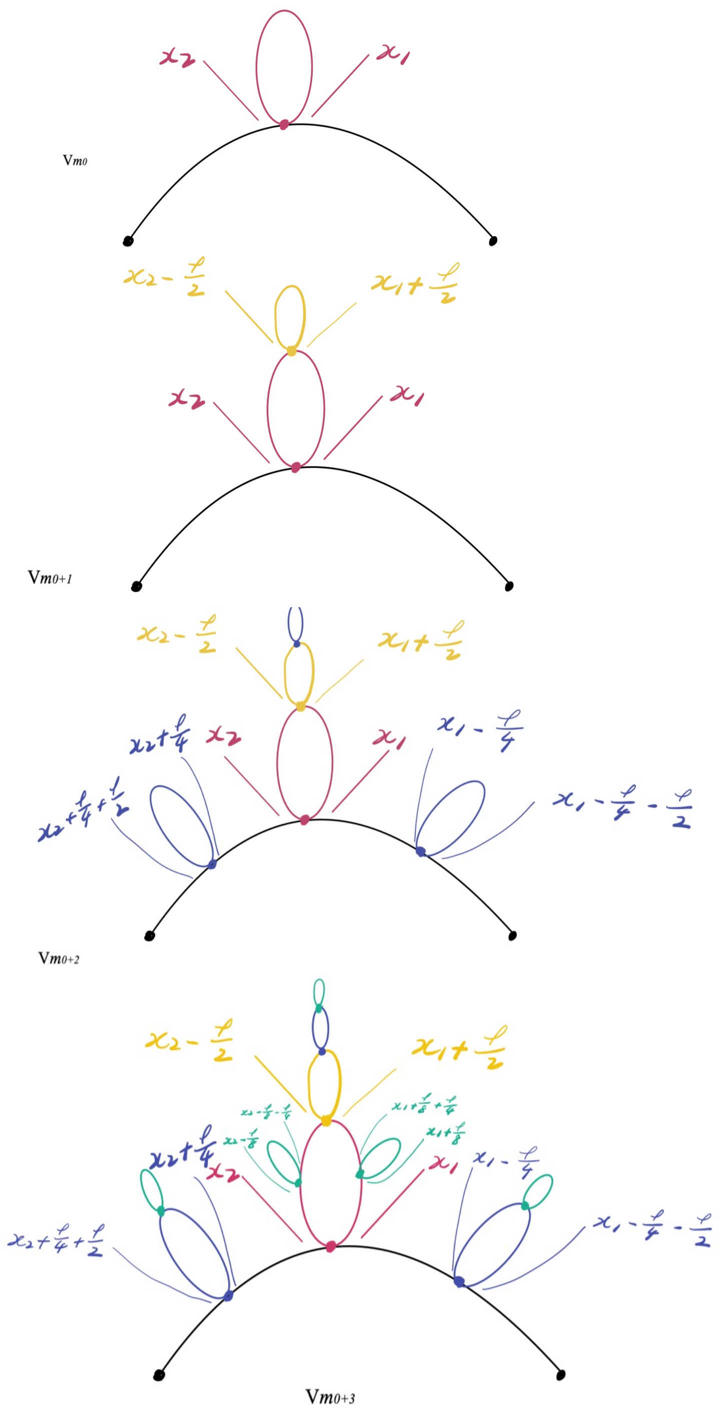



Figure 4.1.3. The formation of the four-neighbor cell for $x$, demonstrated by graph approximation for $\mathcal{J}$.

Since development of the first four-neighbor cell for any positive real in $V_*$ is the same as that of any vertex born with one neighbor, detailed descriptions are spared here, and we only give the ray names of neighbors for the positive reals throughout this process. Let $x \in V_{m_0} \setminus V_{m_0-1}, x \in \mathbb{R}^+$ be any positive real, with its external ray pair $x_1, x_2 \in V'_{m_0}$, $x_1 < x_2$, the interval between which written as $[x_2, x_1]$. As previously said when we discuss the difference in external ray naming patterns, $x$ is born with the single neighbor named $x_1 + \frac{1}{2^{m_0}}\frac{1}{3}, x_2 - \frac{1}{2^{m_0}}\frac{1}{3}$. Let us still denote the length $\frac{1}{2^{m_0}}\frac{2}{3}$ with $2l$ for the sake of simplicity, then the first neighbor of $x$ is named $x_1 + l, x_2 - l$. Notice how this naming is different from its counterpart for a vertex that is not positive real and born with only one neighbor, which would have the name $x_1 - l, x_2 + l$. Transitioning into $V_{m_0+1}$ brings another neighbor in the cell of $x$ while retaining the original one. The length of the interval between the newly inserted identified pair is $l$ and is $\frac{l}{2}$ of the two shorter, newly appearing intervals adjacent to it, completely identical to the situation discussed before for non positive-reals born with one neighbor. The ray names of this new-born neighbor at $V_{m_0+1}$ are $x_1 - \frac{l}{2}, x_2 + \frac{l}{2}$. Upon $V_{m_0+2}$, while the newly appearing neighbor named $x_1 - \frac{l}{2}, x_2 + \frac{l}{2}$ from the previous level remains in the cell of $x$, we substitute the original neighbor of $x$ with two new vertices appearing in the current level, named, in terms of external ray parametrization, $x_1 + \frac{l}{4} + \frac{l}{2}, x_1 + \frac{l}{4}$, and $x_2 - \frac{l}{4}, x_2 - \frac{l}{4} - \frac{l}{2}$. These two neighbors stay in the cell during the passage into $V_{m_0+3}$, where the neighbor emerging at $V_{m_0+1}$ external ray named $x_1 - \frac{l}{2}, x_2 + \frac{l}{2}$ are replaced by yet another two new vertices appearing in the current level, named $x_1 - \frac{l}{8}, x_1 - \frac{l}{8} - \frac{l}{4}$, and $x_2 + \frac{l}{8}, x_2 + \frac{l}{8} + \frac{l}{4}$.

Starting from this paragraph, we move on to the formation of the first four-neighbor cell for vertices born with two neighbors. We would also like to compare the development of the first cell of these vertices with the that of the other type, which is born with only one neighbor. Suppose $x \in V_{m_0}$ is born with two neighbors and named $x_1, x_2 \in V'_{m_0}$, $x_1 < x_2$. Given that the vertex $x$ has two distinct neighbors, its corresponding ray pair should not lie between another identified pair from the previous approximation level. Therefore, suppose the interval $[x_1, x_2]$ lies within the $[y', z']$, where $y', z' \in V'_{m_0-1}$, $y' < z'$ and $y' \sim z'$, $y', z'$ not identified in $V_{m_0-1}$. Again, let $2l = \frac{1}{2^{m_0}} \frac{2}{3}$, length of the interval $[x_1, x_2]$, for a simplicity sake. We can first write out that $y' = x_1 - l$, $z' = x_2 + l$.

Transitioning into the next approximation level, while $y'$ and $z'$ remain as two neighbors of $x$ at level $m_0 + 1$, a new identified ray pair with length $l$ is laid between $x_1, x_2$, subdividing $[x_1, x_2]$ into three sub-intervals with alternating lengths. The other two shorter intervals have the same length $\frac{l}{2}$. We name this newly inserted identified pair $\{x_1 + \frac{l}{2}, x_2 - \frac{l}{2}\}$, whose identification in $\Gamma_{m_0+1}$ is the third neighbor of $x$. Note that intervals have length either $l$ or $\frac{l}{2}$ at the current level; thus, during the following transition, each interval with length $l$ will be considered a long interval and subdivided into three sub-intervals.

Since $|y' - x_1| = |z' - x_2| = l$, the intervals $[x_1, y']$ and $[x_2, z']$ were considered short and not subdivided during the passage from $V_{m_0}$ to $V_{m_0+1}$. Yet they are each subdivided by a pair of newly inserted identified points into three sub-intervals during the passage from $m_0 + 1$ to $m_0 + 2$, the longer sub-interval with length $\frac{l}{2}$ and two shorter ones $\frac{l}{2^2}$. We write out the names for the identified pair inserted within $[x_1, y']$ as $\{x_1 - \frac{l}{2^2} - \frac{l}{2}, x_1 - \frac{l}{2}\}$. Names for the other pair between $x_2, z'$ are $\{x_2 + \frac{l}{2^2}, x_2 + \frac{l}{2^2} + \frac{l}{2}\}$. These two identified pairs substitute the neighbors $x$ was originally born with and become two new neighbors of $x$. It's worth noticing that the naming pattern of these two neighbors is exactly the same as their counterparts for a vertex that is not positive real and born with only one neighbor. Also similar to the one-neighbor situation, there is another newly inserted identified pair within the interval $[x_1 + \frac{l}{2}, x_2 - \frac{l}{2}]$ during the current passage from $m_0 + 1$ to $m_0 + 2$. This pair fails to be identified into a neighbor of $x$ given its non-adjacency to either $x_1$ or $x_2$.

