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Staircase Packings of Integer Partitions

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Abstract

An integer partition is a weakly decreasing sequence of positive integers. We study the family of packings of integer partitions in the triangular array of size n, where successive partitions in the packings are separated by at least one zero. We prove that these are enumerated by the Bell-Like number sequence (OEIS A091768), and investigate its many recursive properties. We also explore their poset (partially ordered set) structure. Finally, we characterize various subfamilies of these staircase packings, including one restriction that connects back to the original patterns of the whole family.

Contents

Chapter 1

Introduction

Combinatorics is a branch of mathematics that counts and enumerates discrete structures. We investigate a specified family of lists of integer partitions and characterize some of its patterns and properties. We offer an overview of the paper and some of the conclusions we will draw through our investigation.

In this chapter, we define some combinatorics terms that will be used throughout the paper, including integer partitions and posets. For any other terms or concepts that are not defined, we recommend *Introductory Combinatorics* by Richard Brualdi [7]. We also give some background on previous works that have informed the research done for this paper. Then we define the main problem being investigated: staircase packings of integer partitions, which we denote by $\mathcal{M}_n$. This family is counted by an integer sequence called the *Bell-Like number sequence*.

In Chapter 2, we will go into some background about our subfamily, as well as the interpretation of the Bell-Like number sequence as it relates to our subfamily.

In Chapter 3, we will prove that there are various bijections within the family of $\mathcal{M}_n$.

In Chapter 4, we will discuss the poset structure of $\mathcal{M}_n$. We give formulas for the number of maximal chains of minimum length and of maximum length.

In Chapter 5, we will observe what happens when further restrictions are added to our subfamily. Here we will prove the bijections between these restrictions, including one that relates back to $\mathcal{M}_n$.

Finally, Chapter 6 will conclude the paper by summarizing the main ideas and discussing future research that can be done with this problem.

1.1 Combinatorics Definitions

In this section, we will begin by defining and giving examples of some combinatorics terms that will be relevant for this paper.

1.1.1 Integer Partitions

Definition 1.1. A *partition of a positive integer* n is a representation of n as a sum of one or more positive integers, listed in weakly decreasing order, called *parts*.

Definition 1.2. The number of integer partitions of n is given by the *partition function* $p(n)$.

Figure 1.1 shows the examples for the partitions of $1, 2, 3, 4$, and 5.

$1;$

$2, 1+1;$

$3, 2+1, 1+1+1;$

$4, 3+1, 2+2, 2+1+1, 1+1+1+1;$

$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1;$

Figure 1.1 Examples for the partitions of $1, 2, 3, 4$, and 5. Here we can see that $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7$.

This paper will refer to integer partitions throughout the paper, as it will focus on counting the staircase packings of integer partitions.

1.1.2 Partially Ordered Sets

In order to define a partially ordered set, we must first define partial order.

Definition 1.3. A *partial order* $\preceq$ on a set is a binary relation that is reflexive, transitive and antisymmetric. Therefore, it is an arrangement of a set such that, for certain pairs of elements, one precedes the other.

Definition 1.4. A *partially ordered set* (or more simply, a *poset*) is a set X along with a partial order, $\preceq$. We denote this poset by $(X, \preceq)$.

We can visualize a poset and its structure using a Hasse diagram. Before defining a Hasse diagram, we must first define a covering relation.

Definition 1.5. If $(X, \preceq)$ is a poset with elements $a, b \in X$, then a is *covered* by b provided that $a \preceq b$ and no element x can be squeezed between a and b; that is, there does not exist an element x such that both $a \preceq x$ and $x \preceq b$ hold.

Definition 1.6. A *Hasse diagram* of a finite poset $(X, \preceq)$ is a directed graph whose vertices are labeled by elements of X where (x_1, x_2) is an arc if and only if x_2 covers x_1.

Typically we visualize the Hasse diagram by positioning the point x_1 below the point x_2 when $x_1 \preceq x_2$, and connecting x_1 and x_2 by a line segment if and only if x_1 is covered by x_2.

Take set X such that $X = \{1, 2, 3\}$. For every element $x \in P(X)$, we can define a covering relation such that $x_1, x_2 \in P(X), x_1 \preceq x_2$ if and only if $x_1 \subseteq x_2$. Figure 1.2 shows an example of the Hasse diagram for this set.

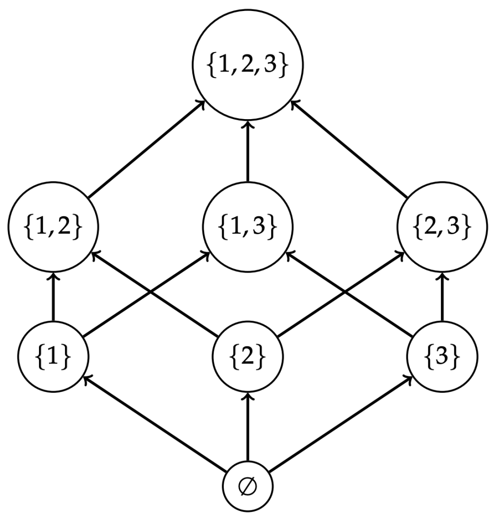


Figure 1.2 Hasse diagram of the poset $(P(X), \preceq)$ where $X = \{1, 2, 3\}$ and for $x_1, x_2 \in P(X)$, we have $x_1 \preceq x_2$ if and only if $x_1 \subseteq x_2$.

1.2 Bijections

The families we study in this paper will have relationships within themselves and other families. We want to describe one of these relationships, which are bijections.

Definition 1.7. A *bijection* is a function $f : A \to B$ between two sets A and B such that

1. For all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$, and

2. For all $b \in B$, there exists an $a \in A$ such that $f(a) = b$

If there is a bijection between two sets then there is a correspondence between their elements. This means that bijections are invertible. Also note that when two finite sets are in bijection, they have the same number of elements.

1.3 Staircase Packings of Integer Partitions

We now introduce the combinatorial family that we will investigate.

Definition 1.8. The set $\mathcal{M}_n$ of *staircase packings of integer partitions* is the collection of sequences $m = (m_1, m_2, \ldots, m_n)$ such that

- We have $0 \leq m_i \leq i$ for $1 \leq i \leq n$
- If $m_i < m_{i-1}$ then $m_i = 0$ for $2 \leq i \leq n$.

The first condition states that the ith element is less than or equal to i. The second condition states that whenever the sequence decreases, the next value must be zero. The sets $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ are shown in Figure 1.3. Note that $|\mathcal{M}_0| = 1$, $|\mathcal{M}_1| = 2$, $|\mathcal{M}_2| = 6$ and $|\mathcal{M}_3| = 22$.

We enumerated larger sets using Python code, and discovered that the sequence of sizes

$$1, 2, 6, 22, 92, 426, 2150, 11708, 68282, \ldots$$

matches sequence A091768 in the Online Encyclopedia of Integer Sequences [11].

Definition 1.9. *Bell-Like numbers* (OEIS sequence A091768) are the numbers in the sequence:

$$1, 2, 6, 22, 92, 426, 2150, 11708, 68282, \ldots$$

n	$\mathcal{M}_n$					
0	$\emptyset$					
1	(0)	(1)				
2	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
3	$(0,0,0)$	$(0,0,1)$	$(0,0,2)$	$(0,0,3)$	$(0,1,0)$	$(0,1,1)$
	$(0,1,2)$	$(0,1,3)$	$(0,2,0)$	$(0,2,2)$	$(0,2,3)$	$(1,0,0)$
	$(1,0,1)$	$(1,0,2)$	$(1,0,3)$	$(1,1,0)$	$(1,1,1)$	$(1,1,2)$
	$(1,1,3)$	$(1,2,0)$	$(1,2,2)$	$(1,2,3)$		

Figure 1.3 Staircase packings of integer partitions $\mathcal{M}_k$ for $0 \leq k \leq 3$.

which are given by the formula

$$b_1 = 1,$$
$$b_{n+1} = \frac{1}{n+1}\binom{2n}{n} + \sum_{k=0}^{n-1} \frac{k+2}{n+1}\binom{2n-k-1}{n-k-1} \quad \text{for} \quad n \geq 1.$$

We will prove that $\mathcal{M}_n$ is enumerated by the Bell-Like numbers. Furthermore, our family $\mathcal{M}_n$ of staircase packings of integer partitions is simpler than any of the examples given in the OEIS. *So our family may be the most elementary example of structures enumerated by this sequence.*

We now explain what we mean by a “staircase packing of integer partitons.” When $(m_1, m_2, \ldots, m_n)$ is viewed in reverse order, we have a packing of integer partitions that fit in the staircase shape $1, 2, \ldots, n-1, n$. Figure 1.4 shows an example. The sequence

$$m = (1,2,3,0,0,1,3,0,2,5,7,9)$$

becomes

$$(9,7,5,2,0,3,1,0,0,3,2,1)$$

which corresponds to a list of three integer partitions

$$9+7+5+2 \qquad 3+1 \qquad 3+2+1,$$

each separated by one or more zeros. The reversed list is elementwise smaller than the staircase shape

$$s = (12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1),$$

so we can view m as a packing of three integer partitions that fits in staircase shape s. In such a packing, we require that the integer partitions are separated by one or more zeros.

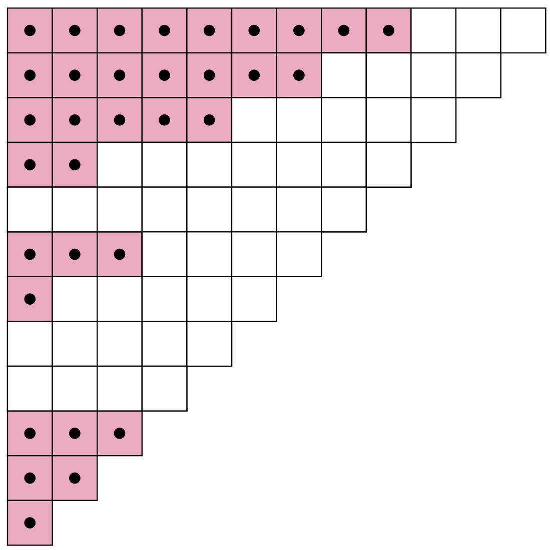
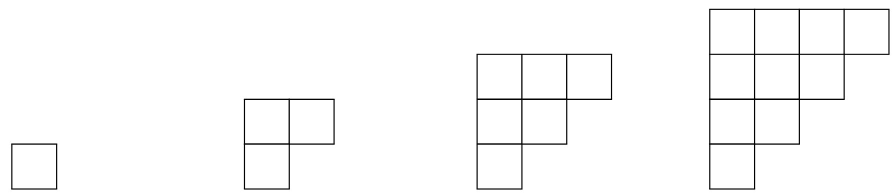


Figure 1.4 An example of an interpretation for $\mathcal{M}_n$. This is the visualization for the sequence of $m \in \mathcal{M}_{12}$ where $m = (1, 2, 3, 0, 0, 1, 3, 0, 2, 5, 7, 9)$.

1.3.1 Understanding $\mathcal{M}_n$

Having a strong understanding of $\mathcal{M}_n$ is important for this paper, so we will further investigate $\mathcal{M}_n$.

For $\mathcal{M}_n$, the bounding staircase shape will have size n. Figure 1.5, shows the empty staircase shapes for $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$. The pattern continues for all sizes n of $\mathcal{M}_n$.

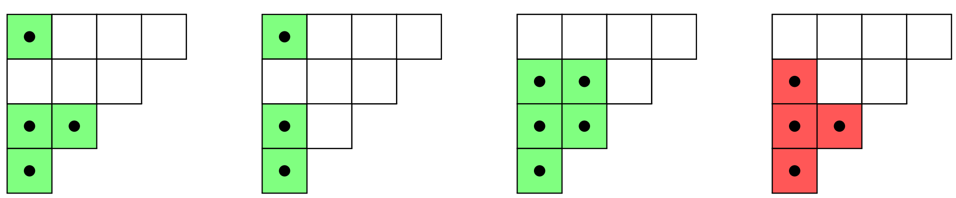


a. $\mathcal{M}_1$ **b.** $\mathcal{M}_2$ **c.** $\mathcal{M}_3$ **d.** $\mathcal{M}_4$

Figure 1.5 Empty staircase shapes for $\mathcal{M}_1$ through $\mathcal{M}_4$.

The definition of $\mathcal{M}_n$ specifies the rules for packing the staircase shape. The main restriction states that if $m_i < m_{i-1}$ then $m_i = 0$. In the corresponding staircase packing, this means that if the ith row from the bottom is smaller than the $(i-1)$th row from the bottom, then the ith row must be empty.

More intuitively, in the staircase packing, the row sizes must weakly decrease to zero as you move down (which forms an integer partition). The next non-zero row is the start of another integer partition. In other words, there are one or more buffer (empty) rows between the integer partitions in the staircase packing. Figure 1.6 illustrates what is allowed in $\mathcal{M}_n$, as well as an example of a packed staircase that would not follow the rules of $\mathcal{M}_n$.



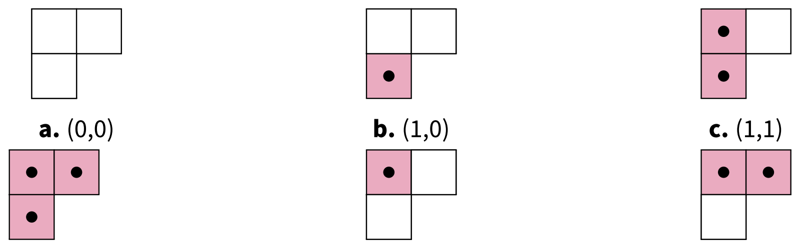
a. Allowed **b.** Allowed **c.** Allowed **d.** Not allowed

Figure 1.6 The first three are elements of $\mathcal{M}_4$. The last one is not an element of $\mathcal{M}_n$, because $1 + 2 + 1$ is not a valid integer partition.

It is through these packings, for every size of $\mathcal{M}_n$, that the numbers from the Bell-Like number sequence appear. For $n = 1$ there are 2 ways to pack the staircase shape (Figure 1.7). For $n = 2$ there are 6 ways for the staircase to be packed (Figure 1.8). For $n = 3$ there are 22 ways for the staircase to be packed (Figure 1.9). The pattern continues with $n = 4$ having 92 ways, $n = 5$ having 426 ways, $n = 6$ having 2150 ways, and so on.

a. (0) **b.** (1)

Figure 1.7 All elements of $\mathcal{M}_1$.



a. (0,0) **b.** (1,0) **c.** (1,1)

d. (1,2) **e.** (0,1) **f.** (0,2)

Figure 1.8 All elements of $\mathcal{M}_2$.

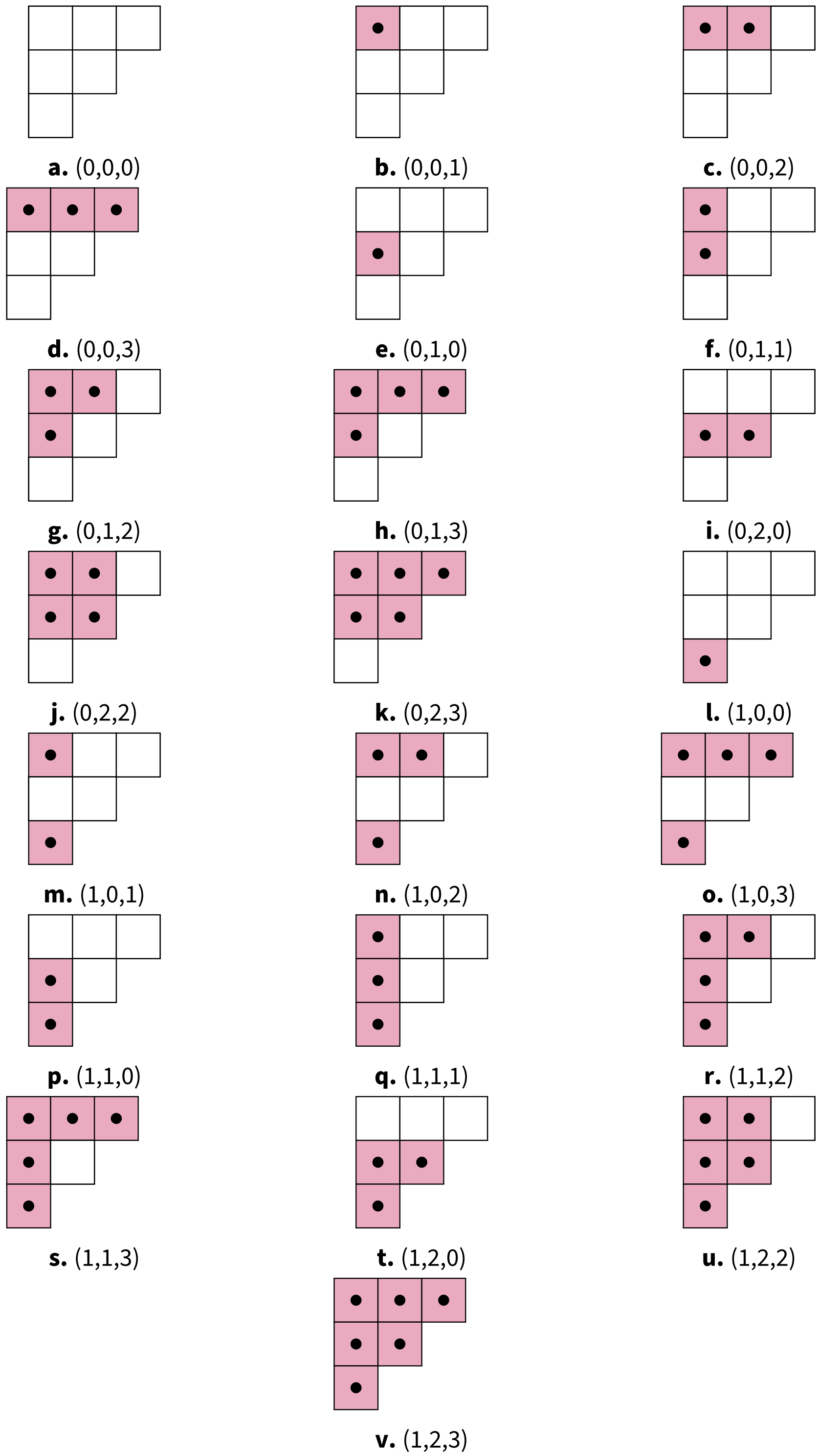


Figure 1.9 All elements of $\mathcal{M}_3$.

