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A Brascamp-Lieb–Rary of Examples

Anina Peersen
Macalester College, apeersen@macalester.edu

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A Brascamp-Lieb–Rary of Examples

Anina Peersen

Taryn Flock, Advisor

Lisa Naples, Reader

Vittorio Addona, Reader

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Macalester College

Department of Mathematics, Statistics and Computer Science

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Abstract

This paper focuses on the Brascamp-Lieb inequality and its applications in analysis, fractal geometry, computer science, and more. It provides a beginner-level introduction to the Brascamp-Lieb inequality alongside related inequalities in analysis and explores specific cases of extremizable, simple, and equivalent Brascamp-Lieb data. Connections to computer science and geometric measure theory are introduced and explained. Finally, the Brascamp-Lieb constant is calculated for a chosen family of linear maps.

Contents

Chapter 1

Introduction

The Brascamp-Lieb inequality was first published in 1976 by Herm Brascamp and Elliott Lieb. The inequality itself grew out of the Cauchy-Schwarz inequality, and is a generalized form of both the Loomis-Whitney inequality and Hölder's inequality. Since its publication, the inequality has had far-reaching impacts across analysis, fractal geometry, Fourier analysis, computer science, and more.

In very broad strokes, the Brascamp-Lieb inequality puts an upper bound on the size of a function. It allows us to generally know how large the product of functions is, even when we cannot directly–or at least not easily–calculate the size. However, after this short explanation, we can no longer talk in broad strokes. We must start asking what do we mean by size? What kind of functions we are dealing with? And why does the inequality matter overall? These questions (and hopefully many more) will be answered in this paper.

To give a basic road map of what to expect, we will start with a brief technical introduction to the Brascamp-Lieb inequality. Then, to strengthen our understanding, we will go through some special cases of the Brascamp-Lieb inequality namely the Cauchy-Schwarz, Hölder's, and Loomis-Whitney inequalities in Chapter 2. Then, in Chapter 3, we will dive back into the Brascamp-Lieb inequality and go more in depth. This detailed return will also highlight the rank-one case as well as different types of Brascamp-Lieb data. After this extended introduction to the inequality we will move onto applications and examples of the inequality in action: a computer science example in Chapter 4 and a geometric example in Chapter 5. In the following chapter, we will go over my work calculating the Brascamp-Lieb constant for a chosen family of linear maps. Then, in Chapter 7, we will wrap the whole exploration together and highlight the significance of the inequality. So, let us dive in; first to a brief technical introduction to the Brascamp-Lieb inequality.

1.1 What is a Brascamp-Lieb Inequality?

The Brascamp-Lieb inequality takes the form

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(B_i x)^{c_i} \leq C \prod_{i=1}^{m} \Big(\int_{\mathbb{R}^{n_i}} f_i \Big)^{c_i}$$

where each f_i is a non-negative function and each B_i is a surjective linear map from $\mathbb{R}^n \to \mathbb{R}^{n_i}$. Furthermore n, m, and n_i are positive integers and c_i is restricted to $[0, 1]$. Essentially, we are saying that the integral of the product of positive functions raised to some powers c, across all of $\mathbb{R}^n$ will be at most equal to some constant C times the product of the integrals of those functions across $\mathbb{R}^{n_i}$ raised again to the power c.

The Brascamp-Lieb inequality itself is incredibly interesting, but is also a generalized, zoomed-out view. Because it is a generalized form, it is hard to pick apart and fully grasp on first (or even seventh) glance. In my own initial understanding, it was incredibly helpful to look at some of the special cases of the Brascamp-Lieb inequality, namely the Cauchy-Schwarz, Hölder's, and Loomis-Whitney inequalities.

Chapter 2

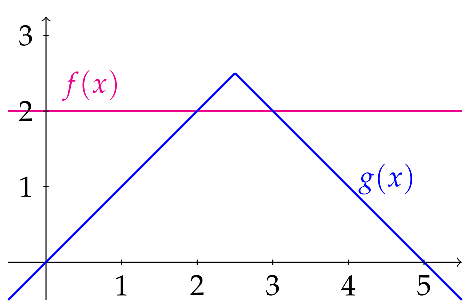
Related Inequalities

The Cauchy-Schwarz, Hölder's, and Loomis-Whitney inequalities are special cases of the Brascamp-Lieb inequality, but all predate the Brascamp-Lieb inequality itself. In essence, these inequalities are nested underneath the Brascamp-Lieb inequality family and were important starting points for Brascamp and Lieb when making the generalized inequality. These are easier introductions to the Brascamp-Lieb inequality.