The passage from $V_{m_0+2}$ to $V_{m_0+3}$ brings another two new neighbors and completes the formation of the first four-neighbor cell of $x$. During this passage, all intervals from $\Gamma_{m_0+2}$ with length $\frac{l}{2}$ are considered long and subdivided into three sub-intervals, the longer one with length $\frac{l}{2^2}$ and two shorter ones with the same length $\frac{l}{2^3}$. Note that external ray names of the three neighbors of $x$ at level $m_0+2$ are $\{x_1 + \frac{l}{2}, x_2 - \frac{l}{2}\}$, born at $m_0 + 1$, and $\{x_1 - \frac{l}{2^2} - \frac{l}{2}, x_1 - \frac{1}{2^2}\}$, $\{x_2 + \frac{l}{2^2}, x_2 + \frac{l}{2^2} + \frac{l}{2}\}$, which were born at the previous level $m_0 + 2$. Hence, during this transition, the identification of the pair $\{x_1 + \frac{l}{2}, x_2 - \frac{l}{2}\}$ in $V_{m_0+1}$ ceases to be $x$'s neighbor. Instead, within each of the intervals $[x_1, x_1 + \frac{l}{2}]$, $[x_2 - \frac{l}{2}, x_2]$ a new identified ray pair with length $\frac{l}{2^2}$ is inserted. We write out the external ray names of these two pairs as $\{x_1 + \frac{l}{2^3}, x_1 + \frac{l}{2^3} + \frac{l}{2^2}\}$, and $\{x_2 - \frac{l}{2^3}, x_2 - \frac{l}{2^3} - \frac{l}{2^2}\}$. Again, these names are the same as their counterparts for vertices that are not positive reals and born with one neighbor, which leads us to claim that the naming of the first four neighbors of a vertex is independent of the number of neighbor(s) this vertex is born with, but is solely conditioned on whether the vertex is positive real. The whole forming process of the four-neighbor cell of $x$ is also demonstrated by Figure 4.1.4 and Figure 4.1.5, in parametrized and actual graph approximations of $\mathcal{J}$.

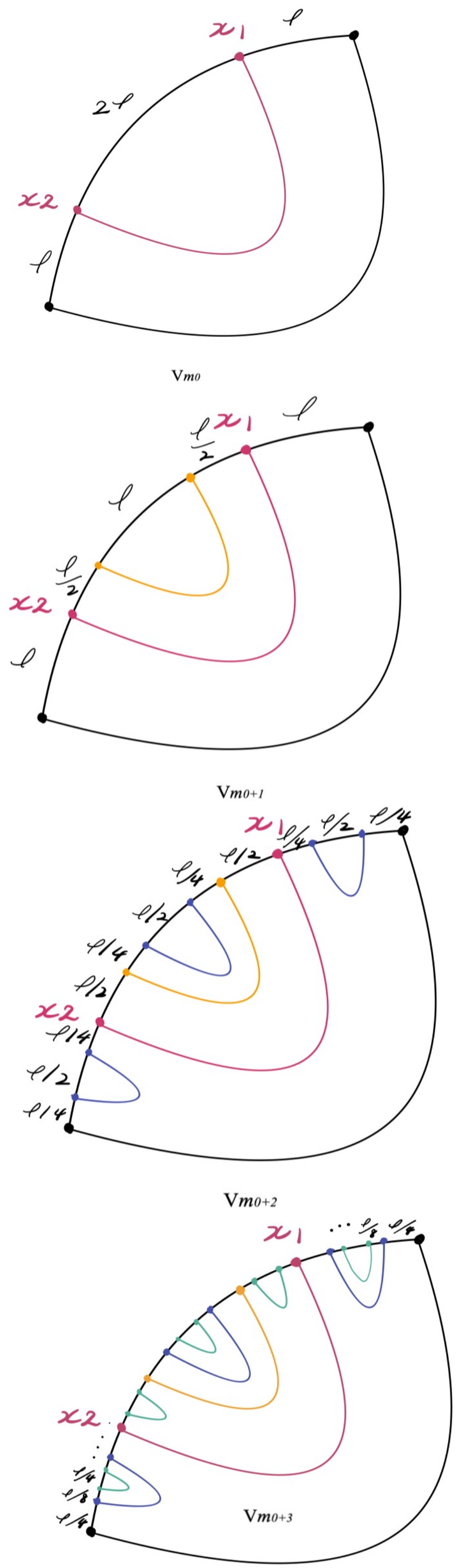



Figure 4.1.4. Demonstration of the formation of a four-neighbor cell for $x \in V_{m_0} \setminus V_{m_0-1}$ throughout the passage from $V_{m_0}$ to $V_{m_0+3}$, in terms of the external ray parametrization, where $x$ is born with two neighbors.

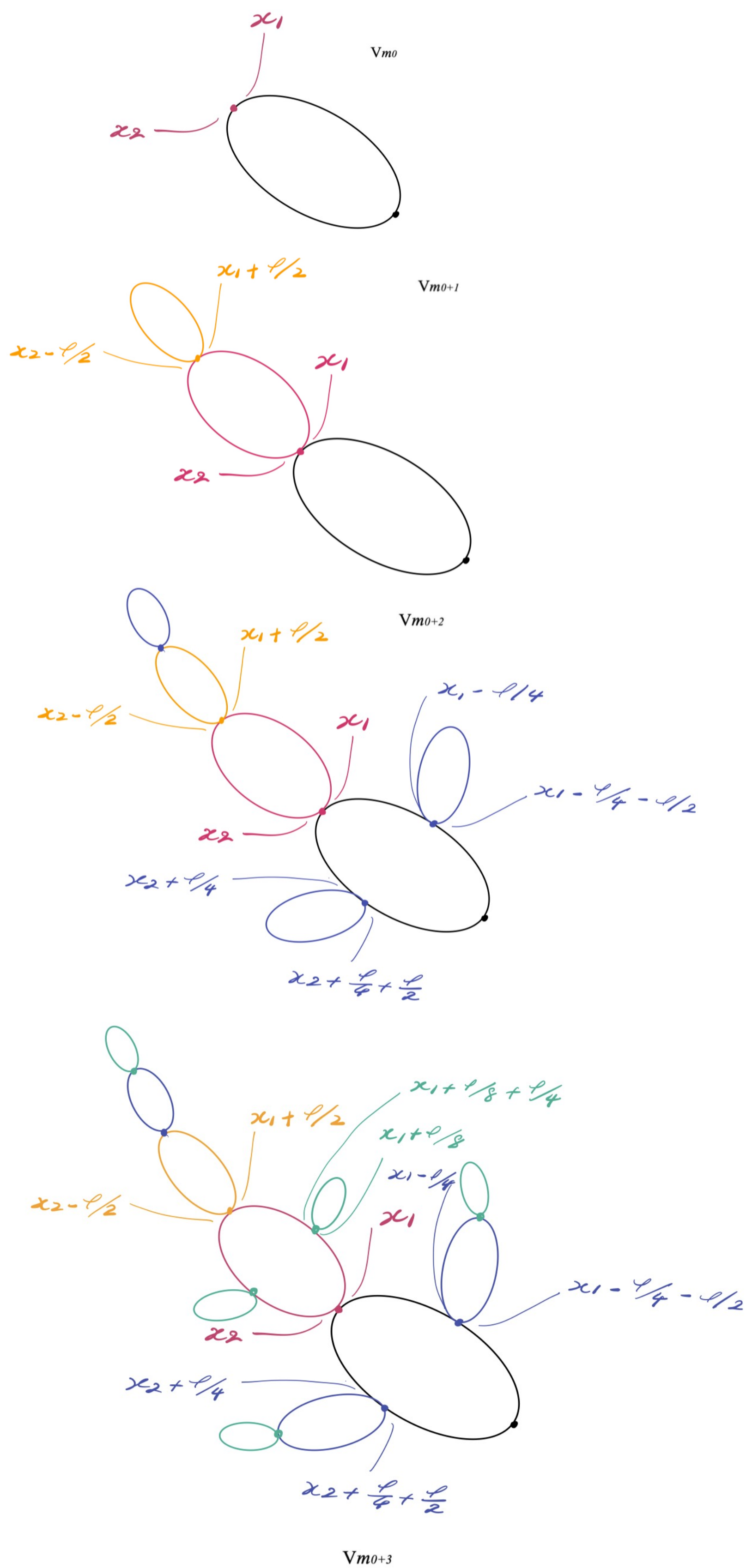



Figure 4.1.5. The formation of the four-neighbor cell for $x$, demonstrated by graph approximation for $\mathcal{J}$.

