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Aaron Cieslicki

"On the Categorical Status of Second Order Logic: An Examination of the Debate Between Quine and Boolos"

Introduction

This paper is concerned with the categorical status of what is generally known as second order logic. Whereas first order logic quantifies only over variables (objects), second order logic is the result of quantifying over predicates (relations). For example, of the statements below,

(1) $\{x: \sim(x=x)\}$

(2) $(F)\ [\ [(F(O)\ .\ ((n)F(n) \rightarrow F\ (n+1))] \rightarrow (n)Fn]$

the first, Frege's definition of zero, is a first order statement, while the second, a formalization of mathematical induction, is second order.

The debate rages as to whether second order statements are to be regarded as pertaining to logic proper or to some other discipline, such as set theory. Both sides agree that there are important differences between statements in first and second order logic. The argument is really: to statements in which discipline are second order statements most similar? In point of scope, Boolos points out (p. 509) that the ramifications of deciding one way or the other is another subject, treating of issues such as the epistemological primacy of the differing branches of the philosophy of mathematics. Accordingly, these issues will not be taken up here.

In order to shed light on this debate, the views of two philosophers, Quine and Boolos, will be examined. Quine concludes that second order logic is a creature other than logic proper; Boolos, whose article ("On Second Order Logic") is written in response to Quine, maintains that it is not.

Debate

Quine has, in effect, two arguments. First, the quantification of predicate letters is simply wrong; it is a misappropriation of significant symbolism of first order logic. Second, if philosophers insist on using this notation, it should be realized that they are doing set theory, regardless of the fact that it is called "second order logic." With this and the claim that Quine makes that set theory and logic are separate disciplines, he concludes that "second order logic" does not belong to the domain of logic proper.

We shall deal first with Quine's second argument and the solution he proposes for those who continue to quantify predicate letters. Second, Quine's first argument will be examined. We will then move on to Boolos' response to Quine's arguments, and end in support of Boolos' critiques.

Quine characterizes logic as comprised of two parts: lexicon and constructions (p. 22). The lexicon is a finite list containing predicates (F, G, H), be they one, two, three or *n* place predicates, and variables (x, y, z, x', y', z', x", etc.). All constructions are one of two types. The first is predication, or joining a predicate with its variable(s). The second type is constructing sentences with sentences by means of the particles (., ~, v) or existential quantification (∃) .[1] Particles are the words necessary for the second type of constructions, but are not a part of the lexicon.

The language of pure set theory is identical to the language of logic just described, save that it contains only one predicate (ε) of membership. This predicate, however, can be reduced to a particle (Quine, 64). To do this, it is important to note, as was just mentioned above, that particles are not in the lexicon. Quine's criterion for lexicon membership (Quine, 29) is that it be interchangeable *salva congruitate.*[2] Since 'ε' is the only predicate, there is nothing with which it is interchangeable *salva congruitate,* and therefore can be regarded as a particle.

At first glance, the accounts just given of logic and set theory would seem to indicate that they are extremely similar, if not identical. They are significantly similar as formal languages. However, when the status of predicates is closely examined, the difference is quite clear. The result of the predication construction for a one place predicate such as 'walks' and an arbitrary variable 'x' is 'Fx,' which is in turn an open sentence: 'x walks.' It is termed 'open' because of the presence of the variable. On its own it is neither true nor false, but only true or false for certain values of the variable (Quine, 23). When Frege claimed to have derived arithmetic from logic alone, it was with the help of a set theory that assumed that logic and set theory were identical. This set theory was based on the idea that an open sentence determines a set "if the sentence is true for all and only the members of the set" (Quine, 45). Russell's paradox shows this idea to be false: the sentence ~(xεx) does not determine a set. If such a set exists, it would be a member of itself if and only if it were not a member of itself. One of the set theorist's jobs is to decide which open sentences determine sets. So, while logic deals with the possible forms of all open sentences, set theory's scope is limited to a certain number of those open sentences, and on that ground Quine distinguishes them. With that, Quine has, if not proved, at least substantiated the claim which makes the first premise of his second argument.

As for his second premise, that in doing "second order logic" philosophers are actually doing set theory, Quine says: "Followers of Hilbert have continued to quantify predicate letters, obtaining what they call a higher-order predicate calculus. The values of these variables are in effect sets; and this way of presenting set theory gives it a deceptive resemblance to logic" (p. 68). This presentation is not only deceptive, it is also dangerous. Quine has us consider the formula

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1 Universal quantification is unnecessary theoretically, as it can be defined in terms of negation and existential quantification.