For each inequality we will first go through its formal statement, then go over any necessary definitions or possible points of confusion. Next we will return to the formal statement and meaning of the inequality. We will only prove the Cauchy-Schwarz inequality, as the proofs for the other two are quite complex. However, before we dive into these inequalities, we must introduce p-norms.

2.0.1 P-Norms

As the Brascamp-Lieb inequality puts a bound on the size of a function, we must first discuss what we mean by size. Given the following two functions, which is bigger?



We can clearly see that, on the interval [0, 5] $\max |g| \geq \max |f|$, but on the same interval $\int |f| \geq \int |g|$. How can we make a definitive statement about size when we can easily come up with an example where either function could be larger? We have to use p-norms!

The formal definition of a p-norm is

$$||f||_p = (\int |f|^p)^{\frac{1}{p}}$$

where p is any positive real number. If we look at this for a second, we can see that if $p = 1$, we have

$$||f||_1 = \int |f|$$

that is, for positive functions, just the integral. Furthermore, through some more complex math that we don't need to go into specifically, we know

$$||f||_\infty = \lim_{p \to \infty} (\int |f|^p)^{\frac{1}{p}} = \max |f|$$

So in general, p-norms are a comparable notion of size of a function across many dimensions for "sufficiently nice" (i.e. continuous) functions.

With this is mind, let's return to the example using f and g on the interval $[0,5]$. If we set $p = 1$, $\int f > \int g$ so f is larger; if we think about $p \to \infty$, $\max g > \max f$ so g is larger. So we didn't really have an issue determining which function is larger, we had an issue determining what value of p, and therefore what notion of size, to use.

2.1 Cauchy-Schwarz Inequality

In technical terms, the Cauchy-Schwarz inequality is as follows:

For all vectors $\vec{u}$ and $\vec{v}$ of an inner product space, it is true that

$$| < \vec{u}, \vec{v} > |^2 \leq < \vec{u}, \vec{u} > \cdot < \vec{v}, \vec{v} >$$

where $< \cdot, \cdot >$ is the inner product. The norm $||\vec{u}|| := \sqrt{< \vec{u}, \vec{u} >}$ is related to the inner product with the defining condition $||\vec{u}||^2 =< \vec{u}, \vec{u} >$. By taking the square root of both sides, it can be rewritten

$$| < \vec{u}, \vec{v} > | \leq ||\vec{u}||||\vec{v}||$$

Now you probably have several questions: what is the inner product? What is this saying? How can we use it? Upon my first reading of the Cauchy-Schwarz inequality I had those same questions. Let us start by taking this apart piece by piece. Since the Brascamp-Lieb inequality was born from Cauchy-Schwarz, we want to make sure it is clear.

The inner product is a form of the dot product most commonly used for functions. Either the dot product or the inner product can be used for any number of or even infinite dimensions, and both provide a way to multiply vectors or functions and return a scalar. Furthermore, the triangle inequality–the notion that the length of a triangle's hypotenuse will be less than the sum of the lengths of the remaining sides–is a consequence of Cauchy-Schwarz in any inner product space.

Now, let us take the formal statement apart. We start with any two vectors $\vec{u}$ and $\vec{v}$. For simplicity's sake, let's think about this in 2D and with the dot product. We already know by properties of vectors that $\sqrt{|\vec{u} \cdot \vec{v}|^2}$ will be at most $|\vec{u} \cdot \vec{v}|$. Furthermore, we know that the magnitude–in this proof called the norm–of a vector $\vec{v}$ is equal to $\sqrt{(\vec{v} \cdot \vec{v})}$. Knowing these two properties, we can derive that the absolute value of the dot product of our two vectors (i.e. the distance between them) will be at most the product of the vectors' magnitudes. Interestingly, we can also prove that the two sides will be equal if and only if $\vec{u}$ and $\vec{v}$ are linearly dependent, meaning that in 2D space it no longer forms a triangle but congruent line segments.