## 4.2 Approaching Behaviors of a Neighborhood

With the method of external ray and its induced graph approximations, a vertex in $V_*$ won't be able to have more than four neighbors. After examining the process a vertex gains its first four neighbors, we become more interested in how neighbors behave in the following graph approximation. We found out that the choice of external ray parametrization of $\mathcal{J}$ causes an asymmetric approaching behavior of neighborhood. In particular, neighbors approach alternately to a vertex throughout the sequence of graph approximations. During the passage from any approximation level $m$ to the next level $m+1$, only a pair of neighbors approach closer to the vertex, while the other pair stays. The transition from $m+1$ to $m+2$ would cause the pair that stays from the previous passage to approach, while the other pair stays. That is, only going up by two consecutive levels will all four neighbors of a vertex approach closer to it. Figure 4.2.1 demonstrates this behavior for neighbors of $x \in V_m \setminus V_{m-1}$, throughout the passage from $V_{m+3}$ to $V_{m+5}$. For the sake of simplicity, note that the figure only shows the cell, namely the smallest neighborhood of a vertex for the current approximation level, of $x$ along these graph approximations, while not specifying the development of neighbors and cells for its neighbors.

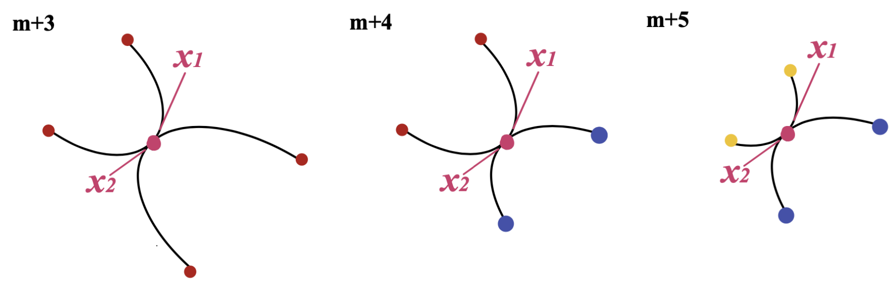
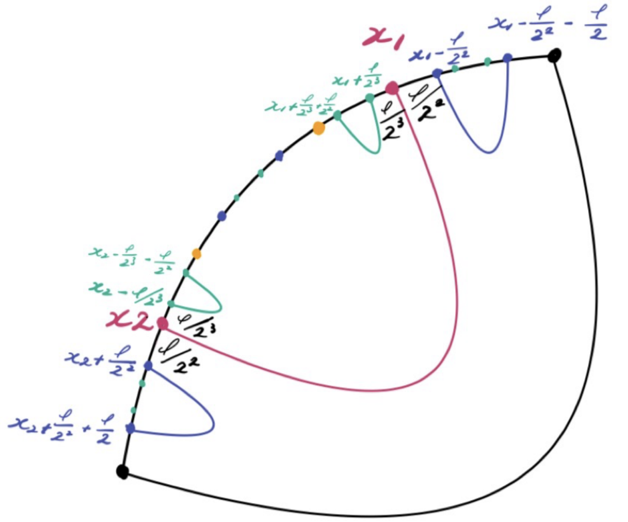



Figure 4.2.1. Demonstration of neighborhood behaviors for $x \in V_m \setminus V_{m-1}$ throughout the passage from $V_{m+3}$ to $V_{m+5}$, in actual graph approximations of $\mathcal{J}$.

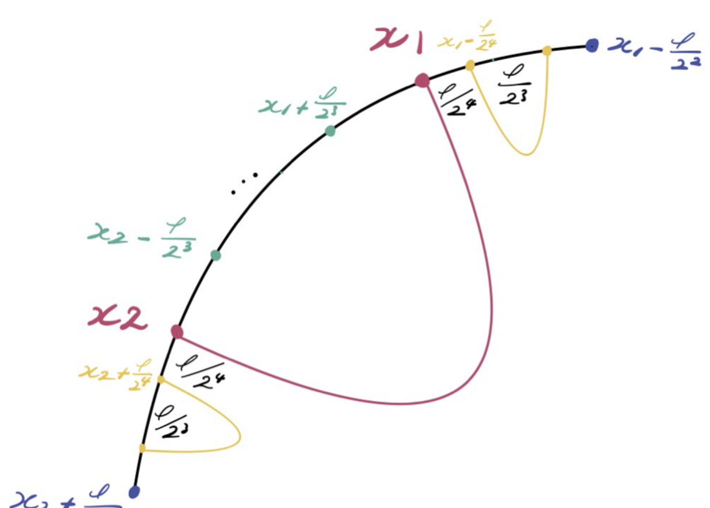
With full detail, we are about to delineate how the method of external ray causes the asymmetric approaching behavior. Consider the first four-neighbor cell of any vertex $x \in V_m \setminus V_{m-1}$, with its two external ray names $x_1, x_2 \in V'_m$, $x_1 < x_2$. Let's say $x$, without loss of generality, is born with two neighbors: We've stated in section 4.1 that even though neighbors of the positive reals are named differently compared to all other vertices, structure of the first four-neighbor cell of any vertex will be the same regardless of the number of neighbors they were born with and whether they are positive reals. Therefore, discussing in cases is not necessary. We will use the example of a non positive real vertex throughout this section, and give the formula for external ray names for neighbors of the positive reals separately later in the section. Let $l = \frac{1}{2^m}\frac{1}{3}$, length of a short interval for $\Gamma'_m$. Based on section 4.1, the corresponding external ray pairs of $x$'s four neighbors at graph approximation level $m+3$ are $\{x_1 - \frac{l}{2^2} - \frac{l}{2}, x_1 - \frac{l}{2^2}\}$, $\{x_2 + \frac{l}{2^2}, x_2 + \frac{l}{2^2} + \frac{l}{2}\}$, two pairs born at level $m+2$, and $\{x_1 + \frac{l}{2^3}, x_1 + \frac{l}{2^3} + \frac{l}{2^2}\}$, and $\{x_2 - \frac{l}{2^3}, x_2 - \frac{l}{2^3} - \frac{l}{2^2}\}$, pairs newly born at level $m+3$. The two points in each ray pair are identified into one neighbor of $x$ in $\Gamma_{m+3}$, the graph approximation of $\mathcal{J}$ at level $m+3$.

Before delving into approaching behaviors of neighbors of $x$, we would like to revisit the method of external ray discussed in section 2. Take $\Gamma'_m$, the parametrization of $\mathcal{J}$ at any graph approximation level $m$. We said that there are $2^{m+1}$ points in $\Gamma'_m$, dividing the unit circle into $2^m$ cycles of intervals, each cycle including two intervals with length $\frac{1}{2^m}\frac{1}{3}$ and $\frac{1}{2^m}\frac{2}{3}$, one having half the length as the other does. Intervals with different lengths are laid alternately on the unit circle; namely, a long interval with length $\frac{1}{2^m}\frac{2}{3}$ has to be adjacent to two short intervals with half the length, and vice versa. For the passage from $V'_m$ to $V'_{m+1}$, we insert two new identified points within each long interval of $\Gamma'_m$, subdividing it into three sub-intervals. The longer sub-interval becomes one of the long intervals for $\Gamma'_{m+1}$, having length $\frac{1}{2^{m+1}}\frac{2}{3}$, half the length as the interval being divided and the same length as short intervals in $\Gamma'_m$. Recall that, as emphasized in section 2, the long sub-interval appears always between an identified pair. Two shorter sub-intervals each has length $\frac{1}{2^{m+1}}\frac{1}{3}$, fourth the length as the original interval, and are considered short for $\Gamma'_{m+1}$. Notice that any interval between a newly added point to the current level and its neighbor not in the same identified pair is short. With this knowledge refreshed, also recall from the last paragraph that corresponding ray pairs of the four neighbors of $x$ is $\{x_1 - \frac{l}{2^2} - \frac{l}{2}, x_1 - \frac{l}{2^2}\}$, $\{x_2 + \frac{l}{2^2}, x_2 + \frac{l}{2^2} + \frac{l}{2}\}$, $\{x_1 + \frac{l}{2^3}, x_1 + \frac{l}{2^3} + \frac{l}{2^2}\}$, and $\{x_2 - \frac{l}{2^3}, x_2 - \frac{l}{2^3} - \frac{l}{2^2}\}$, with $l = \frac{1}{2^m}\frac{1}{3}$.