2 "Without violating agreement," i.e., any predicate of *n* places is interchangeable *salva congruitate* with any other predicate of *n* places. Variables are interchangeable *salve congruitate* only with variables.

(3) $\exists y(x)(x\varepsilon y = Fx)$

the central hypothesis of naive set theory.[3] This formula assumes a set determined by any open sentence 'Fx.' This claim is written in second order notation as

(4) $\exists(G)(x)(Gx=Fx)$

which follows from the first order tautology

(5) $(x)(Fx=Fx)$ (Quine, 68)

Quine finds it unacceptable that a paradox in set theory can be derived from first order logic, which has been proved consistent. This, however, is not much of an objection. In order to make it, Quine is assuming that it is permissible to make a derivation from first order logic into second order logic. I do not believe that it should be allowed.

Quine's solution is simple: if set theory is what one is after, one should use set theoretic notation (Quine, 67). Predications such as 'Fx', 'Fy', Gx', should be changed to '$x\varepsilon\alpha$,' '$y\varepsilon\alpha$,' '$x\varepsilon\beta$.' The values of the variables 'α','β' are to be sets. This allows quantification such as (α), $\exists(\beta)$, where quantification of predicates (F), $\exists$(G) are disallowed because predicates are not value taking variables (Quine, 52).

With that, Quine has justified his second premise, and as the argument is formally valid (~(L=S), (S=2) $\therefore$ (L=2)), he needs only to justify the statement in the solution to the problem presented by the second argument, that quantification of predicate letters should not be allowed. Such is the content of his first argument.

Quine's first argument rests on a rigid understanding of the grammaticality of predicate letters. As this argument is presented in a straightforward manner and in less space than a page, it seems appropriate to quote Quine *in extenso.*

> What is important is that in writing *'F'* and *'Fx'* we are just schematically simulating sentences and their parts, we are not referring to predicates or other strings of signs, nor are we referring to attributes or sets.... The confusion begins as a confusion of sign and object; a confusion between mentioning a sign and using it. Instead of seeing *'F'* steadfastly as *standing in place* of an unspecified predicate, our confused logician sees it half the time as *naming* an unspecified predicate. Thus *'F'* gains noun status....

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3 This is the existential claim that there is a set of all sets, which, if not qualified in some way, leads to a paradox in set theory.

> Consider first some ordinary quantifications: "(∃x)(xwalks)," "(x)(x walks)," "(∃x)(x is prime)." The open sentence after the quantifier shows 'x' in a position where a name could stand; a name of a walker, for instance, or of a prime number. The quantifications do not mean that names walk or are prime; what are said to walk or to be prime are things that could be named by names in these positions. To put the predicate letter 'F' in a quantifier, then, is to treat predicate positions suddenly as name positions, and hence to treat predicates as names of entities of some sort. The quantifier '(∃F)' or '(F)' says not that some or all predicates are thus and so, but that some or all entities of the sort named by predicates are thus and so (Quine, 66-67).

In essence, 'F' is a representation of a predicate, and is joined with a variable in the construction of predication. To treat 'F' as a variable is to change its grammatical categorization (from verbal to nominal). The difference, Quine concludes, is "between schematically *simulating* predicates and quantificationally talking about predicates" (p. 67): variables take values, predicates do not. It is grammatically incorrect to put a predicate in a value-taking position. "Variables eligible for quantification therefore do not belong in predicate positions. They belong in name positions" (*ibid.*). Predicates are not names, and should not be quantified over.

Boolos remarks in the opening of his article that it is commonly supposed that the preceding arguments are decisive (p. 509). He goes on to dismantle these arguments in a systematic, complete and compelling manner. In examining Quine's argument just given above, Boolos finds that Quine's dismissal of second order quantification "on the ground that predicates are not *names* of their extensions" (Boolos, 510) is a mistaken argument by analogy. Quine supposes that because variables in first order quantifiers always occur in positions where a name, but not a predicate, can occur, this must hold of every sort of variable. But, Boolos writes,

> '(∃F)' does not have to be taken as saying that some entities of the sort named by predicates are thus and so; it can be taken to say that some of the entities (extensions) had by predicates contain thus and such. So some variables eligible for quantification might well belong in predicate positions and not in name positions. And taking 'Fx' to be true if and only if that which 'x' names is in the extension of 'F' in no way commits us to supposing that 'F' names anything at all (Boolos, 511).