1.4 Previous Works

We give an overview of some work related to staircase packings of integer partitions.

1.4.1 Approval Ballot Triangles

The family $\mathcal{M}_n$ can be viewed as as subfamily of approval ballot triangles (ABTS).

Definition 1.10. An *approval ballot triangle (ABT)* of order n is a binary triangular array $A(i,j)$ for $1 \leq j \leq i \leq n-1$ satisfying the following row condition

$$\sum_{k=j}^{k} A(i,k) \leq \sum_{k=j}^{i+1} A(i+1,k) \text{ for } 1 \leq j \leq i \leq n-2.$$

We denote this family $\mathcal{A}_n$.

Note that an ABT of order n has $n-1$ rows. Intuitively, a binary triangle is an ABT when row $i+1$ ends with at least as many ones as row i.

It is straight-forward to convert a staircase packing $m \in \mathcal{M}_n$ into an ABT $A \in \mathcal{A}_{n+1}$. Given $m = (m_1, \ldots, m_n)$, let A be the ABT whose $(n-k)$th column contains $n-k-m_k$ zeros on top of m_k ones. Since each column is weakly increasing, the triangle A satisfies the ABT row condition.

Approval ballot triangles (ABTs) are in bijection with totally symmetric self-complementary plane partitions (TSSCPPs). A TSSCPP of order n is a plane partition in a $2n \times 2n \times 2n$ box with with maximum possible symmetry. Andrews [1] proved that the number of TSSCPPs of order n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!},$$

see OEIS A005130 [11]. See Bressoud [6] for a recounting of the history and mathematical connections of this remarkable formula.

Beveridge and Calaway [4] recently introduced the family of approval ballot triangles. ABTs are a binary encoding of a nest of lattice paths obtained from the fundamental domain of a TSSCPP. This was proven in [4], but equivalent encodings appear in [8] and [12].

Beveridge and Calaway show that ABTs encode an approval voting process with $n-1$ ballots in which candidate i never trails candidate j whenever $1 \leq i < j \leq n$. This generalizes many known ballot problems, including the famous Bertrand Ballot Problem for two candidates. See Barton and Mallows [3], Takàcs [14], and Renault [9] for surveys of ballot problems.

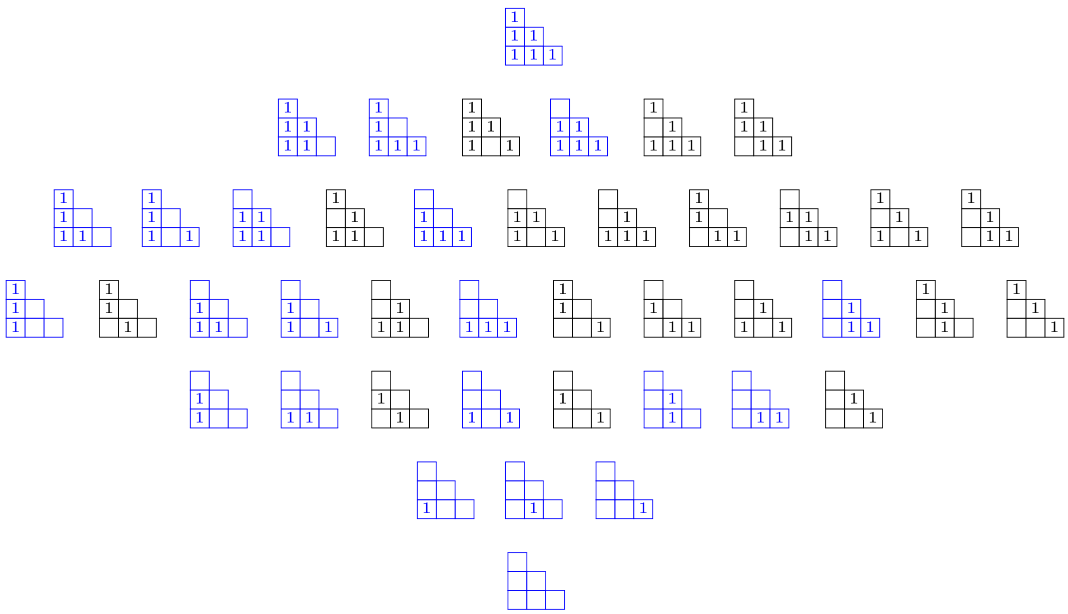


Figure 1.10 The 42 approval ballot triangles of order 4. The zero entries are rendered blank for visual clarity. The blue arrays are the 22 staircase packings of integer partitions in a triangular array of size 3.

The 42 ABTS of order 4 are shown in Figure 1.10. The 22 ABTS that correspond to staircase packings of integer partitions are shown in blue. Approval ballot triangles have many other natural subfamilies that are in bijection with famous combinatorial families, including permutations, set partitions and Catalan numbers. See [2] for an extensive list of these subfamilies and their structures.

1.4.2 Robertson's Binary Triangles

In her 2022 Macalester honors project [10], Robertson explored another family of binary triangles, this one with weakly increasing columns. We describe these triangles using the sequences corresponding to their column sums.

A binary triangle is a left-justified binary array $T(i,j)$ for $1 \leq j \leq i \leq n-1$. Robertson [10] defines a subfamily $\mathcal{L}_n$ of the binary triangle family with weakly increasing columns, T_n. If $r(i)$ is the sum of columns of i of $t \in T_n$, then this subfamily that can be defined as the following:

Definition 1.11. Let $\mathcal{L}_n = \{t \in T_n | r(i) > 0 \mapsto r(i) \geq r(i-1)$.

Definition 1.12. Let $\mathcal{R}_n$ be the set of sequences $(r_1, r_2, \ldots, r_n)$ such that

- $0 \leq r_i \leq i$
- if $r_i > r_{i-1}$ then $r_{i-1} = 0$.

Note that Robertson's second condition is the opposite of ours (if $m_i < m_{i-1}$ then $m_i = 0$). So her sequences consist of weakly increasing blocks, separated by one or more zeros.

We define a partial ordering on $\mathcal{R}$ in the natural way.

Definition 1.13. For $r, s \in \mathcal{R}_n$, we have $r \succeq s$ when $r_i \leq s_i$ for $1 \leq i \leq n$.

Having the partial order allows for the investigation of the set's maximal elements. Robertson showed that these maximal elements are counted by the Fibonacci numbers. Robertson also counted the number of maximal elements with maximum weight $w(r) = \sum_{i=1}^{n} r_i$.

1.4.3 Rectangle Packings of Integer Partitions

In the section, we discuss packings of integer partitions into rectangles of size $r \times s$. The number of such packings was determined by Birmajer, Gil and Weiner [5].

Recall that an integer partition is a weakly decreasing sequence of positive integers. More concretely, the sequence $(a_1, a_2, \ldots, a_r)$ where $a_1 \geq a_2 \geq \cdots \geq a_r$ is an integer partition of $n = \sum_{k=1}^{r} a_r$ into r parts and whose largest part is size a_1.

Rather than fixing an integer n and considering all of its integer partitions, we start with a different question:

Question 1.14. *How many integer partitions are there with at most r parts and whose largest part has size at most s?*

The answer to this question will be revealed by drawing a Ferrers diagram for each of these integer partitions inside an $r \times s$ rectangular array. For example, Figure 1.11 shows the 20 such integer partitions when $r = 3$ and $s = 3$. Furthermore, tracing the lower boundary of these integer partitions gives a bijection to lattice paths from $(0, 0)$ to $(3, 3)$ using only right steps and up steps. This shows that there are $\binom{6}{3}$ such partitions.

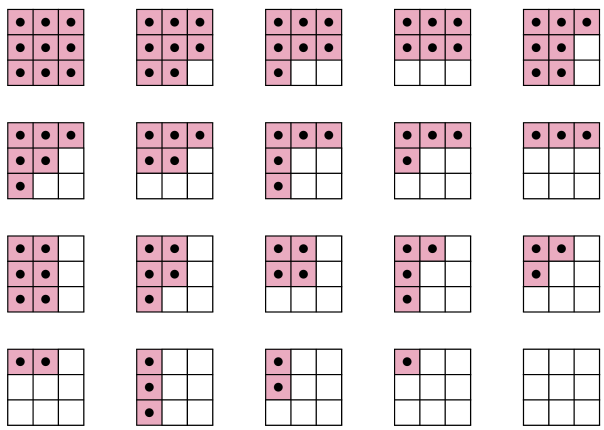


Figure 1.11 Integer partitions with at most 3 parts and whose largest part is at most 3.

The same argument applies to the general case.

Proposition 1.15. *There are* $\binom{r+s}{s}$ *integer partitions with at most* r *parts and whose largest part is at most* s.

Proof. Given such a partition $(a_1, a_2, \ldots, a_k)$, we append $r - k$ zeros so that we have a weakly decreasing sequence of nonnegative integers. The resulting sequences $(a_1, a_2, \ldots, a_r)$ are in bijection with lattice paths from $(0, 0)$ to (r, s). The i-th up step travels from $(i - 1, a_{r+1-i})$ to (i, a_{r+1-i}). □

Inspired by Figure 1.11, we can turn this into a packing problem by allowing for two generalizations.

- We can start with empty rows, if we so choose.
- We can add more integer partitions inside the rectangle, when there is room to do so. Our restriction will be that *there must be at least one empty row between integer partitions.*

Figure 1.12 shows the additional 21 valid integer partition packings in a 3×3 array. So there are a total of 41 ways to pack integer partitions in a 3×3 rectangle.

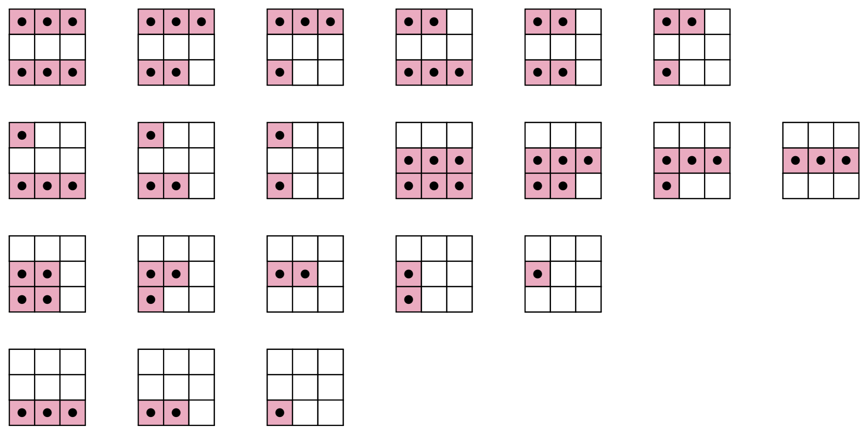


Figure 1.12 Integer partition packings in a 3×3 that either have two blocks, or one block that starts in the second or third row.

This leads to the following definition.

Definition 1.16. A *packing of integer partitions inside an $r \times s$ rectangle* is a sequence $(a_1, a_2, \ldots, a_r)$ where

- $0 \le a_i \le s$ for $1 \le i \le r$, and
- if $a_i > 0$ then $a_{i+1} \le a_i$ for $1 \le i < r$.

Adapting the work of Birmajor, Gil and Weiner [5] shows that the number of integer packings in an $r \times s$ rectangle is

$$\sum_{k=1}^{r} \binom{r + (s-1)k}{r - k + 2}.$$

Chapter 2

Bell and Bell-Like Numbers

2.1 Bell Numbers

The main result of this project is that $\mathcal{M}_n$ is enumerated by the sequence of Bell-Like numbers. This sequence is called 'Bell-Like' because they are generated by a recurrence that is similar to a recurrence for the Bell numbers, OEIS sequence A000110.

Definition 2.1. *Bell numbers* are the number of ways to partition a set of n labeled elements. The sequence of Bell numbers begins:

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, \ldots$$

We can generate the Bell numbers using a triangular recurrence, as shown in Figure 2.1. Row i contains i entries. The first row has one entry: $B_{1,1} = 1$. Assuming that we have generated row $i - 1$, here is how we generate row i. We set the first entry $B_{i,1} = B_{i-1,i-1}$, the last entry of the previous row. Then for $2 \leq i$, we set

$$B_{i,j} = B_{i,j-1} + B_{i-1,j-1}.$$

The last entry $B_{i,i}$ in row i is the ith Bell number.

$$\begin{array}{lllllll}
1 \\
1 & 2 \\
2 & 3 & 5 \\
5 & 7 & 10 & 15 \\
15 & 20 & 27 & 37 & 52 \\
52 & 67 & 87 & 114 & 151 & 203 \\
203 & 255 & 322 & 409 & 523 & 674 & 877 \\
\vdots
\end{array}$$

Figure 2.1 Triangular recurrence used to generate the Bell numbers, where the Bell numbers are the last entry of each row.

The combinatorial interpretation of the triangular recurrence, as explained by Sun and Wu [13], is as follows: $B_{i,j}$ counts the number of partitions of the set $\{1, 2, ..., i+1\}$, where the element $j+1$ is the only element of its set and each higher-numbered element is in a set of more than one element, meaning $j+1$ is the largest singleton of the partition.

Figure 2.2 shows an example of the interpretation of the triangular recurrence, where $B_{4,2}$ counts the number of partitions of $\{1, 2, 3, 4, 5\}$. In this case, $j+1=3$, so 3 is the largest singleton element and every element greater than 3, cannot be in its own subset. This interpretation can be applied to any number in the triangular recurrence used to generate the Bell numbers.

$$\{1,2,4,5\}\{3\} \quad \{1,5\}\{2,4\}\{3\} \quad \{1\}\{2\}\{4,5\}\{3\} \quad \{1,4,5\}\{2\}\{3\}$$
$$\{1,2\}\{4,5\}\{3\} \quad \{1\}\{2,4,5\}\{3\} \quad \{1,4\}\{2,5\}\{3\}$$

Figure 2.2 $B_{4,2} = 7$ counts the number of partitions of $\{1, 2, 3, 4, 5\}$ where element 3 is the largest element that is in a singleton set.

2.2 Bell-Like Numbers

Recall that the Bell-Like number sequence is:

$$1, 2, 6, 22, 92, 426, 2150, 11708, 68282, \ldots$$

The Bell-Like numbers can be generated in by using a triangular recurrence that is similar to the triangular Bell recurrence. Figure 2.3 illustrates how this new triangular recurrence can generate the numbers in the Bell-Like number sequence: they appear as the last entry of each row.

$$
\begin{array}{lllllll}
1 \\
1 & 2 \\
2 & 4 & 6 \\
6 & 10 & 16 & 22 \\
22 & 32 & 48 & 70 & 92 \\
92 & 124 & 172 & 242 & 334 & 426 \\
426 & 550 & 722 & 964 & 1298 & 1724 & 2150 \\
\vdots
\end{array}
$$

Figure 2.3 Triangular recurrence used to generate the Bell-Like numbers, where the Bell-Like numbers are the last entry of each row.

To generate this triangle, we set $B_{1,1} = 1$. Then, assuming that we have defined row $i - 1$, we set the first entry of row i to be $B_{i,1} = B_{i-1,i-1}$, the last entry of row $i - 1$. The rest of the numbers in a given row are generated by adding all the numbers in the previous column. We have

$$B_{i,j} = \sum_{k=j-1}^{i} B_{k,j-1}. \tag{2.2}$$

This is similar to how we generate Bell numbers, but instead of just adding the last two entries in the previous column, we are adding *all* of the entries from the previous column.

Figure 2.4 shows a comparison of the triangular recurrence used to generate the Bell numbers and the triangular recurrence used to generate the Bell-Like numbers. We can see that the two triangular recurrences are similar to one another.

1					
1	2				
2	3	5			
5	7	10	15		
15	20	27	37	52	
52	67	87	114	151	203
⋮					

a. Triangular recurrence used to generate the Bell numbers

1					
1	2				
2	4	6			
6	10	16	22		
22	32	48	70	92	
92	124	172	242	334	426
⋮					

b. Triangular recurrence used to generate the Bell-Like numbers

Figure 2.4 A comparison of the two triangular recurrences used to generate the Bell numbers and Bell-Like numbers.

2.2.1 Combinatorial Interpretation

Recall that the set $\mathcal{M}_n$ of staircase packings of integer partitions is the collection of the sequences $m = (m_1, m_2, \ldots, m_n)$ such that

- We have $0 \le m_i \le i$ for $1 \le i \le n$
- If $m_i < m_{i-1}$ then $m_i = 0$ for $2 \le i \le n$.

Using the elements of the subsets of $\mathcal{M}_n$, we give a combinatorial interpretation of the triangular recurrence Equation 2.2 used to generate Bell-Like numbers. We will show that $|\mathcal{M}_n| = B_{n+1,n+1}$, and the other $B_{i,j}$ count particular elements of the subsets of $\mathcal{M}_n$.