Now, take one point from the corresponding ray pair of each neighbor of $x$ that is adjacent to either of $x_1, x_2$, the two ray names of $x$. This gives us four points in $V'_{m+3}$: $x_1 - \frac{l}{2^2}, x_2 + \frac{l}{2^2}, x_1 + \frac{l}{2^3}$, and $x_2 - \frac{l}{2^3}$. Let $x'$ be either $x_1$ or $x_2$. Distances between these points and $x'$ they are adjacent to are different. For example, $|(x_1 - \frac{l}{2^2}) - x_1| \neq |(x_2 - \frac{l}{2^3}) - x_2|$. We can group these four points with regard to this distance: The first group contains $x_1 - \frac{l}{2^2}$ and $x_2 + \frac{l}{2^2}$, the other $x_1 + \frac{l}{2^3}$ and $x_2 - \frac{l}{2^3}$. Two points from the second group are both closer to $x'$ they are adjacent to, because they were born at the current level $m + 3$. We know from the last paragraph and section 2 that, for $\Gamma'_{m+3}$, the interval between a newly added point and its neighbor other than the point in its identified pair is always short. The pair in the first group, nevertheless, was born at level $m + 2$ and remains to be adjacent to $x_1$ or $x_2$ at level $m + 3$, hence becoming a long interval in $\Gamma'_{m+3}$. Another way to interpret this situation is that intervals $[x_1 - \frac{l}{2^2}, x_1]$ and $[x_1, x_1 + \frac{l}{2^3}]$ are adjacent, then by the mechanism of the method of external ray, they must have alternate lengths. Same reasoning applies for the other two intervals $[x_2 - \frac{l}{2^3}, x_2]$ and $[x_2, x_2 + \frac{l}{2^2}]$. Given this situation, only two longer intervals of the four, $[x_1 - \frac{l}{2^2}, x_1]$ and $[x_2, x_2 + \frac{l}{2^2}]$, will be each sub-divided by a newly laid identified pair during the passage from $V'_{m+3}$ to $V'_{m+4}$. This means that the two identified pairs including either of these two points $x_1 - \frac{l}{2^2}$, $x_2 + \frac{l}{2^2}$ would no longer remain as neighbors of $x$ in $\Gamma_{m+4}$. In our words, they approach closer to $x$.



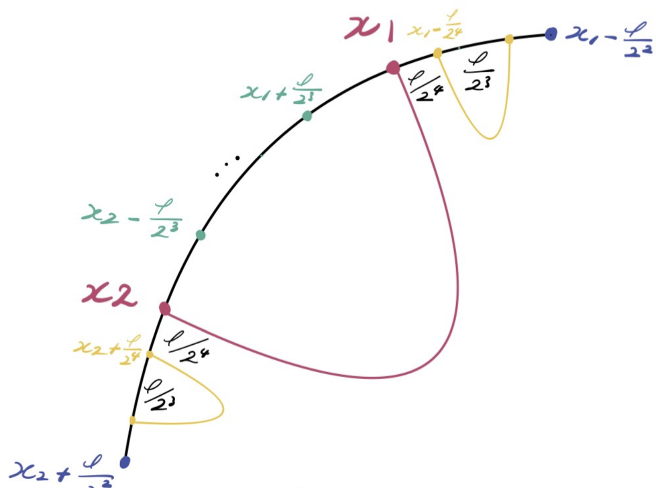
$\mathbf{V}_{m+3}$



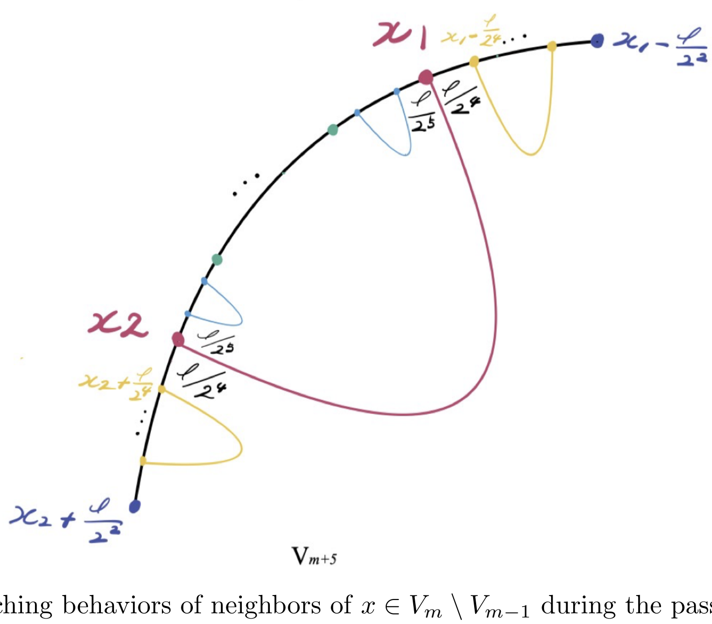
$\mathbf{V}_{m+4}$

Figure 4.2.2. Approaching behaviors of neighbors for $x \in V_m \setminus V_{m-1}$ during the passage from $V_{m+3}$ to $V_{m+4}$, in terms of the external ray parametrization of $\mathcal{J}$. Ray names for each neighbor are labelled. Points not in the cell of $x$ are omitted for the sake of clarity.

We can derive the corresponding external ray pairs of the two new neighbors of $x$ at $V_{m+4}$ by applying the knowledge revisited from section 2. During the passage, the identified pair newly laid between the interval $[x_1 - \frac{l}{2^2}, x_1]$ divides it into three sub-intervals. The longer sub-interval is between the pair and has length $\frac{l}{2^3}$, the same length as the short interval from $\Gamma'_{m+3}$. The two shorter sub-intervals has length $\frac{l}{2^4}$. Hence, this newly inserted ray pair has name $x_1 - \frac{l}{2^4}, x_1 - \frac{l}{2^4} - \frac{l}{2^3}$. Similarly, the ray pair that is laid between the interval $[x_2, x_2 + \frac{l}{2^2}]$ will have name $x_2 + \frac{l}{2^4}, x_2 + \frac{l}{2^4} + \frac{l}{2^3}$. At this point, the four vertices adjacent to either $x_1$ or $x_2$ are the two newly born $x_1 - \frac{l}{2^4}, x_2 + \frac{l}{2^4} \in V'_{m+4}$, and the two vertices remaining to be adjacent to $x'$ from $V'_{m+3}$, $x_1 + \frac{l}{2^3}, x_2 - \frac{l}{2^3}$, because the interval between each of them and either $x'$ they are adjacent to is considered short in $\Gamma'_{m+3}$. However, given that intervals in $\Gamma'_{m+4}$ have length either $\frac{l}{2^3}$ or $\frac{l}{2^4}$, the intervals $[x_1, x_1 + \frac{l}{2^3}]$ and $[x_2 - \frac{l}{2^3}, x_2]$ are no longer short in $\Gamma'_{m+4}$ and will be each sub-divided by an newly laid identified ray pair, transitioning from $V'_{m+4}$ into $V'_{m+5}$. The ray pair between $[x_1, x_1 + \frac{l}{2^3}]$ has name $x_1 + \frac{l}{2^5}, x_1 + \frac{l}{2^5} + \frac{l}{2^4}$, and the pair laid between $[x_2 - \frac{l}{2^3}, x_2]$ has name $x_2 - \frac{l}{2^5}, x_2 - \frac{l}{2^5} - \frac{l}{2^4}$. Hence, at the current level, the four vertices in $V'_{m+5}$ adjacent to $x'$ are the remaining two from the previous level, $x_1 - \frac{l}{2^4}, x_2 + \frac{l}{2^4} \in V'_{m+4} \setminus V'_{m+3}$, and the newly born other two, $x_1 + \frac{l}{2^5}, x_2 - \frac{l}{2^5} \in V'_{m+5} \setminus V'_{m+4}$. The passage from $V'_{m+4}$ to $V'_{m+5}$ is illustrated in Figure 4.2.3. Again, we dismiss pairs not adjacent to $x'$, any ray name of $x$, for a convenient sake. By saying that an identified ray pair is adjacent to $x'$, we mean one point from pair is adjacent to $x'$.



$V_{m+4}$



$V_{m+5}$

Figure 4.2.3. Approaching behaviors of neighbors of $x \in V_m \setminus V_{m-1}$ during the passage from $V_{m+4}$ to

$V_{m+5}$, in terms of the external ray parametrization of $\mathcal{J}$.

Note that at the current graph approximation level, $m+5$, the four points born in $V'_{m+3}$ that are adjacent to $x'$ have been all substituted exactly once. This also indicates that the four neighbors born in $V_{m+3}$ belonging to the first stable cell of $x$ have all been replaced, by vertices in $V_{m+5}$ corresponding to the identified ray pair adjacent to $x'$. In other words, all four neighbors approach closer to $x$ once at the graph approximation level $m+5$. This is the most crucial observance from the whole section. Just as stated at the beginning of this section, it takes two consecutive levels for all four neighbors of any vertex in $V_*$ to approach closer to the vertex once. Neighbors approach alternately due to the very way external ray parametrization operates: Starting from the graph approximation level where the vertex gains its first four neighbors, two of these neighbors approach closer to the vertex first during the passage to next approximation level. Then, the other two approach closer during the following passage, while the two which have already approached in the previous passage stay. This asymmetric approaching behaviors of neighbors results in our classification of even and odd levels of approximations, the definitions of which we have introduced at the beginning of the section and will not repeat. Within each classification of graph approximation levels, the passage from any level along the sequence to the next level would cause all four neighbors of a vertex to approach closer once. Whence, and most importantly for this project, we differentiate even and odd derivatives at $V_*$ for continuous functions defined on $\mathcal{J}$, with respect to whether the current graph approximation level is even or odd for any point of interest. Our definitions of derivatives will be introduced in the next section. Lastly, we make this differentiation only to be in accordance with the classification of approximation levels: We will also explain in the next section that only with such a differentiation would harmonic functions have constant derivatives, as they are supposed to, another pivotal motivation for our definitions of normal derivatives.