Boolos is here agreeing with Quine in one respect: predicates are not names of their extensions. However, to quantify over predicates is not, as Quine asserts, to treat them as names. And while it is true that in first order logic predicates cannot be quantified, it is just this which separates first order logic from second order logic.

Once he has established that second order quantification is acceptable, Boolos points out that Quine's suggestion to re-write second order formulae in a set theoretic notation isn't much of a solution. Not only does the notation of second order logic represent aspects of logical form "in a most striking way" (Boolos, 512), while the set theoretic analogue language does not, the use of set theoretic notation also can result in the loss of validity or implication. He gives for examples two pairs of statements:

(6) (a) $\exists F\ \forall x\ Fx$ (b) $\exists\alpha\ \forall x\ x\varepsilon\alpha$

(7) (a) $\forall Y\ (Yx \rightarrow Yz)$ (b) $\forall\alpha\ (x\varepsilon\alpha \rightarrow z\varepsilon\alpha)$

(6a) is valid, but (6b) is not, and x=z is implied by (7a), but not by (7b) (*ibid.*).

As for Quine's assertion that using second order notation hides set theory's "staggering existential assumptions," Boolos responds: "all set theorists agree that there is a set containing at least two objects, but [this claim in second order logic] $\exists X\exists x\exists y(Xx\&Xy\&x\neq y)$ is not valid" (Boolos, 514).[4] Indeed, second order logic has only one existential commitment, and that only to the empty set. So much for Quine's equivocation of second order logic and set theory.

To finish his destruction of Quine's arguments, Boolos claims that second order logic is not so dissimilar from first order logic. They are different, it is true. While universal quantification in first order logic is quite rightly reckoned as pertaining to all objects, the same reckoning in second order logic leads philosophers to the biggest set-theoretic problem, the set of all sets. Second order logic is commonly regarded as being about sets. As such, all universally quantified second order statements must be understood as pertaining only to a proscribed domain, which cannot be as large as the entire logical universe. We cannot "assume that [$\exists X\ \forall x\ Xx$] ranges over all the sets that there are, for then it would be true if there were a set of all sets" (Boolos, 515).

This difference, one of a few, is not insignificant, but minimal. If we examine validity and implication, we find tremendous continuity between first order logic and second order logic. A sentence is valid in second order logic when it is true under all its interpretations. A sentence follows from others when it is true under all its interpretations under which all the others are true.

> What, then, is an interpretation of a second-order sentence? If we are considering "standard" second order logic in which second-order quantifiers are regarded as ranging over *all* subsets of, or relations on, the range of first-order quantifiers, we may answer: exactly the same sort of thing an interpretation of a first-order sentence is (Boolos, 513).

4 The sentence is false in all one-membered interpretations.

> There is a standard account of the concepts of validity and consequence for first-order sentences, and there is an obvious, straightforward, non-ad hoc way of extending that account to second-order sentences (Boolos, 514).

So, as concerns validity and implication, first order logic and second order logic are quite similar, and nearly indistinguishable. There are still small differences. Boolos cites one logic text which gives nineteen laws of validity for first order logic, all of which save one also hold for second order logic (*ibid.*).

That these minor differences should be regarded as minimal is supported by Boolos. Full first order logic can have predicates of indefinitely many places. Only up to monadic logic (predicates of one place) is first order logic decidable. Some quantified sentences containing a predicate with $n > 1$ places are undecidable. Decidability is an important property, yet first order logic as a unified system contains parts which are decidable and parts which are not (Boolos, 523).

I would add one other point. As Boolos has shown that rewriting second order statements in set theory is troublesome, and having argued that second order statements are acceptable in general, it is important to consider the place of second order logic in mathematics. Statement (2), the formalization of mathematical induction (see introduction), is central to the philosophy of mathematics, something Quine can't deny. His proposals would essentially cripple the philosophy of math by rejecting or invalidating many of the integral statements and theorems of the philosophy of mathematics.

Conclusion

Quine's argument, that logic and set theory are different and second order logic is set theory, and therefore not to be regarded as logic, has been forcefully argued against by Boolos. Quine's suggestion that the notation of second order logic be re-written in set-theoretic notation has been shown to be problematic. I conclude that Boolos is correct, and even Quine himself should be convinced upon reading Boolos.

Appendix: Symbolism

(x), (F), ∀	For all x, F--universal quantification
∃(x), ∃(F)	There exists x, F--existential quantification
.	and
v	or
~	not
→	material implication
ε	set membership

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