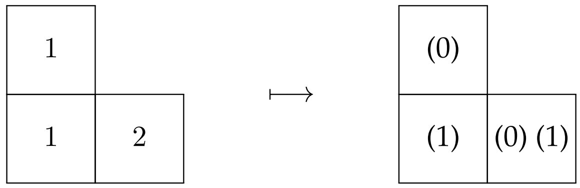
We start by looking at some examples. The subset $\mathcal{M}_1$ and its elements are represented by the triangle

1	
1	2

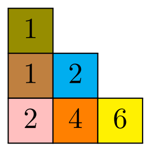
We have $|\mathcal{M}_1| = B_{2,2} = 2$, and indeed

$$\mathcal{M}_1 = \{(0), (1)\}.$$

Next, $B_{2,1} = 1$ counts the elements of $\mathcal{M}_1$ that **do not** end in **0**. In this case the only element would be (1). Finally, $B_{1,1}$ counts the elements of $\mathcal{M}_1$ that **do** end in 0. The only element would be (0). Visually, our interpretation is



The subset $\mathcal{M}_2$ and its elements are represented by the following triangular recurrence of Figure 2.5.



a. $B_{3,3} = 6$

$B_{i,j}$	Interpretation	$\mathcal{M}_2$ elements					
$B_{3,3}$	All of $\mathcal{M}_2$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$B_{3,2}$	end in **not 0**		$(0,1)$	$(0,2)$		$(1,1)$	$(1,2)$
$B_{2,2}$	end in 0	$(0,0)$			$(1,0)$		
$B_{3,1}$	end in **not 1** and **not 0**			$(0,2)$			$(1,2)$
$B_{2,1}$	end in (**not 1**,1)		$(0,1)$				
$B_{1,1}$	end in $(1,1)$					$(1,1)$	

b. $B_{3,3}$ counts *all* the elements of $\mathcal{M}_2$

Figure 2.5 Showing how this triangle counts the sequences of $\mathcal{M}_2$.

Here is how to interpret this triangle, as illustrated in Figure 2.5. We describe the triangle, column by column, starting from the right.

- $B_{3,3}$ counts *all* the sequences of $\mathcal{M}_2$.
- $B_{3,2}$ counts all of the sequences of $\mathcal{M}_2$ that end in **not 0**
- $B_{2,2}$ counts all of the sequences of $\mathcal{M}_2$ that end in 0.
- $B_{3,1}$ counts all of the sequences of $\mathcal{M}_2$ that end in **not 1** and **not 0**. In this case, this is the same as counting sequences that end in 2.
- $B_{2,1}$ counts all of the sequences of $\mathcal{M}_2$ that end in (**not 1**, 1). In this case, this is the same as counting sequences that end in $(0,1)$.

- $B_{1,1}$ counts all of the sequences of $\mathcal{M}_2$ that end in $(1,1)$. .

With these interpretations, we can see why each entry is the sum of the entries in the previous column, in the same row and above.

- $B_{3,3} = B_{2,3} + B_{2,2}$ because the set of all sequences counted by $B_{3,3}$ consists of the sequences that do not end in 0 (which are counted by $B_{3,2}$ and sequences that do end in 0 which are counted by $B_{2,2}$).
- $B_{3,2} = B_{3,1} + B_{2,1} + B_{1,1}$ because the set of sequences that do not end in 0 can be partitioned into three subsets: those that do not end in either 0 or 1 (that is, sequences that end in 2), those that end in $(0,1)$, and those that end in $(1,1)$.

2.2.2 Interpretation Notation

Now that we have a basis for the interpretation of the triangular recurrence used to generate the Bell-Like numbers, we can generalize the interpretation for any size of $\mathcal{M}_n$. We define the general pattern for the subsets of $\mathcal{M}_n$ that are counted by entries of the triangle. Each of these sets specifies constraints on the last entry (or entries) of a sequence $m \in \mathcal{M}_n$.

Definition 2.3. We define the following subsets of $\mathcal{M}_n$.

- $\mathcal{M}(n,k) = \{m \in \mathcal{M}_n : m_n = k\}$
- $\mathcal{M}(n,\widetilde{0}) = \{m \in \mathcal{M}_n : m_n \neq 0\}$
- $\mathcal{M}(n,\widetilde{1}) = \{m \in \mathcal{M}_n : m_n = n\}$
- $\mathcal{M}(n,\widetilde{k}) = \{m \in \mathcal{M}_n : \text{either } 1 \leq m_n \leq k-1 \text{ or } m_n = n\}$
- $\mathcal{M}(n,\widetilde{k}k) = \{m \in \mathcal{M}_n : m_n = k \text{ and } m_{n-1} \neq k\}$
- $\mathcal{M}(n,\widetilde{k}kk) = \{m \in \mathcal{M}_n : m_n = m_{n-1} = k \text{ and } 0 \leq m_{n-2} \leq k-1\}$

Here, we are using the $\widetilde{k}$ notation in two slightly different ways for $m = (m_1, m_2, \ldots, m_n)$.

- The symbol $\widetilde{0}$ means that the last entry m_n is nonzero.
- For $k > 0$, the symbol $\widetilde{k}$ means that the corresponding entry m_j of the sequence is either $1 \leq m_j \leq k-1$, or that $m_j = n$ (which is only allowed when $j = n$ because $m_j \leq j$ by definition).

More generally, we can define $\mathcal{M}(n, \widetilde{k}k^r)$, which specifies the last $r + 1$ entries of $m \in \mathcal{M}_n$.

Definition 2.4. We define

$$\begin{aligned}\mathcal{M}(n, \widetilde{k}k^r) \quad = \quad \{m \in \mathcal{M}_n : \quad & m_i = k \text{ for } i > n - r \\ & \text{and } m_{n-r} \neq k\}.\end{aligned}$$

Note that

$$\mathcal{M}(n, \widetilde{k}k^0) = \mathcal{M}(n, \widetilde{k}),$$

and that

$$\mathcal{M}(n, \widetilde{k}k^{n-k+1}) = \mathcal{M}(n, k^{n-k+1})$$

because m_{k-1} cannot be larger than $k - 1$.

Similarly, we can also define $\mathcal{M}(n, k^r)$, which specifies the last r entries of $m \in \mathcal{M}_n$.

Definition 2.5. We define

$$\mathcal{M}(n, k^r) = \{m \in \mathcal{M}_n : m_i = k \text{ for } i > n - r\}.$$

In the previous section, we gave an explanation of our interpretation (see Figure 2.5). Now we can apply this new notation to that interpretation, meaning that this notation is also our interpretation of $B_{i,j}$ on the sequences of $\mathcal{M}_n$. Figure 2.6 shows the notation used to indicate the sequences of $\mathcal{M}_1$ and Figure 2.7 shows the notation used to indicate the sequences of $\mathcal{M}_2$.

$B_{i,j}$	$\mathcal{M}(1, \widetilde{k}k^r)$	Interpretation	$\mathcal{M}_1$ sequences
$B_{2,2}$	$\mathcal{M}_1$	All of $\mathcal{M}_1$	(0) (1)
$B_{2,1}$	$\mathcal{M}(1, \widetilde{0})$	ends in **not 0**	(1)
$B_{1,1}$	$\mathcal{M}(1, 0)$	ends in 0	(0)

Figure 2.6 $\mathcal{M}(1, \widetilde{k}k^r)$ notation being used to indicate the sequences of $\mathcal{M}_1$ counted by $B_{i,j}$.

$B_{i,j}$	$\mathcal{M}(2, \widetilde{k}k^r)$	Pattern	$\mathcal{M}_2$ sequences
$B_{3,3}$	$\mathcal{M}_2$	All of $\mathcal{M}_2$	$(0,0)$ $(0,1)$ $(0,2)$ $(1,0)$ $(1,1)$ $(1,2)$
$B_{3,2}$	$\mathcal{M}(2, \widetilde{0})$	ends in **not 0**	$(0,1)$ $(0,2)$ $(1,1)$ $(1,2)$
$B_{2,2}$	$\mathcal{M}(2, 0)$	ends in 0	$(0,0)$ $(1,0)$
$B_{3,1}$	$\mathcal{M}(2, \widetilde{1})$	ends in **not 1** and **not 0**	$(0,2)$ $(1,2)$
$B_{2,1}$	$\mathcal{M}(2, \widetilde{1}1)$	ends in (**not 1**, 1)	$(0,1)$
$B_{1,1}$	$\mathcal{M}(2, 11)$	ends in (1, 1)	$(1,1)$

Figure 2.7 $\mathcal{M}(2, \widetilde{k}k^r)$ notation being used to indicate the sequences of $\mathcal{M}_2$ counted by $B_{i,j}$.

Now that we have this notation, we can apply it to any $\mathcal{M}_n$. Figure 2.8 shows the notation for the sequences of $\mathcal{M}_3$ that are counted by every $B_{i,j}$. The interpretation, and pattern, are similar to that of $\mathcal{M}_1$ and $\mathcal{M}_2$.

$B_{i,j}$	$\mathcal{M}(3, \widetilde{k}k^r)$	Pattern	$\mathcal{M}_3$ sequences
$B_{4,4}$	$\mathcal{M}_3$	All of $\mathcal{M}_3$	$(0,0,0)$ $(0,0,1)$ $(0,0,2)$ $(0,0,3)$ $(0,1,0)$ $(0,1,1)$ $(0,1,2)$ $(0,1,3)$ $(0,2,0)$ $(0,2,2)$ $(0,2,3)$ $(1,0,0)$ $(1,0,1)$ $(1,0,2)$ $(1,0,3)$ $(1,1,0)$ $(1,1,1)$ $(1,1,2)$ $(1,1,3)$ $(1,2,0)$ $(1,2,2)$ $(1,2,3)$
$B_{4,3}$	$\mathcal{M}(3, \widetilde{0})$	ends in **not 0**	$(0,0,1)$ $(0,0,2)$ $(0,0,3)$ $(0,1,1)$ $(0,1,2)$ $(0,1,3)$ $(0,2,2)$ $(0,2,3)$ $(1,0,1)$ $(1,0,2)$ $(1,0,3)$ $(1,1,1)$ $(1,1,2)$ $(1,1,3)$ $(1,2,2)$ $(1,2,3)$
$B_{3,3}$	$\mathcal{M}(3, 0)$	ends in 0	$(0,0,0)$ $(0,1,0)$ $(0,2,0)$ $(1,0,0)$ $(1,1,0)$ $(1,2,0)$
$B_{4,2}$	$\mathcal{M}(3, \widetilde{2})$	ends in **not 2**, and **not 0**	$(0,0,1)$ $(0,0,3)$ $(0,1,1)$ $(0,1,3)$ $(0,2,3)$ $(1,0,1)$ $(1,0,3)$ $(1,1,1)$ $(1,1,3)$ $(1,2,3)$
$B_{3,2}$	$\mathcal{M}(3, \widetilde{2}2)$	ends in (**not 2**, 2)	$(0,0,2)$ $(0,1,2)$ $(1,0,2)$ $(1,1,2)$
$B_{2,2}$	$\mathcal{M}(3, 22)$	ends in (2, 2)	$(0,2,2)$ $(1,2,2)$
$B_{4,1}$	$\mathcal{M}(3, \widetilde{1})$	ends in **not 1**, **not 2**, and **not 0**	$(0,0,3)$ $(0,1,3)$ $(0,2,3)$ $(1,0,3)$ $(1,1,3)$ $(1,2,3)$
$B_{3,1}$	$\mathcal{M}(3, \widetilde{1}1)$	ends in (**not 1**, 1)	$(0,0,1)$ $(1,0,1)$
$B_{2,1}$	$\mathcal{M}(3, \widetilde{1}11)$	ends in (**not 1**, 1, 1)	$(0,1,1)$
$B_{1,1}$	$\mathcal{M}(3, 111)$	ends in (1, 1, 1)	$(1,1,1)$

Figure 2.8 $\mathcal{M}(3, \widetilde{k}k^r)$ notation being used to indicate the sequences of $\mathcal{M}_3$ counted by $B_{i,j}$.

Since $\mathcal{M}_4$ has 92 elements, we will not explicitly enumerate how all the subsets are counted by the triangular recurrence. However, Figure 2.9 shows which sequences would be counted by the triangular recurrence using our new notation. Recall that every element of $\mathcal{M}_4$ can be written as $m = (m_1, m_2, m_3, m_4)$.

$$\begin{array}{lllll}
\mathcal{M}(4,1111) & & & & \\
\mathcal{M}(4,\widetilde{1}111) & \mathcal{M}(4,222) & & & \\
\mathcal{M}(4,\widetilde{1}11) & \mathcal{M}(4,\widetilde{2}22) & \mathcal{M}(4,33) & & \\
\mathcal{M}(4,\widetilde{1}1) & \mathcal{M}(4,\widetilde{2}2) & \mathcal{M}(4,\widetilde{3}3) & \mathcal{M}(4,0) & \\
\mathcal{M}(4,\widetilde{1}) & \mathcal{M}(4,\widetilde{2}) & \mathcal{M}(4,\widetilde{3}) & \mathcal{M}(4,\widetilde{0}) & \mathcal{M}_4
\end{array}$$

a. Notation for $\mathcal{M}_4$.

$$\begin{array}{lllll}
(1,1,1,1) & & & & \\
(\widetilde{1},1,1,1) & (m_1,2,2,2) & & & \\
(m_1,\widetilde{1},1,1) & (m_1,\widetilde{2},2,2) & (m_1,m_2,3,3) & & \\
(m_1,m_2,\widetilde{1},1) & (m_1,m_2,\widetilde{2},2) & (m_1,m_2,\widetilde{3},3) & (m_1,m_2,m_3,0) & \\
(m_1,m_2,m_3,\widetilde{1}) & (m_1,m_2,m_3,\widetilde{2}) & (m_1,m_2,m_3,\widetilde{3}) & (m_1,m_2,m_3,\widetilde{0}) & (m_1,m_2,m_3,m_4)
\end{array}$$

b. Sequences of $\mathcal{M}_4$ that correspond to the notation in Figure 2.9a.

Figure 2.9 $\mathcal{M}(4,\widetilde{k}k^r)$ notation being used to indicate the sequences of $\mathcal{M}_4$

To fully understand this pattern, we can analyze what the notation represents for $\mathcal{M}_4$.

- $\mathcal{M}(4,\widetilde{0})$ represents the subset of all elements of $\mathcal{M}_4$ that end in **not 0** and $\mathcal{M}(4,0)$ represents all subsets of $\mathcal{M}_4$ that end in 0.
- $\mathcal{M}(4,\widetilde{3})$ represents the subset of all elements of $\mathcal{M}_4$ that end in **not 3** and **not 0**. This means that an element in $\mathcal{M}(4,\widetilde{3})$ can end in $1, 2$, or 4.
- $\mathcal{M}(4,\widetilde{3}3)$ represents the subset of all elements of $\mathcal{M}_4$ that end in (**not 3**, 3).
- $\mathcal{M}(4,33)$ represents the subset of all elements of $\mathcal{M}_4$ end in $(3,3)$.

- $\mathcal{M}(4,\widetilde{2})$ represents the subset of all elements of $\mathcal{M}_4$ that end in **not 2** and **not 3** and **not 0**. This means that an element in $\mathcal{M}(4,\widetilde{2})$ can end in 1 or 4.

- $\mathcal{M}(4,\widetilde{2}2)$ represents the subset of all elements of $\mathcal{M}_4$ that end in (**not 2**,2).

- $\mathcal{M}(4,\widetilde{2}22)$ represents the subset of all elements of $\mathcal{M}_4$ that end in (**not 2**,$2,2$).

- $\mathcal{M}(4,222)$ represents the subset of all elements of $\mathcal{M}_4$ that end in $(2,2,2)$.

- $\mathcal{M}(4,\widetilde{1})$ represents the subset of all elements of $\mathcal{M}_4$ that end in **not 1** and **not 2** and **not 3** and **not 0**. This means that an element in $\mathcal{M}(4,\widetilde{2})$ can end in 4.

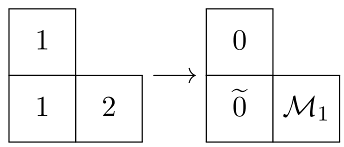
- $\mathcal{M}(4,\widetilde{1}1)$ represents the subset of all elements of $\mathcal{M}_4$ that end in (**not 1**,1).

- $\mathcal{M}(4,\widetilde{1}11)$ represents the subset of all elements of $\mathcal{M}_4$ that end in (**not 1**,$1,1$).

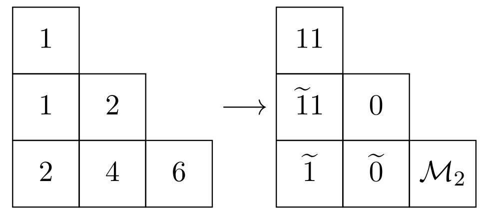
- $\mathcal{M}(4,\widetilde{1}111)$ represents the subset of all elements of $\mathcal{M}_4$ that end in (**not 1**,$1,1,1$).

- $\mathcal{M}(4,1111)$ represents the subset of all elements of $\mathcal{M}_4$ that end in $(1,1,1,1)$.

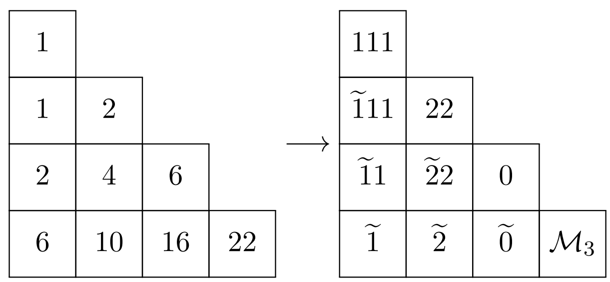
Through the generalized pattern and interpretation for $\mathcal{M}_n$ we can see that $\mathcal{M}_n$ is enumerated by the Bell-Like number triangular recurrence and in turn, the Bell-Like numbers themselves. Figure 2.10 shows how the triangular recurrence relates to the pattern for $\mathcal{M}_1$ through $\mathcal{M}_4$.



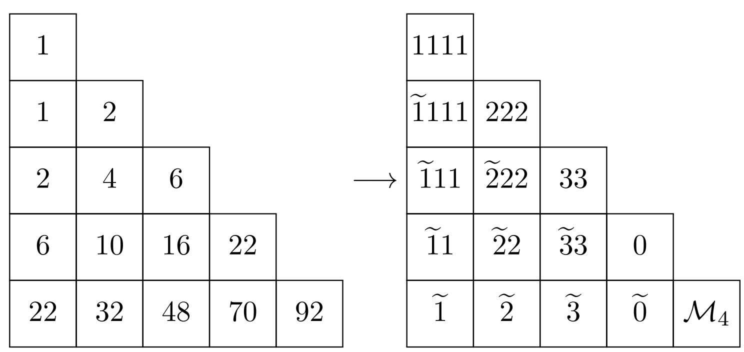
a. $B_{i,j}$ mapping to the interpretation of the subsets of $\mathcal{M}_1$



b. $B_{i,j}$ mapping to the interpretation of the subsets of $\mathcal{M}_2$



c. $B_{i,j}$ mapping to the interpretation of the subsets of $\mathcal{M}_3$



d. $B_{i,j}$ mapping to the interpretation of the subsets of $\mathcal{M}_4$

Figure 2.10 The triangular recurrence mapped to the interpretation of the subsets of $\mathcal{M}_1$ through $\mathcal{M}_4$.