Lastly, we write out the general formula for the external ray names of neighbors of any vertex $x$ for all levels after the level where $x$ first gains its four neighbors. For $x \notin \mathbb{R}^+$ with its two external ray names $x_1, x_2, x_1 < x_2$, for any even approximation level $m^E$, the four neighbors of $x$ have external ray names $x_1 + \frac{1}{3} \cdot \frac{1}{2^{m^E}}$, $x_2 - \frac{1}{3} \cdot \frac{1}{2^{m^E}}$, $x_1 - \frac{2}{3} \cdot \frac{1}{2^{m^E}}$, and $x_2 + \frac{2}{3} \cdot \frac{1}{2^{m^E}}$. For any odd approximation level $m^O$, the four neighbors of $x$ have external ray names $x_1 + \frac{2}{3} \cdot \frac{1}{2^{m^O}}$, $x_2 - \frac{2}{3} \cdot \frac{1}{2^{m^O}}$, $x_1 - \frac{1}{3} \cdot \frac{1}{2^{m^O}}$, and $x_2 + \frac{1}{3} \cdot \frac{1}{2^{m^O}}$. For $x \in \mathbb{R}^+$, the situation is reverse: All external ray names of neighbors for even levels for the other type of points are the names for odd levels, and vice versa.

## 5 Normal Derivatives

With a concept of the neighborhood, or cell, for a point in $V_*$ at arbitrary graph approximation levels, and an understanding of how neighbors behave when approaching arbitrarily close to the point of our interest, we are now prepared to define normal derivatives for a function continuous on $\mathcal{J}$ at any point $x \in V_*$. Given that any vertex would ultimately acquire, and without gaining more than, four neighbors each distinctly approaching to the vertex, and that the values of the function at these neighbors can be arbitrarily assigned, we define four derivatives $\forall\, x \in V_*$. We differentiate even and odd derivatives regarding the classification of graph approximation levels, as mentioned before. Also, given the different external ray naming pattern for neighbors of vertices in $\mathbb{R}^+$ than all others, we present separate formulas for derivatives at all positive reals.

**Definition 5.1.** Let $x \in V_*$ and $x_1, x_2 \in V'_*$ be the two external Ray names of $x$, $x_1 < x_2$. Let $u$ be a continuous function of $\mathcal{J}$, $u \in dom\Delta_\mu$. Given that $x \in V_{m_0} \setminus V_{m_0-1}$ and $x_1, x_2 \in V'_{m_0} \setminus V'_{m_0-1}$, the *even normal derivatives* of $u$ at $x$, $\partial^E u(x)$, are defined by

$$\partial^E_{i,i\in\{1,2,3,4\}} u(x) = \begin{cases} q_1 = \lim_{n\to\infty} \frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^m})-u(x_1)}{\frac{1}{2^n}} \\ q_2 = \lim_{n\to\infty} \frac{u(x_2)-u(x_2-\frac{1}{3}\cdot\frac{1}{2^m})}{\frac{1}{2^n}} \\ q_3 = \lim_{n\to\infty} \frac{u(x_1)-u(x_1-\frac{1}{3}\cdot\frac{1}{2^{m-1}})}{\frac{1}{2^{n-1}}} \\ q_4 = \lim_{n\to\infty} \frac{u(x_2+\frac{1}{3}\cdot\frac{1}{2^{m-1}})-u(x_2)}{\frac{1}{2^{n-1}}} \end{cases} \quad, m \in \{m : m = m_0 + 3 + 2n, n \in \mathbb{N}\}$$

Whereas the *odd normal derivatives* $\partial^O u(x)$ are defined as

$$\partial^O_{i,i\in\{1,2,3,4\}} u(x) = \begin{cases} q_1 = \lim_{n\to\infty} \frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^{m-1}})-u(x_1)}{\frac{1}{2^n}} \\ q_2 = \lim_{n\to\infty} \frac{u(x_2)-u(x_2-\frac{1}{3}\cdot\frac{1}{2^{m-1}})}{\frac{1}{2^n}} \\ q_3 = \lim_{n\to\infty} \frac{u(x_1)-u(x_1-\frac{1}{3}\cdot\frac{1}{2^m})}{\frac{1}{2^n}} \\ q_4 = \lim_{n\to\infty} \frac{u(x_2+\frac{1}{3}\cdot\frac{1}{2^m})-u(x_2)}{\frac{1}{2^n}} \end{cases} , m \in \{m : m = m_0 + 3 + (2n+1), n \in \mathbb{N}\}$$

for $x \notin \mathbb{R}^+$, and by

$$\partial^E_{i,i\in\{1,2,3,4\}} u(x) = \begin{cases} q_1 = \lim_{n\to\infty} \frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^{m-1}})-u(x_1)}{\frac{1}{2^{n-1}}} \\ q_2 = \lim_{n\to\infty} \frac{u(x_2)-u(x_2-\frac{1}{3}\cdot\frac{1}{2^{m-1}})}{\frac{1}{2^{n-1}}} \\ q_3 = \lim_{n\to\infty} \frac{u(x_1)-u(x_1-\frac{1}{3}\cdot\frac{1}{2^m})}{\frac{1}{2^n}} \\ q_4 = \lim_{n\to\infty} \frac{u(x_2+\frac{1}{3}\cdot\frac{1}{2^m})-u(x_2)}{\frac{1}{2^n}} \end{cases} , m \in \{m : m = m_0 + 3 + 2n, n \in \mathbb{N}\}$$

$$\partial^O_{i,i\in\{1,2,3,4\}} u(x) = \begin{cases} q_1 = \lim_{n\to\infty} \frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^m})-u(x_1)}{\frac{1}{2^n}} \\ q_2 = \lim_{n\to\infty} \frac{u(x_2)-u(x_2-\frac{1}{3}\cdot\frac{1}{2^m})}{\frac{1}{2^n}} \\ q_3 = \lim_{n\to\infty} \frac{u(x_1)-u(x_1-\frac{1}{3}\cdot\frac{1}{2^{m-1}})}{\frac{1}{2^n}} \\ q_4 = \lim_{n\to\infty} \frac{u(x_2+\frac{1}{3}\cdot\frac{1}{2^{m-1}})-u(x_2)}{\frac{1}{2^n}} \end{cases} , m \in \{m : m = m_0 + 3 + (2n+1), n \in \mathbb{N}\}$$

otherwise, provided the limits exist. Figure 5.1 and 5.2 illustrate more directly how each derivative is indexed for both $\partial^E$ and $\partial^O$, for the two types of vertices.

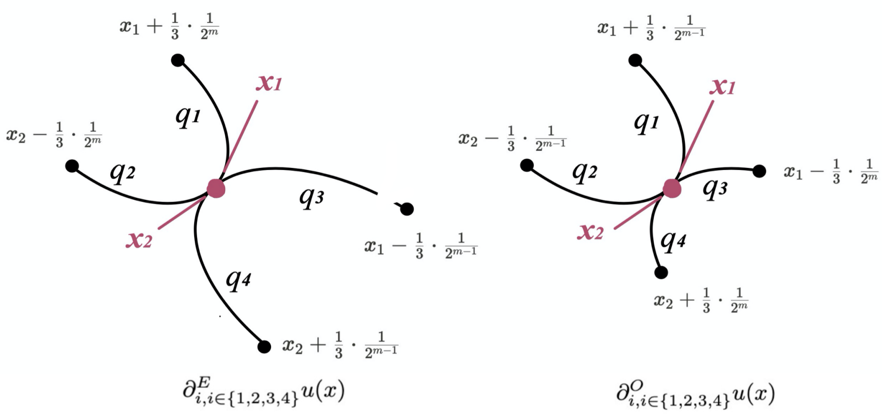



Figure 5.1. Indices of the derivatives at any vertex $x \in V_*$, $x \notin \mathbb{R}^+$.

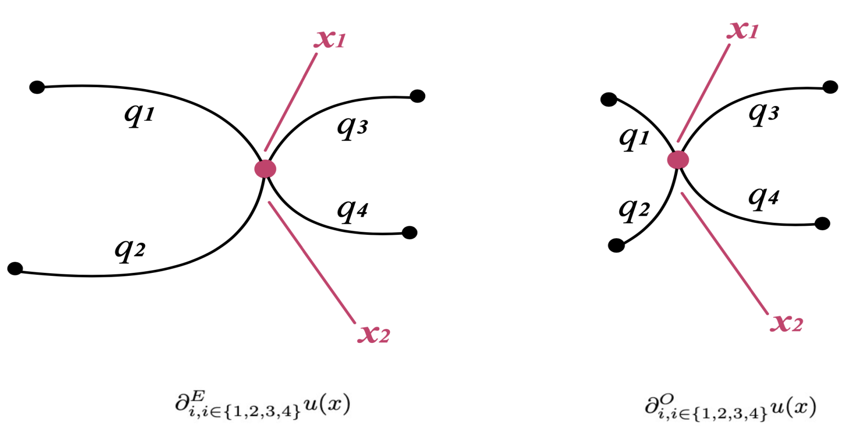



Figure 5.2. Indices of the derivatives at any vertex $x \in V_*$, $x \in \mathbb{R}^+$.

The first thing we would like to clarify is that the function $u$ is defined on the Julia set, meaning that its inputs are points from $V_*$. In Definition 5.1, we use one of the external ray names of the neighbors of any point $x$ as the input of $u$, which is indeed confusing notation, and the actual input of $u$ should be the point in $V_*$ that is mapped to the external ray name. Take the definition of $\partial^E$ for $x \in V_{m_0}$, $x \notin \mathbb{R}^+$ as an example. Still, suppose $u$ is a continuous function of $\mathcal{J}$. At any even level $m$, one of the approximated derivatives $q_1$ is calculated as $\frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^m})-u(x_1)}{\frac{1}{2^n}}$. Note that in numerator, the function $u$, by our confusing notation, takes in external ray names $x_1 + \frac{1}{3} \cdot \frac{1}{2^m}$ and $x_1$. But the actual inputs of $u$ here are the point in $V_*$ mapped to the external ray name $x_1 + \frac{1}{3} \cdot \frac{1}{2^m}$ under the mapping $\phi : \mathcal{J} \to \mathbb{S}$, discussed in the introduction, and the point $x$. Since $u$ takes in points of $\mathcal{J}$, any identified ray pair is assigned the same value. The simplest example is $u(x_1) = u(x_2)$ because both $x_1, x_2$ are mapped to the same vertex, $x$, in the basilica Julia set $\mathcal{J}$.

As said in the last part of section 4, in order to obtain a neighborhood where all neighbors of a point approach closer whenever going up by one level of graph approximation, we differentiate even and odd levels of approximations in terms of each point in $V_*$. Similarly, Definition 5.1 differentiates normal derivatives at any point $x \in V_*$ to be even and odd, $\partial_i^E$ and $\partial_i^O$, based on whether the sequence of quotients approaching to them are from the even or the odd levels of approximations. For example, for any point $x \in V_*$ born at level $m_0$, note that the even derivatives of $x$ only take into consideration of the limits of quotients $q_i^m, i \in \{1, 2, 3, 4\}$ calculated from all even levels; namely, all levels $m$ where the value of $m - (m_0 + 3)$ is even. Similarly, the odd derivatives are the limits of the quotients only from odd graph approximation levels with respect to $x$. Only by making this differentiation are we able to give a formula for the corresponding quotients, or approximated derivatives at graph approximation level $m$, for each level. Given that not all four neighbors approach closer while going up by one approximation level, there is simply no way to give a formula that works for all levels at once. We adapt the normalizer in the denominator of the quotients $q_i$ accordingly. Since we differentiate even and odd derivatives in terms of levels, the notion of distance should not be understood as directly associated with the value of $m-(m_0+3)$, namely the difference in levels after our point $x$ first gaining its four neighbors, but rather with $n = \frac{m-(m_0+3)}{2}$. The reason is obvious: Let the current graph approximation level be even, where $m = m_0 + 3 + 2 \cdot 2$ and $n = 2$. Then, note that the four neighbors of $x$ at level $m$ have approached to $x$ twice, from the first four neighbors of $x$ at level $m_0 + 3$, and the graph approximation $\Gamma_m$ is the third (we start with $n = 0$) along the sequence of graphs at even levels with respect to $x$. Hence it is reasonable to define the distances between four neighbors of $x$ at $m = m_0 + 3 + 2 \cdot 2$ and $x$ to be either $\frac{1}{2^n}$ or $\frac{1}{2^{n-1}}$, the difference resulting from the fact that one pair of neighbors are always closer to $x$ than another pair. It follows that for the even derivatives, $\partial_i^E$, the distance, or the normalizer, in denominator is either $\frac{1}{2^n}$ or $\frac{1}{2^{n-1}}$, depending on whether the corresponding neighbor is closer to $x$.

There is another crucial reason why we do not choose $\frac{1}{2^m}$ or $\frac{1}{2^{m-1}}$ as our normalizers: We would like our derivatives to be in accordance with the neighborhood behaviors of a point. Let's still suppose the function $u$ defined on $\mathcal{J}$ is continuous, and $x \in V_{m_0}$ a vertex gaining its four neighbors before level $m$. By the previous section, the pair of neighbors closer to $x$ at level $m$ will only approach closer again at $m + 2$, which means that this pair remains still at $m + 1$. Then, at $m + 1$, given that this pair doesn't approach closer and still is the neighbor of $x$, the difference between their assigned values by $u$ and $u(x)$ should be the same as at $m$, as well as the distance between this pair of neighbors and $x$. Therefore, the approximated derivatives at this pair of neighbors should be the same, for level $m$ and $m + 1$. If we utilize $\frac{1}{2^m}$ or $\frac{1}{2^{m-1}}$ to represent the notion of distance in the denominator of quotients in $\partial^E$, however, the approximated derivative $q_i^{m+1}$ for this pair of neighbors of $x$ would not be the same for $q_i^m$, given that the denominator would change from $\frac{1}{2^m}$ to $\frac{1}{2^{m+1}}$. Adapting $\frac{1}{2^n}$ as the distance would resolve this problem effectively. For example, for the pair of neighbors of $x$ that only approach at even levels, going from an even level $m$ to $m + 1$ won't change the value of $n$, hence the distance between the pair and $x$. The value of $n$ will only be added by 1 when going from $m$ to $m + 2$, while the pair of neighbors also approach closer.

The definition of odd derivatives $\partial_i^O$ is slightly different in terms of the denominators of the quotients. It is indeed very strange at the first glance, due to the same denominator for all four quotients. We make this choice because we would like the even and odd derivatives to be coherent. By Definition 5.1, for even derivatives of function $u$ at any point $x \notin \mathbb{R}^+$, $\partial^E u(x)$, and any even level $m = m_0+3+2n$, the quotients $q_1$, $q_2$ represent the even derivatives calculated from the two directions where the corresponding neighbors will always be closer to $x$, with an actual distance to $x$, in external ray names,

$\frac{1}{3} \cdot \frac{1}{2^m}$. We know that neighbors from these directions are closer to $x$ because intervals for $\Gamma'_m$ have either length $\frac{2}{3}\frac{1}{2^m}$ or $\frac{1}{3}\frac{1}{2^m}$, the points $x_1 + \frac{1}{3} \cdot \frac{1}{2^m}$ and $x_2 - \frac{1}{3} \cdot \frac{1}{2^m}$ must be newly born at $m$ due to their distance to $x_1$ or $x_2$ is the length of a shorter interval in $\Gamma'_m$. It follows that they will not approach during the following passage from $m$ to $m+1$. The quotient, or approximated derivative, at the same point has to be the same. Then it must be true that $q_1^m = q_1^{m+1}$ and $q_2^m = q_2^{m+1}$. The definition of $\partial^O$ reflects this fact. For example, by Definition 5.1, $q_1^m = \frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^m})-u(x_1)}{\frac{1}{2^n}}$. By what has been stated, it should be satisfied that $q_1^m = q_1^{m+1}$. Now, by definition of odd normal derivatives, $q_1^{m+1} = \frac{u(x_1+\frac{1}{3}\cdot\frac{1}{2^{(m+1)-1}})-u(x_1)}{\frac{1}{2^n}} = q_1^m$. A similar reasoning works for all other quotients and is spared here. The key fact to remember here is that, quotients at direction 1 and 2 for any odd level $m_0+3+2n+1$, $q_{\{1,2\}}^{m_0+3+2n+1}$, are the same as those for its previous even level $m_0+3+2n$, because neighbors of any point that is non-positive real at these two directions only approach closer at even levels, and remain still in the following odd levels. Similarly, quotients at directions 3 and 4 for any even level are the same as those for its previous odd level. It is the reverse situation for points $y$ that are positive reals, given that corresponding neighbors from directions 3 and 4 are always closer to $y$ at even levels, and from directions 1 and 2 closer to $y$ at odd levels.