Later, we will discuss the different properties of the columns of $\mathcal{M}_n$. Therefore, we will provide these columns with their own notation.

Definition 2.6. Let the columns of $\mathcal{M}_n$ be defined as

$$\mathcal{M}_n^0 = \mathcal{M}(n, \widetilde{0}) \cup \mathcal{M}(n, 0)$$

and for $1 \leq k \leq n-1$, we define

$$\mathcal{M}_n^k = \mathcal{M}(n, \widetilde{k}) \cup \mathcal{M}(n, \widetilde{k}k) \cup \mathcal{M}(n, \widetilde{k}kk) \cup \cdots$$

Here is an example of how we would use this column notation for $\mathcal{M}_4$:

- $\mathcal{M}_4^0$ is the column that includes $\mathcal{M}(4, \widetilde{0})$ and $\mathcal{M}(4, 0)$.
- $\mathcal{M}_4^3$ is the column that includes $\mathcal{M}(4, \widetilde{3}), \mathcal{M}(4, \widetilde{3}3)$, and $\mathcal{M}(4, 33)$.
- $\mathcal{M}_4^2$ is the column that includes $\mathcal{M}(4, \widetilde{2}), \mathcal{M}(4, \widetilde{2}2), \mathcal{M}(4, \widetilde{2}22)$, and $\mathcal{M}(4, 222)$.
- $\mathcal{M}_4^1$ is the column that includes $\mathcal{M}(4, \widetilde{1}), \mathcal{M}(4, \widetilde{1}1), \mathcal{M}(4, \widetilde{1}11), \mathcal{M}(4, \widetilde{1}111)$, and $\mathcal{M}(4, 1111)$.

Figure 2.11 shows the column notation being used for $\mathcal{M}_4$.

1111				
$\widetilde{1}111$	222			
$\widetilde{1}11$	$\widetilde{2}22$	33		
$\widetilde{1}1$	$\widetilde{2}2$	$\widetilde{3}3$	0	
$\widetilde{1}$	$\widetilde{2}$	$\widetilde{3}$	$\widetilde{0}$	$\mathcal{M}_4$
$\mathcal{M}_4^1$	$\mathcal{M}_4^2$	$\mathcal{M}_4^3$	$\mathcal{M}_4^0$	$\mathcal{M}_4$

Figure 2.11 The column notation for the interpretation of the subsets $\mathcal{M}_4$.

Chapter 3

Bijections Within the Bell-Like Sequence Interpretation

In this section we will prove $\mathcal{M}_n$ is counted by the Bell-Like number sequence from Section 2.2. First, we discuss some of the properties that come from the interpretation of the triangular recurrence as it relates to the subsets of $\mathcal{M}_n$. With this interpretation there are various bijections that appear. We will start by studying some of these bijections.

3.1 Column and Ends In 'Not' Mapping

Recall that $\mathcal{M}(n, \widetilde{k+1})$ are the sequences $m \in \mathcal{M}_n$ such that

$$m = (m_1, m_2, \ldots, m_{n-1}, m_n)$$

where either $1 \leq m_n \leq k$ or $m_n = n$.

Also recall that $\mathcal{M}_n^k$ represents the column k of $\mathcal{M}_n$, where the column includes $\mathcal{M}(n, \widetilde{k}) \cup \mathcal{M}(n, \widetilde{k}k) \cup \mathcal{M}(n, \widetilde{k}kk) \cup \cdots$.

We will show that

$$\mathcal{M}(n, \widetilde{k+1}) = \mathcal{M}_n^k.$$

We first provide an example of the bijection between these sets: Figure 3.1 shows how to bijectively map $\mathcal{M}(3, \widetilde{2})$ to $\mathcal{M}_3^1$. The column $\mathcal{M}_3^1$ is the first column where all the entries are highlighted.

111			
$\widetilde{111}$	22		
$\widetilde{11}$	$\widetilde{22}$	0	
$\widetilde{1}$	$\widetilde{2}$	$\widetilde{0}$	all

Figure 3.1 $\mathcal{M}(3,\widetilde{2}) = \mathcal{M}_3^1$.

Figure 3.2 shows all the sequences of $\mathcal{M}(3,\widetilde{2})$ and all the sequences of the column $\mathcal{M}_2^1$. In this example we can see that every sequence of $\mathcal{M}(3,\widetilde{2})$ is also a sequence in $\mathcal{M}_2^1$.

$\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(3,\widetilde{1})$	$\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(3,111)$
$(0,0,3) \mapsto (0,0,3)$	$(1,1,1) \mapsto (1,1,1)$
$(0,1,3) \mapsto (0,1,3)$	
$(0,2,3) \mapsto (0,2,3)$	$\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(3,\widetilde{111})$
$(1,0,3) \mapsto (1,0,3)$	$(0,1,1) \mapsto (0,1,1)$
$(1,1,3) \mapsto (1,1,3)$	
$(1,2,3) \mapsto (1,2,3)$	$\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(3,\widetilde{11})$
	$(0,0,1) \mapsto (0,0,1)$
	$(1,0,1) \mapsto (1,0,1)$

Figure 3.2 Explicitly showing how $\mathcal{M}(3,\widetilde{2}) = \mathcal{M}_3^1$.

Now that we see that

$$\mathcal{M}(3,\widetilde{2}) = \mathcal{M}(3,\widetilde{1}) \cup \mathcal{M}(3,\widetilde{11}) \cup \mathcal{M}(3,\widetilde{111}) \cup \mathcal{M}(3,111) = \mathcal{M}_3^1,$$

we can generalize this statement to apply to all n, k, and $k+1$.

Lemma 3.1. *We have*

$$\mathcal{M}(n, \widetilde{k+1}) = \mathcal{M}_n^k = \bigcup_{r=0}^{n-k+1} \mathcal{M}(n, \widetilde{k}k^r).$$

Proof. Pick any $m \in \mathcal{M}(n, \widetilde{k+1})$. We have $m = (m_1, m_2, m_3, \ldots, m_n)$ where $1 \leq m_n \leq k$ or $m_n = n$. We look at how many k's appear at the tail end of m. This partitions $\mathcal{M}(n, \widetilde{k+1})$ into disjoint sets.

- If $m_n \neq k$ then $m \in \mathcal{M}(n, \widetilde{k})$.
- If $m_n = k$ and $0 \leq m_{n-1} \leq k-1$ then $m \in \mathcal{M}(n, \widetilde{k}k)$.
- If $m_n = m_{n-1} = k$ and $0 \leq m_{n-2} \leq k-1$ then $m \in \mathcal{M}(n, \widetilde{k}kk)$.
- If $m_n = m_{n-1} = \cdots = m_{n-r+1} = k$ and $0 \leq m_{n-r} \leq k-1$ then $m \in \mathcal{M}(n, \widetilde{k}k^r)$.
- If $m_i = k$ for $n-r+1 \leq i \leq n$ and $0 \leq m_{n-r} \leq k-1$ then $m \in \mathcal{M}(n, \widetilde{k}k^r)$. This holds for $0 \leq r \leq n-k+1$

This covers all possible suffixes for $m \in \mathcal{M}(n, \widetilde{k+1})$, and we are done. □

An example of what was proven above can be seen in Figure 3.1 Here we see that $\mathcal{M}(3, \widetilde{2}) = \mathcal{M}_3^1$. Figure 3.2 shows how every m in $\mathcal{M}(3, \widetilde{2})$ is equal to every restriction from the previous column, $\mathcal{M}_3^1$.

3.2 All Sequences End in 0 or End in Not 0

Next we want to show that all sequences of $\mathcal{M}_n$ map to the column $\mathcal{M}_n^0$. Recall that $\mathcal{M}_n^0 = \mathcal{M}(n, \widetilde{0}) \cup \mathcal{M}(n, 0)$. So the claim that $\mathcal{M}_n$ maps to the column $\mathcal{M}_n^0$ is saying that every sequence in $\mathcal{M}_n$ ends in either not 0 or 0.

Let us start with an example. Take $\mathcal{M}_3$ and $\mathcal{M}_3^0$, as shown in Figure 3.3.

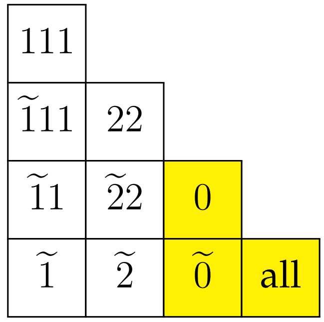


Figure 3.3 $\mathcal{M}_3 = \mathcal{M}_3^0$

The mapping is given in Figure 3.4. Clearly, $\mathcal{M}(3, \widetilde{0})$ and $\mathcal{M}(3, 0)$ are disjoint subsets of $\mathcal{M}_3$, and each sequence in $\mathcal{M}_3$ belongs to one of these two sets.

$$\mathcal{M}_3 \mapsto \mathcal{M}(3, \widetilde{0})$$

$(0,0,1) \mapsto (0,0,1)$	$(1,0,1) \mapsto (1,0,1)$
$(0,0,2) \mapsto (0,0,2)$	$(1,0,2) \mapsto (1,0,2)$
$(0,0,3) \mapsto (0,0,3)$	$(1,0,3) \mapsto (1,0,3)$
$(0,1,1) \mapsto (0,1,1)$	$(1,1,1) \mapsto (1,1,1)$
$(0,1,2) \mapsto (0,1,2)$	$(1,1,2) \mapsto (1,1,2)$
$(0,1,3) \mapsto (0,1,3)$	$(1,1,3) \mapsto (1,1,3)$
$(0,2,2) \mapsto (0,2,2)$	$(1,2,2) \mapsto (1,2,2)$
$(0,2,3) \mapsto (0,2,3)$	$(1,2,3) \mapsto (1,2,3)$

$$\mathcal{M}_3 \mapsto \mathcal{M}(3, 0)$$

$(0,0,0) \mapsto (0,0,0)$	$(1,0,0) \mapsto (1,0,0)$
$(0,1,0) \mapsto (0,1,0)$	$(1,1,0) \mapsto (1,1,0)$
$(0,2,0) \mapsto (0,2,0)$	$(1,2,0) \mapsto (1,2,0)$

Figure 3.4 Explicitly showing how $\mathcal{M}_3 = \mathcal{M}_3^0$

We can show this will always be the case for any $\mathcal{M}_n$ and $\mathcal{M}_n^0$.

Lemma 3.2. $\mathcal{M}_n = \mathcal{M}_n^0$

Proof. Pick any $m \in \mathcal{M}_n$. We have $m = (m_1, m_2, m_3, \ldots, m_n)$ where $0 \leq m_n \leq n$. We can partition $\mathcal{M}_n$ into two disjoint sets: sequences with $m_n = 0$ and sequences with $m_n \neq 0$. These are preccisely the sets $\mathcal{M}(n, 0)$ and $\mathcal{M}(n, \widetilde{0})$.Therefore $\mathcal{M}_n = \mathcal{M}_n^0$. □

3.3 Bijection Between Two Sizes

We now explore the bijections that occur between $\mathcal{M}_n$ and $\mathcal{M}_{n+1}$.

To begin, we explore the mappingsbetween $\mathcal{M}_n$ and $\mathcal{M}(n+1, 0)$. We start with the example $\mathcal{M}_3$ and $\mathcal{M}(4, 0)$ as shown in Figure 3.5.

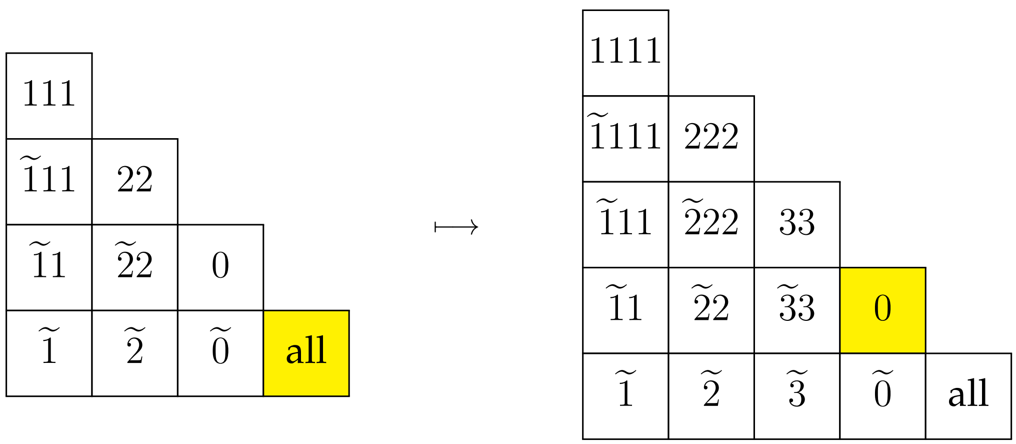


Figure 3.5 $\mathcal{M}_3 \mapsto \mathcal{M}(4, 0)$.

Recall that $\mathcal{M}(4, 0)$ is all the sequences of $m \in \mathcal{M}_4$ that end in 0, which look like $m = (m_1, m_2, m_3, 0)$. On the other hand, $\mathcal{M}_3$ are all the sequences of $m \in \mathcal{M}_3$, which look like $m = (m_1, m_2, m_3)$.

We map a sequence in $\mathcal{M}_3$ to a sequence in $\mathcal{M}(4, 0)$ by appending a 0. We exhibit this mapping in Figure 3.6. For example, the sequence $(1, 2, 3) \in \mathcal{M}_3$ maps to $(1, 2, 3, 0) \in \mathcal{M}(4, 0)$.

$$\mathcal{M}_3 \mapsto \mathcal{M}(4,0)$$

$(0,0,0) \mapsto (0,0,0,0)$	$(0,2,0) \mapsto (0,2,0,0)$	$(1,1,1) \mapsto (1,1,1,0)$
$(0,0,1) \mapsto (0,0,1,0)$	$(0,2,2) \mapsto (0,2,2,0)$	$(1,1,2) \mapsto (1,1,2,0)$
$(0,0,2) \mapsto (0,0,2,0)$	$(0,2,3) \mapsto (0,2,3,0)$	$(1,1,3) \mapsto (1,1,3,0)$
$(0,0,3) \mapsto (0,0,3,0)$	$(1,0,0) \mapsto (1,0,0,0)$	$(1,2,0) \mapsto (1,2,0,0)$
$(0,1,0) \mapsto (0,1,0,0)$	$(1,0,1) \mapsto (1,0,1,0)$	$(1,2,2) \mapsto (1,2,2,0)$
$(0,1,1) \mapsto (0,1,1,0)$	$(1,0,2) \mapsto (1,0,2,0)$	$(1,2,3) \mapsto (1,2,3,0)$
$(0,1,2) \mapsto (0,1,2,0)$	$(1,0,3) \mapsto (1,0,3,0)$	
$(0,1,3) \mapsto (0,1,3,0)$	$(1,1,0) \mapsto (1,1,0,0)$	

Figure 3.6 Explicitly showing how $\mathcal{M}_3 \mapsto \mathcal{M}(4,0)$.

Now that we can see the mapping between $\mathcal{M}_3$ and $\mathcal{M}(4,0)$, we can generalize this mapping to all n and $n+1$.

Lemma 3.3. *There exists a bijection between $\mathcal{M}_n$ and $\mathcal{M}(n+1,0)$.*

Proof. We define a mapping $f : \mathcal{M}_n \to \mathcal{M}(n+1,0)$. We simply append a zero to the sequence. Our mapping is

$$f : (m_1, m_2, m_3, \dots, m_n) \mapsto (m_1, m_2, m_3, \dots, m_n, 0).$$

This mapping is well-defined, and it is invertible. The inverse mapping is $g : \mathcal{M}(n+1,0) \to \mathcal{M}_n$ where

$$g : (b_1, b_2, b_3, \dots, b_n, 0) \mapsto (b_1, b_2, b_3, \dots, b_n).$$

Indeed, we see that $g \circ f$ is the identity mapping on $\mathcal{M}_n$ and that $f \circ g$ is the identity mapping on $\mathcal{M}(n+1,0)$.

□

3.4 Ends In Not 0 Maps to (not n, n)

We want to prove that $\mathcal{M}(n, \widetilde{0}) \mapsto \mathcal{M}(n+1, \widetilde{n}n)$.

We start with an example: we give a bijection $\mathcal{M}(3,\widetilde{0}) \mapsto \mathcal{M}(4,\widetilde{3}3)$, as depicted in Figure 3.7.

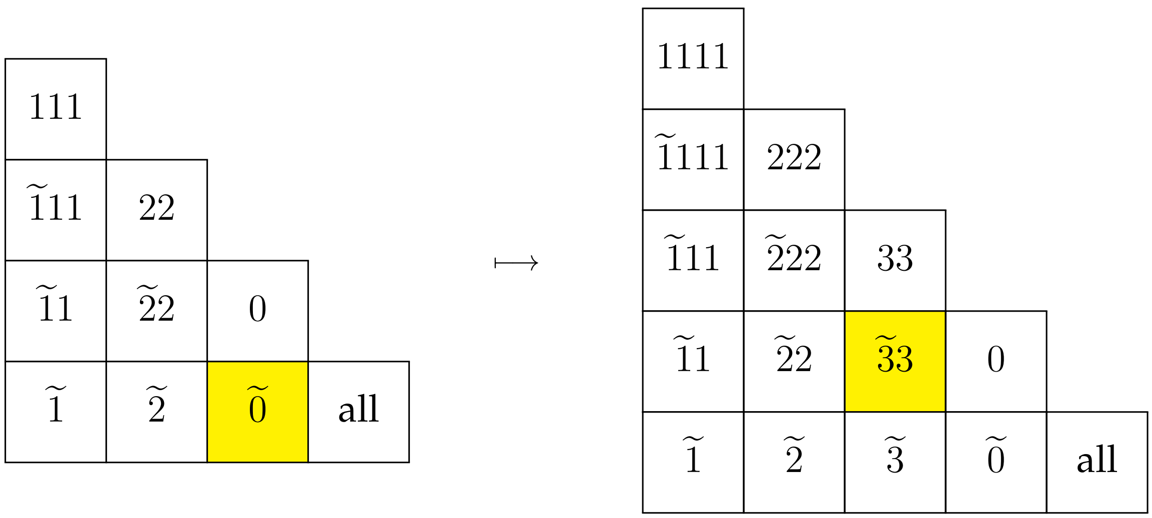


Figure 3.7 $\mathcal{M}(3,\widetilde{0}) \longmapsto \mathcal{M}(4,\widetilde{3}3)$.

The set $\mathcal{M}(3,\widetilde{0})$ contains sequences in $\mathcal{M}_3$ that end in not 0, which look like

$$m = (m_1, m_2, m_3) \quad \text{where} \quad 1 \le m_3 \le 3.$$

We also have $\mathcal{M}(4,\widetilde{3}3)$ which are the sequences of $\mathcal{M}_4$ that end in $\widetilde{3}, 3$, which look like

$$m' = (m'_1, m'_2, m', 3) \quad \text{where} \quad 0 \le m'_3 \le 2.$$

The mapping between $\mathcal{M}(3,\widetilde{0})$ and $\mathcal{M}(4,\widetilde{3}3)$ is shown in Figure 3.8. There are two ways for the mappings to be accomplished.

The first is when $m \in \mathcal{M}(3,\widetilde{0})$ does not end in 3. We can simply append a 3 to this sequence to obtain a sequence in $\mathcal{M}(4,\widetilde{3}3)$. The resulting sequence follows the condition of membership in $\mathcal{M}(4,\widetilde{3}3)$. An example of this mapping is $(0,0,1) \mapsto (0,0,1,3)$.

The second mapping is slightly more complicated, but still straightforward. This mapping starts with the sequences of $\mathcal{M}(3,\widetilde{0})$ where $m_3 = 3$. The reason that we cannot append a 3 to the end of these sequences for

$\mathcal{M}(4,\widetilde{33})$, is because it breaks the rule of not ending in $3,3$ for $\mathcal{M}(4,\widetilde{33})$. Meanwhile, we have not yet mapped anything to the sequences of $\mathcal{M}(4,\widetilde{33})$ of the form $(m'_1, m'_2, 0, 3)$ because the sequences of $\mathcal{M}(3,\widetilde{0})$ cannot end in 0.

Therefore the second mapping is between these two subsets. It maps a sequence $(m_1, m_2, 3) \in \mathcal{M}(3,\widetilde{0})$ to $(m_1, m_2, 0, 3 \in \mathcal{M}(4,\widetilde{33})$. An example of this mapping is $(1,2,3) \mapsto (1,2,0,3)$. Since $(1,2,3)$ ends in 3, it gets mapped with the sequence $(1,2,0,3)$ which ends in $0,3$.

$$\mathcal{M}(3,\widetilde{0}) \mapsto \mathcal{M}(4,\widetilde{33})$$

$(0,0,1) \mapsto (0,0,1,3)$	$(1,0,1) \mapsto (1,0,1,3)$
$(0,0,2) \mapsto (0,0,2,3)$	$(1,0,2) \mapsto (1,0,2,3)$
$(0,0,3) \mapsto (0,0,0,3)$	$(1,0,3) \mapsto (1,0,0,3)$
$(0,1,1) \mapsto (0,1,1,3)$	$(1,1,1) \mapsto (1,1,1,3)$
$(0,1,2) \mapsto (0,1,2,3)$	$(1,1,2) \mapsto (1,1,2,3)$
$(0,1,3) \mapsto (0,1,0,3)$	$(1,1,3) \mapsto (1,1,0,3)$
$(0,2,2) \mapsto (0,2,2,3)$	$(1,2,2) \mapsto (1,2,2,3)$
$(0,2,3) \mapsto (0,2,0,3)$	$(1,2,3) \mapsto (1,2,0,3)$

Figure 3.8 Explicitly showing how $\mathcal{M}(3,\widetilde{0}) \longmapsto \mathcal{M}(4,\widetilde{33})$.

We can follow this same bijection for all n and $n+1$. It will always be true that we can either append n to the end, or replace a final n with $0, n$.

Lemma 3.4. *There exists a bijection between $\mathcal{M}(n,\widetilde{0})$ and $\mathcal{M}(n+1,\widetilde{nn})$.*

Proof. We define a mapping $f : \mathcal{M}(n,\widetilde{0}) \to \mathcal{M}(n+1,\widetilde{nn})$

$$f : (m_1, m_2, m_3, \ldots, m_{n-1}, n) \mapsto (m_1, m_2, m_3, \ldots, m_{n-1}, 0, n)$$

and

$$f : (m_1, m_2, m_3, \ldots, m_n) \mapsto (m_1, m_2, m_3, \ldots, m_n, n)$$

when $1 \leq m_n \leq n-1$. This mapping is well-defined because the nth entry of the image is not n in either case.

This mapping is invertible. The inverse mapping is $g : \mathcal{M}(n+1,\widetilde{nn}) \to$

$\mathcal{M}(n, \widetilde{0})$ where

$$g : (b_1, b_2, b_3, \ldots, b_{n-1}, 0, n) \mapsto (b_1, b_2, b_3, \ldots, a_{n-1}, n)$$

and

$$g : (b_1, b_2, b_3, \ldots, b_n, n) \mapsto (b_1, b_2, b_3, \ldots, b_n)$$

when $b_n \in [n-1]$.

We see that $g \circ f$ is the identity mapping on $\mathcal{M}(n, \widetilde{0})$ and that $f \circ g$ is the identity mapping on $\mathcal{M}(n+1, \widetilde{n}n)$. □

We generalize this result in the next subsection.

3.5 Row Correspondence Between $\mathcal{M}_n$ and $\mathcal{M}_{n+1}$

We prove that there is always a row correspondence between the last row of $\mathcal{M}_n$ and the second to last row of $\mathcal{M}_{n+1}$.

As always, we start with an example: $\mathcal{M}_3$ and $\mathcal{M}_4$. Figure 3.9 shows where the bijection takes place between $\mathcal{M}_3$ and $\mathcal{M}_4$. We have already proved the following bijections: $\mathcal{M}(3, \widetilde{0}) \mapsto \mathcal{M}(4, \widetilde{3}3)$ and $\mathcal{M}_3 \mapsto \mathcal{M}(4, 0)$. We will now show the mappings $\mathcal{M}(3, \widetilde{2}) \mapsto \mathcal{M}(4, \widetilde{2}2)$, and $\mathcal{M}(3, \widetilde{1}) \mapsto \mathcal{M}(4, \widetilde{1}1)$.

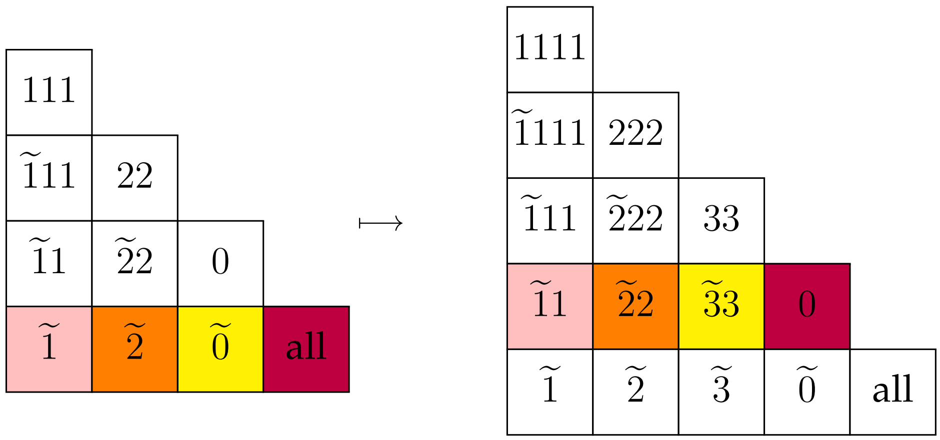


Figure 3.9 The colors indicate the four bijections $\mathcal{M}_3 \mapsto \mathcal{M}(4,0)$, $\mathcal{M}(3,\widetilde{0}) \mapsto \mathcal{M}(4,\widetilde{3}3)$,$\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(4,\widetilde{2}2)$, and $\mathcal{M}(3,\widetilde{1}) \mapsto \mathcal{M}(4,\widetilde{1}1)$.

Let's look at $\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(4,\widetilde{2}2)$. Recall that $\mathcal{M}(3,\widetilde{2})$ is the sequences of $\mathcal{M}_3$ that do not end in either 2 or 0, meaning the sequence can end in 1 or 3. Also recall that $\mathcal{M}(4,\widetilde{2}2)$ is the sequences of $\mathcal{M}_4$ that end in (not $2, 2$), so it can end with either $(0,2)$ or $(1,2)$. The bijection requires two distinct submappings. We can see all the mappings of $\mathcal{M}(3,\widetilde{2}) \mapsto \mathcal{M}(4,\widetilde{2}2)$ in Figure 3.10.

The first mapping handles the sequences of $\mathcal{M}(3,\widetilde{2})$ that end in 1. We use the mapping

$$(m_1, m_2, 1) \mapsto (m_1, m_2, 1, 2).$$

An example of this can be seen with the mapping $(0,0,1) \mapsto (0,0,1,2)$.

The second mapping handles the sequences of $\mathcal{M}(3,\widetilde{2})$ that end in 3. We use the mapping

$$(m_1, m_2, 3) \mapsto (m_1, m_2, 0, 2).$$

An example is $(1,2,3) \mapsto (1,2,0,2)$. These map to the sequences of $\mathcal{M}(4,\widetilde{2}2)$ that end in $(0,2)$. Recall that it will never be the case that a sequence of $\mathcal{M}(4,\widetilde{2}2)$ ends in $(3,2)$, because that does not follow the definition of $\mathcal{M}_n$: we are only allowed to decrease to 0. Therefore all remaining sequences get mapped to one another.

Finally, we exhibit our mapping $\mathcal{M}(3, \widetilde{1}) \mapsto \mathcal{M}(4, \widetilde{1}1)$, which follows a similar pattern. In this case, $\mathcal{M}(3, \widetilde{1})$ is the sequences of $\mathcal{M}_3$ that end in 3 and $\mathcal{M}(4, \widetilde{1}1)$ are the sequences of $\mathcal{M}_4$ that end in (not 1,1). We use the mapping

$$(m_1, m_2, 3) \mapsto (m_1, m_2, 0, 1),$$

which is similar to a mapping from the previous bijection.

$\mathcal{M}(3, \widetilde{2}) \mapsto \mathcal{M}(4, \widetilde{2}2)$

$(0, 0, 1) \mapsto (0, 0, 1, 2)$ $(1, 0, 1) \mapsto (1, 0, 1, 2)$

$(0, 0, 3) \mapsto (0, 0, 0, 2)$ $(1, 0, 3) \mapsto (1, 0, 0, 2)$

$(0, 1, 1) \mapsto (0, 1, 1, 2)$ $(1, 1, 1) \mapsto (1, 1, 1, 2)$

$(0, 1, 3) \mapsto (0, 1, 0, 2)$ $(1, 1, 3) \mapsto (1, 1, 0, 2)$

$(0, 2, 3) \mapsto (0, 2, 0, 2)$ $(1, 2, 3) \mapsto (1, 2, 0, 2)$

a. $\mathcal{M}(3, \widetilde{2}) \mapsto \mathcal{M}(4, \widetilde{2}2)$

$\mathcal{M}(3, \widetilde{1}) \mapsto \mathcal{M}(4, \widetilde{1}1)$

$(0, 0, 3) \mapsto (0, 0, 0, 1)$

$(0, 1, 3) \mapsto (0, 1, 0, 1)$

$(0, 2, 3) \mapsto (0, 2, 0, 1)$

$(1, 0, 3) \mapsto (1, 0, 0, 1)$

$(1, 1, 3) \mapsto (1, 1, 0, 1)$

$(1, 2, 3) \mapsto (1, 2, 0, 1)$

b. $\mathcal{M}(3, \widetilde{1}) \mapsto \mathcal{M}(4, \widetilde{1}1)$

Figure 3.10 Explicitly showing how $\mathcal{M}(3, \widetilde{2}) \mapsto \mathcal{M}(4, \widetilde{2}2)$, and $\mathcal{M}(3, \widetilde{1}) \mapsto \mathcal{M}(4, \widetilde{1}1)$.

This pattern can be generalized to a bijection between $\mathcal{M}(n, \widetilde{k})$ and $\mathcal{M}(n+1, \widetilde{k}k)$ where $k \in [n-1]$.

Theorem 3.5. *There exists a bijection between $\mathcal{M}(n, \widetilde{k})$ and $\mathcal{M}(n+1, \widetilde{k}k)$ where $k \in [n-1]$.*

Proof. We define a mapping $f : \mathcal{M}(n, \widetilde{k}) \to \mathcal{M}(n+1, \widetilde{k}k)$

$$f : (m_1, m_2, \ldots, m_{n-1}, n) \mapsto (m_1, m_2, \ldots, m_{n-1}, 0, k) \in \mathcal{M}(n+1, \widetilde{k}k)$$

and

$$f : (m_1, m_2, \ldots, m_n) \mapsto (m_1, m_2, \ldots, m_n, k) \in \mathcal{M}(n+1, \widetilde{k}k)$$

when $m_n \in [k-1]$. This mapping is well-defined because the nth entry of the image is in $[k-1]$.

This mapping is invertible. The inverse mapping is $g : \mathcal{M}(n+1, \widetilde{k}k) \to$

$\mathcal{M}(n, \widetilde{k})$ where

$$g : (b_1, b_2, \ldots, b_{n-1}, 0, k) \mapsto (b_1, b_2, \ldots, b_{n-1}, n) \in \mathcal{M}(n, \widetilde{k})$$

and

$$g : (b_1, b_2, \ldots, b_n, k) \mapsto (b_1, b_2, \ldots, b_n) \in \mathcal{M}(n, \widetilde{k})$$

when $b_n \in [k-1]$.

We see that $g \circ f$ is the identity mapping on $\mathcal{M}(n, \widetilde{k})$ and that $f \circ g$ is the identity mapping on $\mathcal{M}(n+1, \widetilde{k}k)$. □

This last proof shows that there is a correspondence between the elements of $\mathcal{M}_n$ and $\mathcal{M}_{n+1}$. Meaning that we can see $\mathcal{M}_n$ in the triangular recurrence of $\mathcal{M}_{n+1}$.

Chapter 4

Poset

In this section, we study the poset structure of $\mathcal{M}_n$.

Recall that a poset is a set X on which a partial order, $\preceq$, is denoted by $(X, \preceq)$. We define our poset $(\mathcal{M}_n \preceq)$ in the natural way. We have

$$(a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n) \quad \Longleftrightarrow \quad a_i \leq b_i \text{ for } 1 \leq n.$$

Clearly, $(0, 0, \ldots, 0)$ is the unique minimum element of our poset, and $(1, 2, \ldots, n)$ is the unique maximum element of our poset.

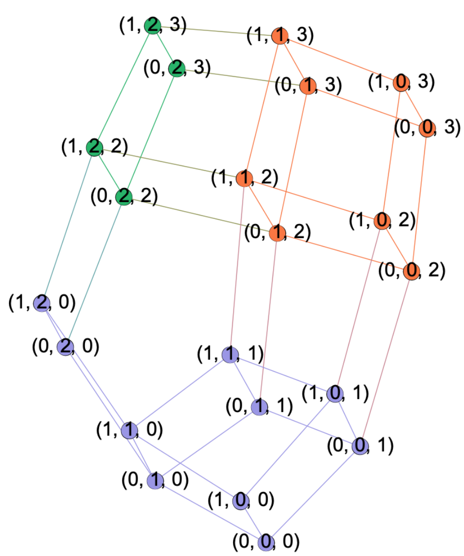


Figure 4.1 Hasse diagram illustrating the poset structure of $\mathcal{M}_3$.

Figure 4.1 shows the Hasse diagram of $\mathcal{M}_3$ and Figure 4.2 shows the Hasse diagram of $\mathcal{M}_4$. We will determine the maximum and minimum length for a maximal chain of the poset $\mathcal{M}_n$.

Definition 4.1. A collection C of elements of poset X is a *chain* provided that elements in C are pairwise comparable. That is, for all $x_1, x_2 \in C$, either $x_1 \succeq x_2$ or $x_2 \succeq x_1$. A *maximal chain* is a chain C such that no superset of C is also a chain.

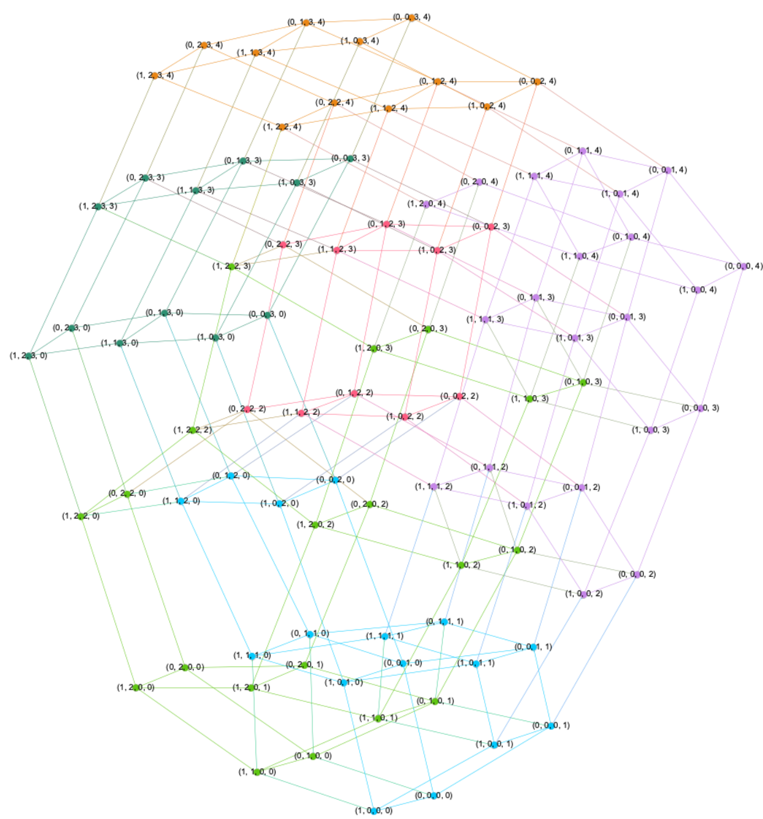


Figure 4.2 Hasse diagram illustrating the poset structure of $\mathcal{M}_4$.