With this fact in mind, note that even if we assume all limits, separately for the even and odd derivatives, exist for Definition 5.1, the existence of any one of the even and odd derivatives will imply the existence of the other.

**Lemma 5.1.** *Let $x \in V_{m_0} \setminus V_{m_0-1}$ and $u$ be a continuous function of $\mathcal{J}$. Suppose all four even derivatives, $\partial^E_{i,i\in\{1,2,3,4\}}u(x)$, of $u$ at $x$ exist. Then, the existence of the four odd derivatives $\partial^O_i u(x)$, $i \in \{1,2,3,4\}$, will simply follow that of $\partial^E_i u(x)$, and vice versa. Moreover, $\partial^O_i = \partial^E_i$.*

The proof for Lemma 5.1 is simple and follows naturally from the fact that, as what have been stated before, for any point that is not positive real, the approximated derivatives at direction 1, 2 at any odd level are the same as those at the previous even level. Similarly, the quotients at direction 3, 4 at any even level are the same as those calculated at its previous odd level. Hence, each of the four sequences approaching to even or odd derivatives actually has the same limits. In other words, though the quotients are different throughout levels of graph approximation, $\partial^O_i = \partial^E_i$. The situation for positive reals is the reverse but still similar, hence the discussion for them is spared. We attempt to further investigate potential features of our derivatives, hence we have also studied the compatibility condition, or the matching condition, of normal derivatives in this project. It would be great if the compatibility condition, meaning that all four derivatives at any given point $x\ inV_*$ sum to 0, is satisfied in our case. We have tried out the proof mechanism by Strichartz of this condition on the Gasket but end up realizing that such a condition is not always met under our setting. Specifically, there is no way to directly related the sum of all four derivatives by Definition 5.1 with the graph Laplacian $\Delta_m u(x)$, the core of Strichartz's proof.

The last thing to discuss in this section is on the derivatives for harmonic functions. We have briefly explained that these functions should have constant first derivatives in section 3. This requirement is satisfied by Definition 5.1. Take the local harmonic function $\psi_x^{(m)}$ as an example, $x \in V_{m_0} \setminus V_{m_0-1}$. Suppose, without loss of generality, that $m = m_0 + 3 + 2n$ is any even level, and points $A, B, C, D$ are four neighbors of $x$ at level $m$, where $A, B$ are corresponding points from direction 1,2 hence closer to $x$. By definition, $\psi_x^{(m)}(\{A,B,C,D\}) = \{0\}$. Then, $q_1^m = \frac{-\psi_x^{(m)}(x)}{\frac{1}{2^n}} = -2^n$. Given that a harmonic function has constant first derivatives, $q_i^m = q_i^{m+1} = \cdots = q_i$. Since vertex $A$ is in the pair of neighbors closer to $x$, it will not approach closer again until the passage from level $m+1$ to $m+2$. Note that the value of $n$ is added by one going from $m$ to $m+2$. Denote the neighbor of $x$ from direction 1 at level $m+2$, or the approached vertex $A$, by $A_1$. By rule of harmonic extension, $\psi_x^{(m)}(A_1) = \frac{\psi_x^{(m)}(A_1)+\psi_x^{(m)}(x)}{2} = \frac{1}{2}$. Then, by Definition 5.1, $q_1^{m+2} = \frac{\psi_x^{(m)}(A_1)-\psi_x^{(m)}(x)}{\frac{1}{2^{n+1}}} = \frac{\frac{1}{2}-1}{\frac{1}{2^{n+1}}} = -\frac{1}{2}\cdot 2^{n+1} = -2^n$, the same as $q_1^m$. A similar reasoning would work for all other neighbors from directions 2,3, and 4. This consideration of constant derivatives for harmonic functions is another pivotal factor of selecting $\frac{1}{2^n}$ and $\frac{1}{2^{n-1}}$ to reflect the distance between a vertex and its neighbors. If we choose directly the current level $m$ instead, or the difference $m - m_0 - 3$, the derivatives of a harmonic function will diverge. Let's continue with the example $\psi_x^{(m)}$. All four neighbors only approach closer once going up two consecutive levels, hence we are normalizing the quotients of approximated derivatives by a factor of $2^2$, going from $m$ to

$m+2$, if the distance in the denominator is directly associated with $m$. However, values of functions at neighbors of $m+2$, by the rule of harmonic extension, are half of the sum of their neighbors; namely, $\psi_x^{(m)}\{A_1, B_1, C_1, D_1\} = \{\frac{1}{2}\}$. If the distance is assessed by the difference in levels directly, then $q_1^m = -2^m$ and $q_1^{m+2} = -\frac{1}{2} \cdot 2^{m+2} = -2^{m+1}$, and the sequence $(q_1^m)$ diverges.

# 6 First Order Tangents

In the traditional setting calculus, computing the first order tangent at a point requires the derivative at the point, and the value of function under examination at the same point. We finally get to the state of investigating first order tangents of functions defined on $V_*$, at any point $x \in V_*$, with a definition of first derivatives at $x$. As explained far back in section 3, we choose harmonic functions as candidates for potential first-order tangents since they have zero Laplacian, whence a constant first derivative. We also state that since the only global harmonic functions on $V_*$ are constant, we are motivated to define local harmonic functions restricted on the cell of a vertex in $V_*$ of any given level, and use them to further explore the definition of local tangents at the vertex.

**Definition 6.1.** Let $u$ be a function defined on $V_m$ and $\tilde{u}$ be its harmonic extension to $V_*$. Suppose $x \in V_{m_0} \setminus V_{m_0-1}$ with its first four neighbors in $V_{m_0+3}$, $m \geq m_0 + 3$. Also suppose the four neighbors of $x$ in graph $\Gamma_m$ are $A, B, C, D$. Let $A', B', C', D'$ be the external ray name of the four neighbors that are adjacent to $x'$, any ray name of $x$. Denote the interval on $\Gamma'_m$ between each of $A', B', C', D'$ and $x'$ as $I_A, I_B, I_C$, and $I_D$. Then, we define the *local harmonic function* of $u$ *restricted on the $m$-level cell of* $x$ as the restriction of $\tilde{u}$ on $\phi(\{I_A, I_B, I_C, I_D\})|_{V_*} \in \mathcal{J}$.

Let's try to parse this definition. First, extend the function $u$ harmonically to $V_*$. Then choose our vertex of interest $x$ and any level $m$ where $x$ already gained four neighbors. Pick the four neighbors of $x$ at level $m$ and four edges connecting these neighbors and $x$. Now, take all vertices born at all later graph approximation levels $n, n > m$ either on these edges, or connected to any point on these edges. A local harmonic function restricted on the $m$-level cell of $x$ is $\tilde{u}$ restricted on these vertices. We also give the name $m$-level local harmonic function.

Before introducing the process of defining the local tangents, we would like to first show that all $m$-level local harmonic functions of $x$ have zero graph Laplacian at any graph approximation level starting at $m+1$, except the points born at or before $m$. Furthermore, with the equilibrium measure $\mu$, the Laplacian $\Delta_\mu$ as defined in the formula (1.10) is zero.

**Theorem 6.1.** *Let $\tilde{u}$ be a local harmonic function of any function $u$ defined on $V_m$ with respect to a vertex $x \in V_{m_0} \setminus V_{m_0-1}$. Let $A, B, C, D$ be the four neighbors of $x$ at level $m$. Any vertex in the domain of $\tilde{u}$, except $A, B, C, D$, and $x$, will have zero graph Laplacian as defined in (1.11). Moreover, the Laplacian with the equilibrium measure $\Delta_\mu$ is also zero for these points.*

Given the way graph Laplacian is defined, the first statement of Theorem 6.1 follows naturally from Theorem 3.1, where we prove that if being harmonically extended, then the value at a point will continue to be the average of its neighbors. The case where a point is born with one neighbor is trivial: The point will be assigned the same value of its first neighbor, and neighborhood of the point at any level will be constant. For the case where a point $x$ is born with two different neighbors, given that the distance between each of the two neighbors and the point is same, the constant normalizer $c_m(x, y)$ is the same, which can be simply factored out. Since we are considering harmonic functions here, the point will be the average of its neighbors. Hence the graph Laplacian at the point for the level of its birth will be zero. The next level brings forth a new neighbor of $x$. Given that this neighbor is born on the self-loop of $x$, it is assigned the same value as that of $x$, hence the graph Laplacian for this level can be calculated just the same as the previous one. For the next level, the two neighbors $x$ is born with approach closer to $x$. Since $x = \frac{a+b}{2} \to x = \frac{\frac{a+x}{2} + \frac{b+x}{2}}{2}$, the graph Laplacian at $x$ is still zero. We can deduce that, if $x$ is average of its two neighbors of different values, then it is still the average of the neighbors after they approach closer, under the rule of harmonic extension. It is a similar situation for the next level where $x$ gains its first four neighbors and all following levels. Among the four neighbors of $x$, two of them always have the same value as $x$ does, and the other two of the same distance to $x$ always sum to twice the value of $x$.