4.1 Maximum Length for a Maximal Chain

We will show that the maximum length for a maximal chain of M_n is $1 + \binom{n+1}{2}$. That is to say that the longest path to create a maximal chain in the poset structure of $\mathcal{M}_n$ contains $1 + \binom{n+1}{2}$ elements of $\mathcal{M}_n$. Figure 4.3 shows the maximum size of subsets to create a maximal chain of $\mathcal{M}_1$ through $\mathcal{M}_5$.

$\mathcal{M}_n$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
Maximum length for a maximal chain: $1 + \binom{n+1}{2}$	2	4	7	11	16

Figure 4.3 Table illustrating the size of the maximum length for a maximal chain for $\mathcal{M}_1$ through $\mathcal{M}_5$.

Figure 4.4 shows the Hasse diagram of $\mathcal{M}_3$. The green highlighted path is an example of the maximum length for a maximal chain of $\mathcal{M}_3$. The maximal chain in this example is:

$$(0,0,0) \prec (1,0,0) \prec (1,0,1) \prec (1,0,2) \prec (1,0,3) \prec (1,1,3) \prec (1,2,3)$$

We can now prove that this statement is true.

Lemma 4.2. *The maximum length for a maximal chain of $\mathcal{M}_n$ is $1 + \binom{n+1}{2}$.*

Proof. We must find the longest path from the minimum element $(0,0,\ldots,0)$ to the maximum element $(1,2,3,\ldots,n)$.

Here is our proposed maximal chain. The chain will always begin with the minimum sequence $(0,0,\ldots,0)$. From here, we increment the entry m_n a total of n times,

$$(0,\ldots,0,0) \longrightarrow (0,\ldots,0,1) \longrightarrow (0,\ldots,0,2) \longrightarrow \cdots \longrightarrow (0,\ldots,0,n).$$

Then we move on to m_{n-1}. We can increment entry m_{n-1} a total of $n-1$ times. This pattern continues: we can increment m_k a total of k times for $1 \leq k \leq n$. Finally, after increasing m_1, we will be at the maximal element at $(1,2,3,4,\ldots,n)$. The total number of steps is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

Therefore the length of the chain is $1 + \binom{n+1}{2}$. Finally, we observe that this is certainly the longest possible chain. Entry m_k can be changed a total of k times, and we have found a valid chain of this length. $\square$

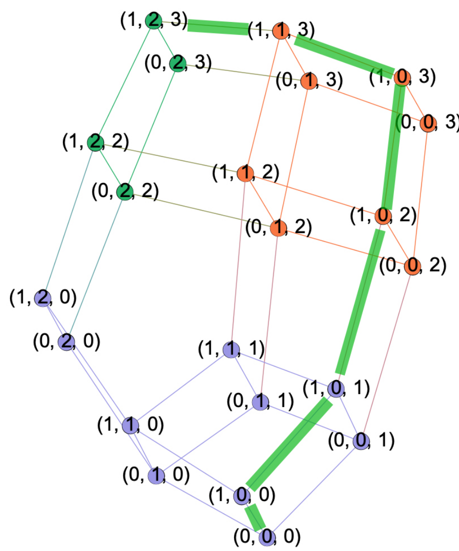


Figure 4.4 Hasse diagram illustrating the poset structure of $\mathcal{M}_3$ where the green, highlighted, path is an example of the maximum length for a maximal chain.

4.2 Minimum Length for a Maximal Chain

Next we can show that the minimum length for a maximal chain of M_n is $2n$. That is to say that the shortest path to create a maximal chain in the poset structure of $\mathcal{M}_n$ contains $2n$ elements.

Figure 4.5 shows the minimum size of subsets to create a maximal chain of $\mathcal{M}_1$ through $\mathcal{M}_5$. Here we can see that the minimum length is $2n$.

$\mathcal{M}_n$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
Minimum length for a maximal chain: $2n$	2	4	6	8	10

Figure 4.5 Table illustrating the size of the minimum length for a maximal chain for $\mathcal{M}_1$ through $\mathcal{M}_5$.

Figure 4.6 shows the Hasse diagram of $\mathcal{M}_3$. The purple highlighted path is an example of the minimum length for a maximal chain of $\mathcal{M}_3$. The maximal chain is

$$(0,0,0) \prec (1,0,0) \prec (1,1,0) \prec (1,2,0) \prec (1,2,2) \prec (1,2,3).$$

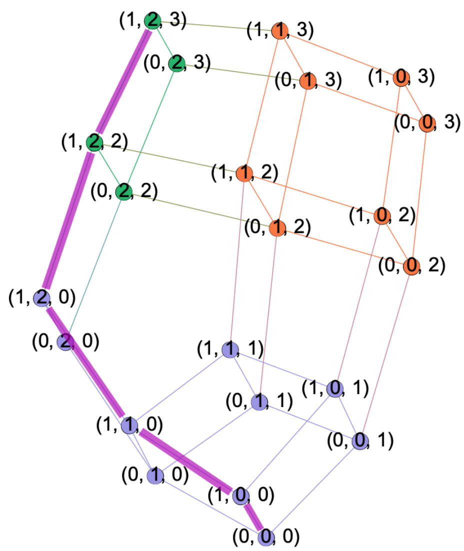


Figure 4.6 Hasse diagram illustrating the poset structure of $\mathcal{M}_3$ where the purple, highlighted, path is an example of the minimum length for a maximal chain.

We can now prove that this statement is true.

Lemma 4.3. *The minimum length for a maximal chain of $\mathcal{M}_n$ is $2n$.*

Proof. We must find the shortest path from $(0,0,\ldots,0)$ to $(1,2,\ldots,n)$. Here is our proposed chain of minimum length.

First, change m_1 from 0 to 1. Then, change m_2 from 0 to 1, and then from 1 to 2. This is where we start moving more quickly. Change m_3 from 0 to 2 (because we can only decrease to 0), and then from 2 to 3. More generally, for $2 \leq k \leq n$, change m_k to $k-1$ and then to k. This takes a total of $1+2(n-1) = 2n-1$ steps, so there are $2n-1+1$ elements in this maximal chain.

We now show that every maximal chain contains at least $2n$ elements. For $k=1$, we change m_1 from 0 to 1. This requires 1 step. Suppose that we are updating the kth entry as we step through our chain. There are two cases we must consider.

Case 1: $m_k < k-1$. Then we update m_k to be $\max\{m_{k-1}, m_k+1\} \leq k-1$. So we must increment this entry at least once more.

Case 2: $m_{k-1} = k-1$. We update the kth entry to be its maximum value k

The two cases show that for $2 \leq k \leq n$, there are at least 2 steps that need to be performed to make $m_k = k$. Since this is the case for $k > 1$, we require at least $2n-2$ steps to achieve $m_k = k$ for $1 < k \leq n$. The total number of steps is at least $1 + (2n-2) = 2n-1$, and the number of elements in the chain is at least $2n$. □

4.2.1 Counting Maximal Chains of Minimum Length

Finally, we can also show that the number of maximal chains of minimum length of M_n is $2n-1$. This means that there are $2n-1$ maximal chains of minimum length in $\mathcal{M}_n$.

Figure 4.7 shows the number of maximal chains of minimum length of $\mathcal{M}_1$ through $\mathcal{M}_5$. Here we can see that the minimum length is $2n-1$.

$\mathcal{M}_n$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
Number of maximal chains of minimum length: $2n-1$	1	3	5	7	9

Figure 4.7 Table illustrating the size of the number of maximum chains of minimum length for $\mathcal{M}_1$ through $\mathcal{M}_5$.

We can prove that this statement is true.

Lemma 4.4. *There are $2n-1$ maximal chains of minimum length of $\mathcal{M}_n$.*

Proof. From the previous lemma, we know that for $2 \leq k \leq n$ we must change entry m_k exactly twice in order to have a minimum length chain.

For $3 \leq k \leq n$, we can only change m_k when $m_{k-1} = k-1$. Here is why:

- The first step is to change $m_k = 0$ to $m_k = k-1$, which can be done because $m_{k-1} = k-1$.
- The second step is to change $m_k = k-1$ to $m_k = k$.

Therefore for $3 \leq k \leq n$, we cannot change m_k before we have $m_{k-1} = k-1$. So the order of these changes is determined: we must increase m_2 to 2, and then increase m_3 to 3, and so on.

It always takes 2 steps to increase m_2 to 2. As noted above, this must happen before we increase m_3. In summary, these $2(n-1)$ steps must occur in order.

Additionally, we still have the case where $m_1 = 0$, which needs to be changed to $m_1 = 1$. Since this requires only one step and no prerequisites from any m_k, we can change m_1 at any given time. There are $1 + 2(n-1) = 2n - 1$ different times that this can happen: it can be the first thing you do, or it can happen after any of the $2(n-1)$ other steps.

Therefore, we can conclude that there are $2n - 1$ maximum chains of minimum length of $\mathcal{M}_n$. $\square$

Chapter 5

Adding Restrictions

In this chapter, we explore some subfamilies of $\mathcal{M}_n$ by adding additional restrictions on the sequences.

5.1 Staircase Packings With Maximum Entry at most 2

The first subfamily we consider are sequences of $\mathcal{M}_n$ whose entries are at most 2.

Definition 5.1. We use

$$\mathcal{T}_n = \{(t_1, t_2, \dots, t_n) \in \mathcal{M}_n : 0 \le t_i \le 2\}$$

to denote the staircase packings whose maximum entry at most 2. We then define $T_n = |\mathcal{T}_n|$.

The subfamilies $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are shown in Figure 5.1. We have $T_1 = 2$, $T_2 = 6$ and $T_3 = 16$.

n	$\mathcal{T}_n$							
1	(0)	(1)						
2	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$		
3	$(0,0,0)$	$(0,0,1)$	$(0,0,2)$	$(0,1,0)$	$(0,1,1)$	$(0,1,2)$	$(0,2,0)$	$(0,2,2)$
	$(1,0,0)$	$(1,0,1)$	$(1,0,2)$	$(1,1,0)$	$(1,1,1)$	$(1,1,2)$	$(1,2,0)$	$(1,2,2)$

Figure 5.1 The staircase packings $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$.

Lemma 5.2. *For $n \geq 3$, we have*

$$T_n = 3T_{n-1} - T_{n-2}.$$

Proof. We partition

$$\mathcal{T}_n = \mathcal{T}_{n,0} \cup T_{n,1} \cup T_{n,2},$$

where $T_{n,k}$ is the set of sequences that end in k. We have $T_{n,0} = T_{n-1}$ because we can append a zero to any sequence in $\mathcal{T}_{n-1}$, and this mapping is a bijection. Similarly, $T_{n,2} = T_{n-1}$ because we can always append a 2 to any sequence in $\mathcal{T}_{n-1}$.

We claim that $T_{n,1} = T_{n-1} - T_{n-2}$. Given a sequence $t = (t_1, \ldots, t_n) \in \mathcal{T}_{n,1}$, we have $t_n = 1$ and therefore we cannot have $t_{n-1} = 2$. In other words,

$$T_{n,1} = T_{n-1} - T_{n-1,2}.$$

As observed in the previous case, we have $T_{n-1,2} = T_{n-2}$. Therefore

$$T_{n,1} = T_{n-1} - T_{n-2}.$$

Finally, we have

$$\begin{aligned} T_n &= T_{n,0} + T_{n,1} + T_{n,2} \\ &= T_{n-1} + (T_{n-1} - T_{n-2}) + T_{n-1} \\ &= 3T_{n-1} - T_{n-2}. \end{aligned}$$

□

5.2 Subsets Ending In k

Next place restrictions on the final entry of the staircase packing

Definition 5.3. Let

$$\mathcal{E}(n,k) = \{(m_1, m_2, \ldots, m_n) \in \mathcal{M}_n : m_n = k\}$$

be the staircase packings of size n whose final entry is k. We use $E(n,k) = |\mathcal{E}(n,k)|$ to denote the size of the corresponding set.

Figure 5.2 shows the families $\mathcal{E}(n,k)$ for $0 < k < n \le 3$.

n	k	$\mathcal{E}(n,k)$
1	0	(0)
1	1	(1)
2	0	$(0,0)$ $(1,0)$
2	1	$(0,1)$ $(1,1)$
2	2	$(0,2)$ $(1,2)$
3	0	$(0,0,0)$ $(0,1,0)$ $(0,2,0)$ $(1,0,0)$ $(1,1,0)$ $(1,2,0)$
3	1	$(0,0,1)$ $(0,1,1)$ $(1,0,1)$ $(1,1,1)$
3	2	$(0,0,2)$ $(0,1,2)$ $(0,2,2)$ $(1,0,2)$ $(1,1,2)$ $(1,2,2)$
3	3	$(0,0,3)$ $(0,1,3)$ $(0,2,3)$ $(1,0,3)$ $(1,1,3)$ $(1,2,3)$

Figure 5.2 Staircase packings in $\mathcal{E}(n,k)$ for $0 \le k < n \le 3$.

The $\mathcal{E}(n,k)$ obey a triangular recurrence. Figure 5.3 shows the structure of this triangle for the sets $\mathcal{E}(n,k)$. Figure 5.4 shows this triangular recurrence for the actual sizes $E(n,k)$ for $0 \le k < n \le 5$. Note that $\mathcal{E}(n,n+1) = \emptyset$ (and therefore $E(n,n+1) = 0$), so these entries are not shown in these figures.

$\mathcal{E}(0,0)$							
$\mathcal{E}(1,0)$	$\mathcal{E}(1,1)$						
$\mathcal{E}(2,0)$	$\mathcal{E}(2,1)$	$\mathcal{E}(2,2)$					
$\mathcal{E}(3,0)$	$\mathcal{E}(3,1)$	$\mathcal{E}(3,2)$	$\mathcal{E}(3,3)$				
$\mathcal{E}(4,0)$	$\mathcal{E}(4,1)$	$\mathcal{E}(4,2)$	$\mathcal{E}(4,3)$	$\mathcal{E}(4,4)$			
$\mathcal{E}(5,0)$	$\mathcal{E}(5,1)$	$\mathcal{E}(5,2)$	$\mathcal{E}(5,3)$	$\mathcal{E}(5,4)$	$\mathcal{E}(5,5)$		
$\vdots$							
$\mathcal{E}(n,0)$	$\mathcal{E}(n,1)$	$\mathcal{E}(n,2)$	$\mathcal{E}(n,3)$	$\mathcal{E}(n,4)$	$\mathcal{E}(n,5)$	$\dots$	$\mathcal{E}(n,n)$

Figure 5.3 The triangular recurrence of $\mathcal{E}(n,k)$.

1					
1	1				
2	2	2			
6	4	6	6		
22	10	16	22	22	
92	32	48	70	92	92

Figure 5.4 The triangular recurrence of $\mathcal{E}(n,k)$ for $0 \le n \le 5$.

With these figures in mind, we will give the triangular recurrence for the sizes $E(n,k)$ and investigate further properties of the families $\mathcal{E}(n,k)$.

5.3 The Triangular Recurrence for $E(n,k)$

We will show that

$$E(n,k) = E(n-1,k) + E(n,k-1)$$

for $1 \le k \le n-1$.

We start by considering an example. We will show that $E(3,2) = E(2,2) + E(3,1)$. Recall that $\mathcal{E}(3,2)$ are the sequences of $\mathcal{M}_3$ that end in 2, $\mathcal{E}(2,2)$ are the sequences of $\mathcal{M}_2$ that end in 2 and $\mathcal{E}(3,1)$ are the sequences of $\mathcal{M}_3$ that end in 1. Figure 5.5 highlights the triangular recurrence $E(3,2) = E(2,2) + E(3,1)$.

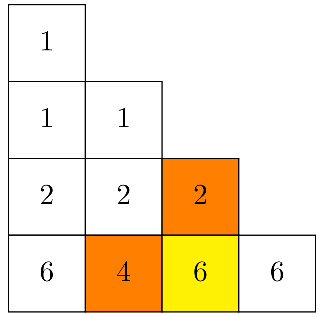


Figure 5.5 Triangular recurrence showing that $E(3,2) = E(2,2) + E(3,1)$.

We define an injection f from $\mathcal{E}(2,2)$ into $\mathcal{E}(3,2)$ by mapping $(m_1, 2) \mapsto (m_1, 2, 2)$. We define an injection g from $\mathcal{E}(3,1)$ into $\mathcal{E}(3,2)$ by mapping $(m_1, m_2, 1) \mapsto (m_1, m_2, 2)$. Next we observe that the images of these mappings are disjoint: the image of f consists of sequences that end with $(2,2)$, while the image of g consists of sequences that end with either $(0,2)$ or $(1,2)$.

Figure 5.6 shows the injection $f : \mathcal{E}(2,2) \to \mathcal{E}(3,2)$ and the injection $g : \mathcal{E}(3,1) \to \mathcal{E}(3,2)$. Observe that every sequence in $\mathcal{E}(3,2)$ is mapped to exactly once, and this proves that $\mathcal{E}(3,2) = \mathcal{E}(2,2) + \mathcal{E}(3,1)$.

$$f : \mathcal{E}(2,2) \mapsto \mathcal{E}(3,2)$$

$$(0,2) \mapsto (0,2,2) \qquad (1,2) \mapsto (1,2,2)$$

$$g : \mathcal{E}(3,1) \mapsto \mathcal{E}(3,2)$$

$$(0,0,1) \mapsto (0,0,2) \qquad (1,0,1) \mapsto (1,0,2)$$

$$(0,1,1) \mapsto (0,1,2) \qquad (1,1,1) \mapsto (1,1,2)$$

Figure 5.6 The injection from $\mathcal{E}(2,2)$ into $\mathcal{E}(3,2)$ and the injection from $\mathcal{E}(3,1)$ into $\mathcal{E}(3,2)$. Taken together, we have a bijection from $\mathcal{E}(2,2) \cup \mathcal{E}(3,1)$ to $\mathcal{E}(3,2)$.

We now prove that this triangular recurrence holds in general.

Lemma 5.4. *For $1 \le k \le n-1$, there is a bijection from $\mathcal{E}(n-1,k) \cup \mathcal{E}(n,k-1)$ to $\mathcal{E}(n,k)$. Therefore*

$$E(n,k) = E(n-1,k) + E(n,k-1)$$

Proof. Let $f : \mathcal{E}(n-1,k) \to \mathcal{E}(3,2)$ be the injection given by the mapping

$$f : (m_1, \ldots, , m_{n-2}, k) \mapsto (m_1, \ldots, , m_{n-2}, k, k).$$

Let $g : \mathcal{E}(n, k-1)$ into $\mathcal{E}(n, k-1)$ be the injection given by the mapping

$$f : (m_1, \ldots, , m_{n-1}, k-1) \mapsto (m_1, \ldots, m_{n-1}, k).$$

We claim that the images of these mappings are disjoint. Indeed, the image of f consists of sequences that end with (k, k), while the image of g consists of sequences that end with (j, k) where $0 \leq j \leq k-1$.

Finally, it is clear that every sequence of $\mathcal{E}(n, k)$ is mapped to by either f or g. Taken together,this composite mapping is a bijection from $\mathcal{E}(n-1, k) \cup \mathcal{E}(n, k-1)$ to $\mathcal{E}(n, k)$. Therefore

$$E(n, k) = E(n-1, k) + E(n, k-1).$$

□

5.4 The First Entry is the Sum of the Previous Row

In this section, we prove that

$$E(n, 0) = \sum_{i=0}^{n-1} E(n-1, i).$$

We start by proving that this is true for

$$\mathcal{E}(3, 0) = \mathcal{E}(2, 0) + \mathcal{E}(2, 1) + \mathcal{E}(2, 2).$$

Figure 5.7 highlights this recurrence: the first entry in row 3 is equal to the sum of the entries in row 2.

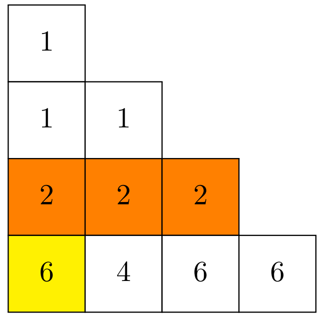


Figure 5.7 Triangular recurrence showing that $\mathcal{E}(3,0) = \mathcal{E}(2,0) + \mathcal{E}(2,1) + \mathcal{E}(2,2)$.

The bijection from $\mathcal{E}(2,0) \cup \mathcal{E}(2,1) \cup E(2,2)$ to $\mathcal{E}(3,0)$ is very simple: append a 0 to the sequence of length 2.

$$\mathcal{E}(2,0) \mapsto \mathcal{E}(3,0)$$
$$(0,0) \mapsto (0,0,0) \qquad (1,0) \mapsto (1,0,0)$$

$$\mathcal{E}(2,1) \mapsto \mathcal{E}(3,0)$$
$$(0,1) \mapsto (0,1,0) \qquad (1,1) \mapsto (1,1,0)$$

$$\mathcal{E}(2,2) \mapsto \mathcal{E}(3,0)$$
$$(0,2) \mapsto (0,2,0) \qquad (1,2) \mapsto (1,2,0)$$

Figure 5.8 Injections showing that $E(3,0) = E(2,0) + E(2,1) + E(2,2)$.

We prove the general case.

Lemma 5.5. *We have*

$$E(n,0) = \sum_{i=0}^{n-1} E(n-1,i).$$

Proof. First, we observe that

$$\mathcal{M}_{n-1} = \mathcal{E}(n-1,0) \cup \mathcal{E}(n-1,1) \cup \cdots \cup E(n-1,n-1).$$

Indeed, we are simply partitioning the set $\mathcal{M}_{n-1}$ according to the value of the final entry of each sequence.

Next, it is clear that the mapping $f : \mathcal{M}_{n-1} \to \mathcal{E}(n, 0)$ given by

$$f : (m_1, \ldots, m_{n-1}) \mapsto (m_1, \ldots, m_{n-1}, 0)$$

is a bijection from $\mathcal{M}_{n-1}$ to $\mathcal{E}(n, 0)$. □

5.5 Partial Row Sum Recurrence

Next, we prove that

$$E(n,k) = \sum_{i=0}^{k} E(n-1, i).$$

The sum of the first k terms of row $n-1$ gives us the kth term of row n. We start with an instructive example: $E(3,1) = E(2,0) + E(2,1)$. Figure 5.10 highlights this recurrence in our triangle.

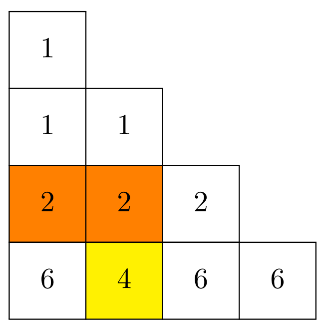


Figure 5.9 Triangular recurrence showing that $\mathcal{E}(3,1) = \mathcal{E}(2,0) + \mathcal{E}(2,1)$.

The injection from $\mathcal{E}(2,0) \cup E(2,1)$ to $\mathcal{E}(3,1)$ is very intuitive: we simply append a 1 to the given sequence. . Figure 5.10 shows how the mapping creates all the subsets of $\mathcal{E}(3,1)$.

$$\mathcal{E}(2,0) \mapsto \mathcal{E}(3,1)$$

$$(0,0) \mapsto (0,0,1) \qquad (1,0) \mapsto (1,0,1)$$

$$\mathcal{E}(2,1) \mapsto \mathcal{E}(3,1)$$

$$(0,1) \mapsto (0,1,1) \qquad (1,1) \mapsto (1,1,1)$$

Figure 5.10 Bijections showing that $\mathcal{E}(3,1) = \mathcal{E}(2,0) + \mathcal{E}(2,1)$.

We now prove the general case.

Lemma 5.6. *We have*

$$E(n,k) = \sum_{i=0}^{k} E(n-1,i).$$

Proof. We define the mapping

$$f : \mathcal{E}(n-1,0) \cup \mathcal{E}(n-1,1) \cup \cdots \cup E(n-1,k) \to \mathcal{E}(n,k)$$

by

$$f : (m_1, m_2, \ldots, m_k) = (m_1, m_2, \ldots, m_k, k).$$

The mapping is well-defined because $0 \le m_k \le k$. It is also clear that the mapping is a bijection. □

5.6 Partial Column Sum Recurrence

Next we prove that

$$E(n,k) = \sum_{i=k}^{n} E(i,k-1).$$

As always, we start with an example. This time, we show that $\mathcal{E}(3,2) = \mathcal{E}(2,1) + \mathcal{E}(3,1)$. Figure 5.11 highlights this recurrence in the triangle.

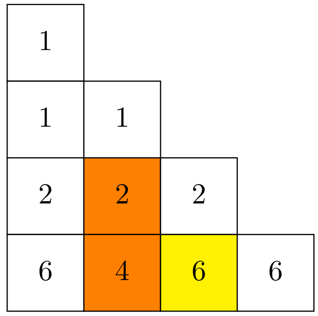


Figure 5.11 Triangular recurrence showing that $\mathcal{E}(3,2) = \mathcal{E}(2,1) + \mathcal{E}(3,1)$.

For each sequence in $\mathcal{E}(2,1)$, we replace the last 1 with $(2,2)$, For each sequence in $\mathcal{E}(3,1)$, we change the last 1 to a 2. Figure 5.12 shows that taken together, these injective mappings give a bijection to $\mathcal{E}(3,2)$.

$$\mathcal{E}(2,1) \mapsto \mathcal{E}(3,2)$$
$$(0,1) \mapsto (0,2,2) \qquad (1,1) \mapsto (1,2,2)$$

$$\mathcal{E}(3,1) \mapsto \mathcal{E}(3,2)$$
$$(0,0,1) \mapsto (0,0,2) \qquad (1,0,1) \mapsto (1,0,2)$$
$$(0,1,1) \mapsto (0,1,2) \qquad (1,1,1) \mapsto (1,1,2)$$

Figure 5.12 Bijections showing how $\mathcal{E}(3,2) = \mathcal{E}(2,1) + \mathcal{E}(3,1)$.

We now prove the general case.

Lemma 5.7. *We have* $E(n,k) = \sum_{i=k}^{n} E(i,k-1)$.

Proof. We define a series of injective mappings

$$f_i : \mathcal{E}(i,k-1) \to \mathcal{E}(n,k)$$

for $k \leq i \leq n$. These mappings are

$$f_i : (m_1, \ldots, m_{i-1}, k-1) \mapsto (m_1, \ldots, m_{i-1}, \underbrace{k, k, \ldots, k}_{n-i+1 \text{ times}}).$$

The resulting sequence is a valid sequence in $\mathcal{E}(n,k)$, and it ends in exactly $n-i+1$ entries equal to k. This is because $m_i \leq k-1$.

The images of these injective mappings are disjoint. Furthermore, every element of $\mathcal{E}(n,k)$ is mapped to exactly once: its preimage is determined by the length of the ultimate sequence of k's. □

5.7 Entrywise Bijection Between the Triangles

There is a lovely correspondence between the sets $\mathcal{M}(n-1, \widetilde{k}k^r)$ and (most of) the sets $\mathcal{E}(n,j)$. Figure 5.13 summarizes this mapping for the sets $\mathcal{M}(3, \widetilde{k}k^r)$ and the sets $\mathcal{E}(4,j)$.

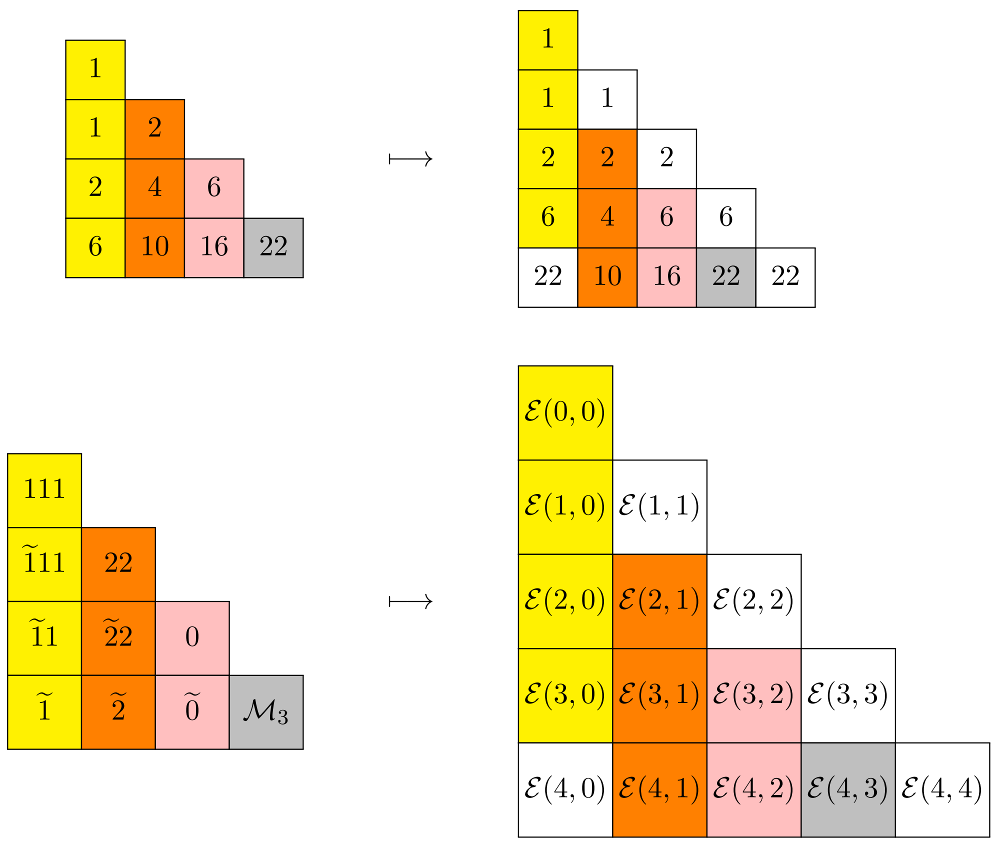


Figure 5.13 The set-by-set bijections between the sets $\mathcal{M}(3, \widetilde{k}k^r)$ and the sets $\mathcal{E}(4,j)$.

We prove each of the results shown.

5.7.1 Bijection between $\mathcal{M}_n$ and $\mathcal{E}(n+1, n)$

We prove that $M_n = E(n+1, n)$. Let's start with an example: $\mathcal{M}_3 = \mathcal{E}(4, 3)$. Figure 5.14 highlights what this looks like in the respective triangles.

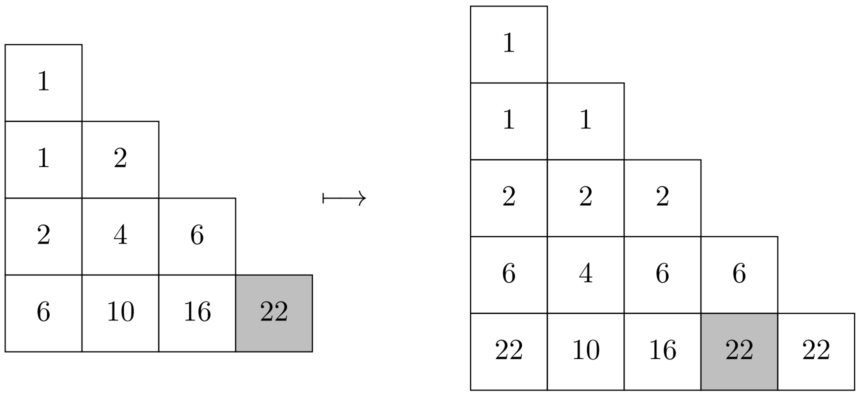


Figure 5.14 Triangular recurrence highlighting the bijection between $\mathcal{M}_3$ and $\mathcal{E}(4, 3)$.

Recall that that $\mathcal{E}(4, 3)$ is the set of staircase packings $(m_1, m_2, m_3, 3)$ of size 4 that end with 3, and that $\mathcal{M}_3$ is equal to all the staircase packings (m_1, m_2, m_3) of size 3. The bijection from $\mathcal{E}(4, 3)$ to $\mathcal{M}_3$ is

$$(m_1, m_2, m_3) \mapsto (m_1, m_2, m_3, 3).$$

Figure 5.15 explicitly shows this mapping.

$$\mathcal{E}(4,3) \mapsto \mathcal{M}_3$$

$(0,0,0,3) \mapsto (0,0,0)$	$(0,0,1,3) \mapsto (0,0,1)$
$(0,0,2,3) \mapsto (0,0,2)$	$(0,0,3,3) \mapsto (0,0,3)$
$(0,1,0,3) \mapsto (0,1,0)$	$(0,1,1,3) \mapsto (0,1,1)$
$(0,1,2,3) \mapsto (0,1,2)$	$(0,1,3,3) \mapsto (0,1,3)$
$(0,2,0,3) \mapsto (0,2,0)$	$(0,2,2,3) \mapsto (0,2,2)$
$(0,2,3,3) \mapsto (0,2,3)$	$(1,0,0,3) \mapsto (1,0,0)$
$(1,0,1,3) \mapsto (1,0,1)$	$(1,0,2,3) \mapsto (1,0,2)$
$(1,0,3,3) \mapsto (1,0,3)$	$(1,1,0,3) \mapsto (1,1,0)$
$(1,1,1,3) \mapsto (1,1,1)$	$(1,1,2,3) \mapsto (1,1,2)$
$(1,1,3,3) \mapsto (1,1,3)$	$(1,2,0,3) \mapsto (1,2,0)$
$(1,2,2,3) \mapsto (1,2,2)$	$(1,2,3,3) \mapsto (1,2,3)$

Figure 5.15 Bijection from $\mathcal{E}(4,3)$ to $\mathcal{M}_3$

We prove the general case.

Lemma 5.8. *There is a bijection from $\mathcal{E}(n+1,n)$ to $\mathcal{M}_n$. Therefore $E(n+1,n) = M_n$.*

Proof. Recall that $\mathcal{E}(n+1,n)$ is the set of staircase packings $(m_1, m_2, \ldots, m_n, n)$ of size $n+1$ that end with n, and that $\mathcal{M}_n$ is equal to all the staircase packings $(m_1, m_2, \ldots, m_n)$ of size n. The bijection from $\mathcal{E}(n+1,n)$ to $\mathcal{M}_n$ is

$$(m_1, m_2, \ldots, m_n, n) \mapsto (m_1, m_2, \ldots, m_n).$$

□

5.7.2 The Penultimate Columns

Next, we give a bijection between $\mathcal{M}_n^0 = \mathcal{M}(n, \widetilde{0}) \cup \mathcal{M}(n,0)$ and $\mathcal{E}(n, n-1) \cup \mathcal{E}(n+1, n-1)$. Therefore

$$M(n,0) = E(n,n-1) + E(n+1,n-1) = E(n+1,n)$$

by Lemma 5.4.

We start with an example. Figure 5.14 highlights the sets $\mathcal{M}(3,\widetilde{0})$, $\mathcal{M}(3,0)$ and $\mathcal{E}(3,2)$, $\mathcal{E}(4,2)$.