The second statement claiming of Theorem 6.1 is also direct to prove. Since $\Delta_m u(x) = 0$ regardless of $m$, it is guaranteed that any tern throughout the sequence is 0. The limit is also 0 given that $\lim_{m\to\infty} \int \psi_x^{(m)} d\mu \to 0$ while $\psi_x^{(m)}(x) = 1$.

We exploit the definition of local harmonic functions to help define local tangents. The first issue to address is the domain of local tangents. Given that there are four derivatives at $x$ regardless of even or odd levels, there should be four tangents going across $x$, each with a slope of one of the four derivatives representing slope of change of different directions. We know that there are four edges connecting $x$ with its neighbors at level $m$, and neighbors will approach closer to $x$ only along these edges due to the method of external ray. Recall from the last paragraph that we denote four neighbors of $x$ at level $m$ as $A, B, C, D$, and the four intervals of $\mathbb{S}$ between $x'$ and $A', B', C', D'$ as $I_A, I_B, I_C, I_D$. Under $\phi$, each of these intervals is mapped to the corresponding edge connecting $x$ with $A, B, C$, or $D$ as well as all the bulb structures on the edge. Thus, it makes sense to define four $m$-level local tangents whose domain are $\phi(I_A), \phi(I_B), \phi(I_C)$, and $\phi(I_D)$, respectively.

To match the traditional definition of a tangent line, our local tangent of any function $u$ going through the point $x$ should have the same slope as $u'(x)$. To meet this condition, we first manually assign values to $A, B, C$, and $D$ with respect to $u(x)$, $u'(x)$, and the distance between these points and $x$. Specifically, the value of $u$ at any of the neighbors is the sum of $u(x)$ and the distance between the neighbor and $x$ multiplied by the corresponding approximated derivative for level $m$. For instance, assume that the current level $m = m_0 + 3 + 2n$ is even, and points $A, B$ are the pair closer to $x$. Let $A, B, C, D$ approach $x$ from direction 1,2,3, and 4, respectively. Suppose that $x \notin \mathbb{R}^+$ for the sake of simplicity. Then, given the four quotients of even derivatives $q_{\{1,2,3,4\}}$, we let $u(A) = u(x) + q_1 \cdot \frac{1}{2^n}$, $u(B) = u(x) - q_2 \cdot \frac{1}{2^n}$. Note that the distance here follows directly the denominator of quotients of even derivatives in Definition 5.1. Since $A, B$ are closer to $x$, their distances to $x$ are both $\frac{1}{2^n}$. For the other pair farther from $x$, we let $u(C) = u(x) - q_3 \cdot \frac{1}{2^{n-1}}$, $u(D) = u(x) + q_4 \cdot \frac{1}{2^{n-1}}$.

Let $u$ be defined only on the $m$-level cell of $x$, and denote its harmonic extension to $V_*$ by $\tilde{u}$. By definition, $\tilde{u}$ is an $m$-level local harmonic function with respect to $x$. Now, break the function $\tilde{u}$ into four separate functions defined on each edge connecting $A, B, C$, and $D$ with $x$ and the connected bulb structures to each edge. In other words, keep the value of $\tilde{u}$, but break its domain into four: $\phi(I_A)$, $\phi(I_B)$, $\phi(I_C)$, $\phi(I_D)$, hence four derived functions. Denote these functions with the four sub-domains, $\phi(I_A)$, $\phi(I_B)$, $\phi(I_C)$, $\phi(I_D)$, by $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ respectively, since points $A, B, C, D$ approach $x$ from direction 1,2,3, and 4 respectively. We now show that $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ satisfy all requirements for local tangents. Given that these are parts of an $m$-level local harmonic function, they should have constant first derivatives on their entire domain, except the point of interest $x$ and its four neighbors of level $m$, $A, B, C, D$. They also have corresponding slopes to $q_i$. To prove this, consider the normal setting under $\mathbb{R}^2$. If we intend to show that a function $f$ is linear, then it is sufficient to claim that any linear approximation of a point $x$ in the domain of $f$ is the same as $f(x)$. Namely, if we arbitrarily pick a point from the domain of $f$, say $a$, and linearly approximate the value of $f$ at any point $f(x)$ with $f(a) + f'(a) \cdot (x - a)$, the approximation should be the same as the actual value $f(x)$. Similarly, to show that $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ are linear local functions, it will suffice to show that the value assigned to any point in their domains through linear approximation is the same as the actual value assigned by the function, or the rule of harmonic extension.

Now, take the $x$ and its $m$-neighbors $A, B, C, D$. We previously suppose that $m$ is even, and that $A, B$ are closer to $x$. Then, during the passage from $m$ to $m + 1$, only $C, D$ approach closer once, as shown in Figure 6.1. Denote the approached neighbors as $C_1, D_1$.

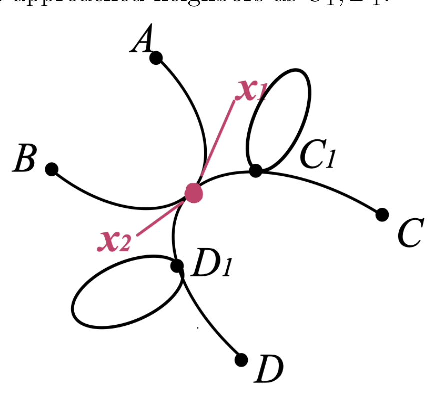


Figure 6.1. The behaviors of points at $m+1$. Two new neighbors of $x$ are born, each between $x$ and the corresponding previous neighbor.

Recall that $\tilde{u}_3$ is the local function defined on the edge connecting $x$ and $C_1$. We want to show that $\tilde{u}_3(C_1)$, assigned by the rule of harmonic extension, is the same as the linear approximation of $\tilde{u}_3$ at $C_1$. Let's first calculate $\tilde{u}_3(C_1)$ by the rule of harmonic extension. $C_1$ is born with two neighbors, $x$ and $C$, hence $\tilde{u}_3(C_1) = \frac{\tilde{u}_3(x)+\tilde{u}_3(C)}{2} = \frac{\tilde{u}_3(x)+\tilde{u}_3(x)-q_3\cdot\frac{1}{2^{n-1}}}{2} = \tilde{u}_3(x) - q_3\cdot\frac{1}{2^n}$. By linear approximation using the derivative at $x$ of the direction where $C$ approaches, $q_3$, given that the distance between $x$ and $C$ is represented by $\frac{1}{2^{n-1}}$, the distance between $x$ and $C_1$ should be written out as $\frac{1}{2^n}$. To calculate the linear approximation at $C_1$, add $\tilde{u}_3(x)$ to the distance between $x$ and $C_1$ multiplied by $q_3$, which gives the same value as $\tilde{u}_3(x) - q_3\cdot\frac{1}{2^n}$. A similar reasoning works for all other points on the edges connecting $A, B, C, D$ and $x$ throughout the graph sequence. Hence, functions $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ are linear.

We now give a formal definition of the four $m$-level local tangents at a vertex $x \in V_*$ of a continuous function $u$ of $\mathcal{J}$.

**Definition 6.2.** Let $u$ be any function defined on $V_*$, and a vertex $x \in V_{m_0} \setminus V_{m_0-1}$. Still, suppose the four neighbors of $x$ at level $m$ are $A, B, C, D$ and let $A', B', C', D'$ be the external ray names of the four neighbors that are adjacent to $x'$, any ray name of $x$. Denote the interval on $\Gamma'_m$ between each of $A', B', C', D'$ and $x'$ as $I_A, I_B, I_C$, and $I_D$. Now, reassign points $A, B, C, D$ the values calculated via linear approximation using the value $u(x)$ and all the four derivatives $q_{\{1,2,3,4\}}$ at $x$. Let $u|_m$ be the restriction of $u$ on $V_m$ with the reassigned values at $A, B, C, D$, and $\tilde{u}$ the $m$-level local harmonic function of $u|_m$ with respect to $x$, where $m \geq m_0 + 3 + 2n$. Define $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ to be $\tilde{u}$ restricted on the domains $\phi(I_A)$, $\phi(I_B)$, $\phi(I_C)$, and $\phi(I_D)$, respectively. We say that $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ are the four *$m$-level local tangents* of $u$ with respect to $x$.

All prime goals of this study have been achieved now: To fully understand the asymmetric neighbor behaviors resulting from the method of external ray, to define normal derivatives at any point in $V_*$ with respect to characteristics of these behaviors, and finally, to define local harmonic functions of any function whose domain is $V_*$, at any vertex in $V_*$ of interest.

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