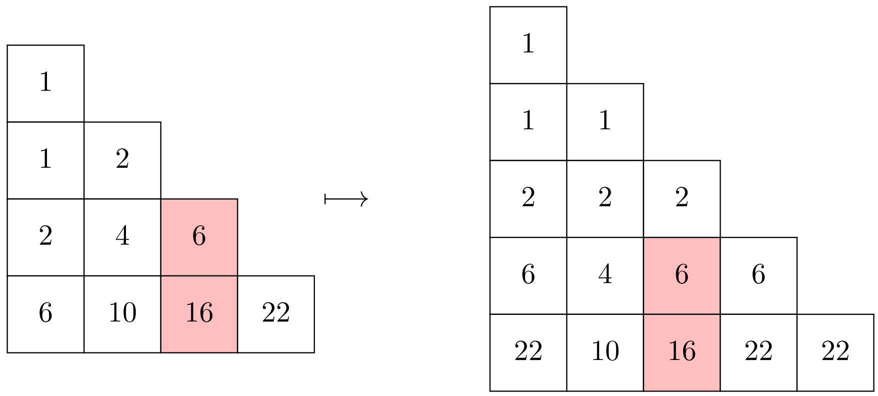


Figure 5.16 Triangular recurrence showing the bijections between $\mathcal{M}_3^0, \mathcal{E}(3,2)$ and $\mathcal{E}(4,2)$.

We describe a bijection $f : \mathcal{E}(3,2) \to \mathcal{M}(3,0)$ and a bijection $g : \mathcal{E}(4,2) \to \mathcal{M}(3,\widetilde{0})$. The bijection from $\mathcal{E}(3,2)$ to $\mathcal{M}(3,0)$ changes the last entry from a 2 to a 0:

$$(m_1, m_2, 2) \to (m_1, m_2, 0).$$

The bijection from $\mathcal{E}(4,2)$ to $\mathcal{M}(3,\widetilde{0})$ has two cases. If the sequence in $\mathcal{E}(4,2)$ ends in $(0,2)$ then we replace these two entries with the single entry 3. Otherwise, we just remove the final 2 of the sequence. In other words

$$\begin{aligned} f &: (m_1, m_2, 0, 2) \to (m_1, m_2, 3), \\ g &: (m_1, m_2, m_3, 2) \to (m_1, m_2, m_3) \quad \text{when } m_3 \neq 0. \end{aligned}$$

Figure 5.17 shows the bijection $f : \mathcal{E}(3,2) \to \mathcal{M}(3,0)$ and the bijection $g : \mathcal{E}(4,2) \to \mathcal{M}(3,\widetilde{0})$.

$$f : \mathcal{E}(3,2) \mapsto \mathcal{M}(3,0)$$

$(0,0,2) \mapsto (0,0,0)$	$(1,0,2) \mapsto (1,0,0)$
$(0,1,2) \mapsto (0,1,0)$	$(1,1,2) \mapsto (1,1,0)$
$(0,2,2) \mapsto (0,2,0)$	$(1,2,2) \mapsto (1,2,0))$

$$g : \mathcal{E}(4,2) \mapsto \mathcal{M}(3,\widetilde{0})$$

$(0,0,1,2) \mapsto (0,0,1)$	$(1,0,1,2) \mapsto (1,0,1)$
$(0,0,2,2) \mapsto (0,0,2)$	$((1,0,2,2) \mapsto (1,0,2)$
$(0,0,0,2) \mapsto (0,0,3)$	$(1,0,0,2) \mapsto (1,0,3)$
$(0,1,1,2) \mapsto (0,1,1)$	$(1,1,1,2) \mapsto (1,1,1)$
$(0,1,2,2) \mapsto (0,1,2)$	$(1,1,2,2) \mapsto (1,1,2)$
$(0,1,0,2) \mapsto (0,1,3)$	$(1,1,0,2) \mapsto (1,1,3)$
$(0,2,2,2) \mapsto (0,2,2)$	$(1,2,2,2) \mapsto (1,2,2)$
$(0,2,0,2) \mapsto (0,2,3)$	$(1,2,0,2) \mapsto (1,2,3)$

Figure 5.17 The bijections showing how $f : \mathcal{E}(3,2) \to \mathcal{M}(3,0)$ and $g : \mathcal{E}(4,2) \to \mathcal{M}(3,\widetilde{0})$.

We prove the general case.

Lemma 5.9. *There are bijections* $f : \mathcal{E}(n,n-1) \to \mathcal{M}(n,0)$ *and* $g : \mathcal{E}(n+1,n-1) \to \mathcal{M}(n,\widetilde{0})$. *Therefore*

$$M(n,0) = E(n,n-1) \quad \textit{and} \quad M(n,\widetilde{0}) = E(n+1,n-1).$$

Proof. We describe a bijection $f : \mathcal{E}(n,n-1) \to \mathcal{M}(n,0)$ and a bijection $g : \mathcal{E}(n+2,n-1) \to \mathcal{M}(n,\widetilde{0})$.

The bijection from $\mathcal{E}(n,n-1)$ to $\mathcal{M}(n,0)$ changes the last entry from $n-1$ to a 0:

$$(m_1,\ldots,m_{n-1},n-1) \to (m_1,\ldots,m_{n-1},0).$$

This mapping is clearly a bijection.

The bijection from $\mathcal{E}(n+1,n-1)$ to $\mathcal{M}(n,\widetilde{0})$ has two cases. If the sequence in $\mathcal{E}(n+1,n-1)$ ends in $(0,n-1)$ then we replace these two entries with the single entry n. Otherwise, we just remove the final $n-1$ of the sequence. In other words

$$\begin{aligned} &f : (m_1,\ldots m_{n-1},0,n-1) \to (m_1,\ldots,m_{n-1},n), \\ &g : (m_1,\ldots,m_n,n-1) \to (m_1,,\ldots,m_n) \qquad \text{when } m_n \neq 0. \end{aligned}$$

It is straight forward to check that this mapping is a bijection. □

5.7.3 The First Columns

Next we describe bijections $f_{n,k} : \mathcal{E}(k,0) \to \mathcal{M}(n,\widetilde{1}\underbrace{11\cdots1}_{n-k})$ which maps between the first columns of the triangles.

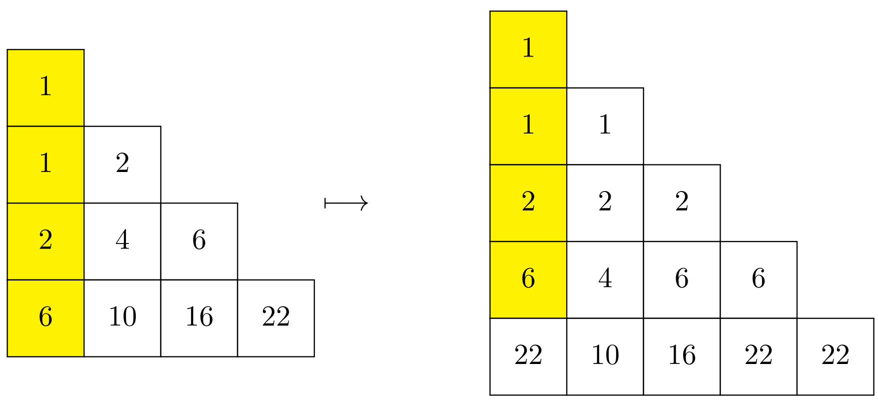


Figure 5.18 The bijection between $\mathcal{M}_3^1$ and $\mathcal{E}(0,0), \mathcal{E}(1,0), \mathcal{E}(2,0)$, and $\mathcal{E}(3,0)$.

We describe bijections:

- $f_{3,3} : \mathcal{E}(3,0) \to \mathcal{M}(3,\widetilde{1})$,
- $f_{2,3} : \mathcal{E}(2,0) \to \mathcal{M}(3,\widetilde{1}1)$,
- $f_{1,3} : \mathcal{E}(1,0) \to \mathcal{M}(3,\widetilde{1}11)$,
- $f_{0,3} : \mathcal{E}(0,0) \to \mathcal{M}(3,111)$.

Our mappings are

$$\begin{array}{lll} f_{3,3} : (m_1, m_2, 0) & \mapsto & (m_1, m_2, 3) \\ f_{2,3} : (m_1, 0) & \mapsto & (m_1, 0, 1) \\ f_{1,3} : (0) & \mapsto & (0, 1, 1) \\ f_{0,3} : \emptyset & \mapsto & (1, 1, 1). \end{array}$$

We remind the reader that $\mathcal{M}(3,\widetilde{1})$ is the set of all sequences that end in 3.

Figure 5.19 shows these bijections.

$$f_{0,3} : \mathcal{E}(0,0) \mapsto \mathcal{M}(3,111)$$

$$\emptyset \mapsto (1,1,1)$$

$$f_{1,3} : \mathcal{E}(1,0) \mapsto \mathcal{M}(3,\widetilde{1}11)$$

$$(0) \mapsto (0,1,1)$$

$$f_{2,3} : \mathcal{E}(2,0) \mapsto \mathcal{M}(3,\widetilde{1}1)$$

$$(0,0) \mapsto (0,0,1) \qquad (1,0) \mapsto (1,0,1)$$

$$f_{3,3} : \mathcal{E}(3,0) \mapsto \mathcal{M}(3,\widetilde{1})$$

$$\begin{array}{lcl} (0,0,0) \mapsto (0,0,3) & & (1,0,0) \mapsto (1,0,3) \\ (0,1,0) \mapsto (0,1,3) & & (1,1,0) \mapsto (1,1,3) \\ (0,2,0) \mapsto (0,2,3) & & (1,2,0) \mapsto (1,2,3) \end{array}$$

Figure 5.19 Bijections showing the mappings between the first columns of our two triangles.

We now prove the general result.

Lemma 5.10. *For* $0 \le k \le n$*, let* $f_{k,n} : \mathcal{E}(k, i-1) \to \mathcal{M}(n, \widetilde{1}\underbrace{11\cdots 1}_{n-k})$ *where*

$$f_{0,n} : \emptyset \to (1,1,\cdots,1)$$

and

$$f_{1,n} : (0) \to (0,1,\cdots,1)$$

and for $2 \le k \le n-1$

$$f_{k,n} : (m_1,\ldots,m_{k-1},0) \to (m_1,\ldots,m_{k-1},0,\underbrace{1,\cdots,1}_{n-k})$$

and

$$f_{n,n} : (m_1,\ldots,m_{n-1},0) \to (m_1,\ldots,m_{n-1},3).$$

Then each of these mappings is a bijection.

Proof. It is clear that each of these mappings is a bijection. We just remind the reader of two definitions. First, the sequences in $\mathcal{M}(n,\widetilde{1})$ must end in n. Second, sequences in $\mathcal{M}(n,\widetilde{1}\underbrace{11\cdots 1}_{r})$ must end in $(0,\underbrace{1,1,\cdots,1}_{r})$. □

5.7.4 Mapping the kth Columns

We now consider the general case. We describe bijections

$$f_{k,n} : \mathcal{E}(k, \ell - 1) \to \mathcal{M}(n, \widetilde{\ell}\underbrace{\ell\ell\cdots\ell}_{n+1-k})$$

which maps between the kth columns of the triangles.

As always, we start with an example. We describe bijections:

- $f_{4,3} : \mathcal{E}(4, 1) \to \mathcal{M}(3, \widetilde{2})$,
- $f_{3,3} : \mathcal{E}(3, 1) \to \mathcal{M}(3, \widetilde{2}2)$,
- $f_{2,3} : \mathcal{E}(2, 1) \to \mathcal{M}(3, \widetilde{2}22)$.

Figure 5.20 highlights these mapping in our triangles.

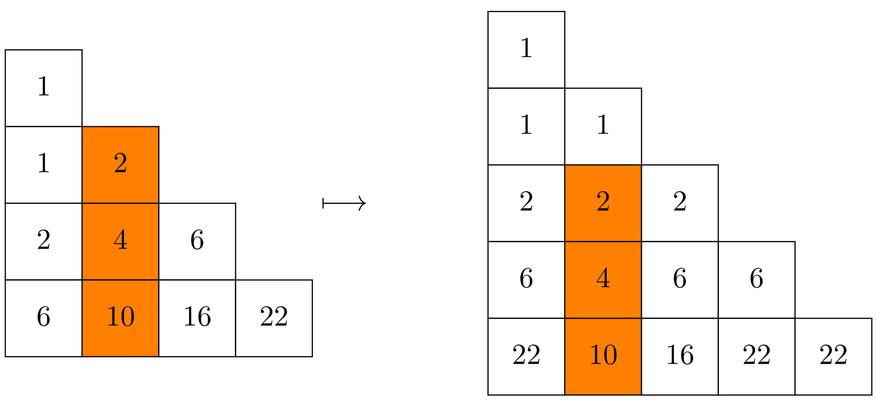


Figure 5.20 The bijection between $\mathcal{M}_3^2$ and $\mathcal{E}(2, 1)$, $\mathcal{E}(3, 1)$, and $\mathcal{E}(4, 1)$.

Our mappings are

$$f_{4,3} : \begin{cases} (m_1, m_2, 0, 1) & \mapsto & (m_1, m_2, 3) \\ (m_1, m_2, m_3, 1) & \mapsto & (m_1, m_2, m_3) \quad \text{for} \quad m_3 \neq 0 \end{cases}$$

$$f_{3,3} : (m_1, m_2, 1) \mapsto (m_1, m_2, 2)$$

$$f_{2,3} : (m_1, 1) \mapsto (m_1, 2, 2)$$

We remind the reader that $\mathcal{M}(3,\widetilde{2})$ is the set of all sequences that end in 1 or 3. Figure 5.21 shows these bijections.

$$f_{2,3}:\mathcal{E}(2,1)\mapsto\mathcal{M}(3,22)$$

$$(0,1)\mapsto(0,2,2) \qquad (1,1)\mapsto(1,2,2)$$

$$f_{3,3}:\mathcal{E}(3,1)\mapsto\mathcal{M}(3,\widetilde{2}2)$$

$$\begin{array}{lcl}(0,0,1)\mapsto(0,0,2) & & (1,0,1)\mapsto(1,0,2)\\ (0,1,1)\mapsto(0,1,2) & & (1,1,1)\mapsto(1,1,2)\end{array}$$

$$f_{4,3}:\mathcal{E}(4,1)\mapsto\mathcal{M}(3,\widetilde{2})$$

$$\begin{array}{lcl}(0,0,1,1)\mapsto(0,0,1) & & (1,0,1,1)\mapsto(1,0,1)\\ (0,0,0,1)\mapsto(0,0,3) & & (1,0,0,1)\mapsto(1,0,3)\\ (0,1,1,1)\mapsto(0,1,1) & & (1,1,1,1)\mapsto(1,1,1)\\ (0,1,0,1)\mapsto(0,1,3) & & (1,1,0,1)\mapsto(1,1,3)\\ (0,2,0,1)\mapsto(0,2,3) & & (1,2,0,1)\mapsto(1,2,3)\end{array}$$

Figure 5.21 Bijections showing how $\mathcal{M}_3^2=\mathcal{E}(2,1)+\mathcal{E}(3,1)+\mathcal{E}(4,1)$.

We now prove the general case.

Lemma 5.11. *For $\ell\le k\le n+1$, let $f_{k,n}:\mathcal{E}(k,\ell-1)\to\mathcal{M}(n,\widetilde{\ell}\underbrace{\ell\ell\cdots\ell}_{n+1-k})$ where*

$$f_{n+1,n}:\begin{cases}(m_1,\ldots,m_{n-1},0,1) & \mapsto & (m_1,\ldots,m_{n-1},n)\\ (m_1,\ldots,m_{n-1},m_n,1) & \mapsto & (m_1,\ldots,m_{n-1},m_n) \quad \textit{for} \quad 0<m_3<\ell\end{cases}$$

$$f_{k,n}:(m_1,\ldots m_{k-1},\ell-1)\mapsto(m_1,\ldots m_{k-1},\underbrace{\ell,\ell,\ldots,\ell}_{n+1-k}) \quad \textit{for} \quad \ell\le k\le n$$

Then each of these mappings is a bijection.

Proof. It is clear that each of these mappings is a bijection. We just remind the reader of two definitions. First, the sequences in $\mathcal{M}(n,\widetilde{\ell})$ is the set of all sequences that end in n or in $1\le k<\ell$. Second, sequences in $\mathcal{M}(n,\widetilde{\ell}\underbrace{\ell\ell\cdots\ell}_{r})$ must end in $(j,\underbrace{\ell,\ell,\cdots,\ell}_{r})$ where $0\le j\le\ell-1$. □

Chapter 6

Conclusion

This paper explored the family of stacked integer partitions $\mathcal{M}_n$. We started by introducing terms used throughout the paper as well as describing previous works that have informed the work in this paper.

Next, we introduced the Bell-Like numbers, and made our claim that $\mathcal{M}_n$ is enumerated by the Bell-Like numbers, and also stated that $\mathcal{M}_n$ is the interpretation for the triangular recurrence created by the Bell-Like numbers. Therefore, we were able to show that $\mathcal{M}_n$ is the combinatorial family counted by this sequence.

We then gave some background about Bell Numbers and how they relate to Bell-Like numbers. We further explored our Bell-Like numbers and the triangular recurrence used to generate the sequence. As well as giving the $\mathcal{M}_n$ interpretation some notation to easily talk about it throughout the paper.

From Chapters 3, 4, and 5 contained various proofs relating to $\mathcal{M}_n$ and its subfamilies. In Chapter 3 we focused on the bijections formed within the interpretation of $\mathcal{M}_n$. Chapter 4, focused on the poset structure of $\mathcal{M}_n$ and explored the minimum and maximum length of maximal chains of $\mathcal{M}_n$. Finally, in Chapter 5 we saw some more bijections when we added restrictions to the subsets of $\mathcal{M}_n$ and created subsets T_n, the subsets of $\mathcal{M}_n$ that have a max of 2 in its subsets, and $\mathcal{E}(n,k)$, the subsets of $\mathcal{M}_n$ that are of size n, ending in k.

The next steps in this research are further investigating the different properties within the family of $\mathcal{M}_n$. Something interesting that can be done is seeing if any new subfamilies appear if more restrictions are placed on $\mathcal{M}_n$.

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