The dot product proof of the Cauchy-Schwarz inequality in n dimensions is as follows. To start, let $\vec{u}$ and $\vec{v}$ be any arbitrary vectors and the arbitrary function $p(t) = ||t\vec{v} - \vec{u}||$ where t is any scalar. Since $p(t)$ is equal to the length of this combination of vectors, it will be positive by the definition of vector length. Therefore, we can start with

$$p(t) = ||t\vec{v} - \vec{u}|| \geq 0$$

Using the definition of vector magnitude and properties of the dot product, we can transform this inequality like so

$$\begin{aligned}(t\vec{v} - \vec{u}) \cdot (t\vec{v} - \vec{u}) &\geq 0 \\ (t\vec{v} \cdot t\vec{v}) - (\vec{u} \cdot t\vec{v}) - (\vec{u} \cdot t\vec{v}) + (\vec{u} \cdot \vec{u}) &\geq 0 \\ t^2(\vec{v} \cdot \vec{v}) - 2t(\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{u}) &\geq 0\end{aligned}$$

Now, for simplicity, let us define $V = (\vec{v} \cdot \vec{v})$, $U = (\vec{u} \cdot \vec{u})$, and $W = 2(\vec{u} \cdot \vec{v})$. This yields

$$t^2V - tW + U \geq 0$$

Since t is any arbitrary scalar, let us set $t = \frac{W}{2V}$.

$$\frac{W^2}{4V} - \frac{W^2}{2V} + U \geq 0$$
$$\frac{W^2 - 2W^2}{4V} + U \geq 0$$
$$U \geq \frac{W^2}{4V}$$
$$4UV \geq W^2$$

Now, we can substitute back in our definitions of V, U, and W and use the definition of vector magnitude, which shows

$$4(\vec{v} \cdot \vec{v})(\vec{u} \cdot \vec{u}) \geq 4(\vec{u} \cdot \vec{v})^2$$
$$||\vec{v}||||\vec{u}|| \geq |\vec{u} \cdot \vec{v}|$$

Thus proving the Cauchy-Schwarz inequality. This relationship holds for all values of t (but includes complex vector algebra). Nevertheless the relationship holds for any arbitrary value, so it suffices to prove one instance.

2.2 Hölder's Inequality

In technical terms, Hölder's inequality is as follows:

Let (S, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all measurable real- or complex-valued functions f, g on S,

$$||fg||_1 \leq ||f||_p ||g||_q$$

Again, the technical statement of this inequality leaves us with questions. Let us start with measure space. Simply put, this is just saying we have a space that is a collection of items which we will find the measure, a generalized notion of volume. A measure space contains three elements: the elements being measured, in this case the set S; a nonempty collection of subsets of the overall set, represented above by Σ; and an idea of measure being applied to the set, here noted as μ. Measure spaces fall into several important classes, one being probability spaces. A probability space is a measure space where the measure of the whole space is 1; for example, a bell curve where 0.68 of the space is within one standard deviation of the mean, 0.95 within two, and 0.997 within three. Another example of a measure space is a cumulative distribution function, defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

where $f(t)$ is a probability density function. Using the formulation of a measure space from Hölder's inequality, $f(t)$ is equivalent to $f(\mu)$. That is to say, the probability of a value less than or equal to x is equal to the area under a probability density function from $-\infty$ to the point of interest x.

Now, the proof for Hölder's inequality is, frankly, gnarly, and the meaning is more significant. However it is important to note that Cauchy-Schwarz is a special case of Hölder's inequality where $p = q = 2$. The proof of the Cauchy-Schwarz inequality is a simplified version of that of Hölder's inequality given this condition.

So, now onto the all-important question: what does Hölder's inequality actually mean? With this understanding, we see that Hölder's inequality is really saying the p-norm of a product of functions will be less than the product of the p-norms of each individual function, with constraints on the values of p for each p-norm. More simply put, the size of the product of functions will be less than the product of the sizes of the individual functions.

This is eerily similar to the idea of a Brascamp-Lieb inequality: to put a bound on the size of a function. This is because Hölder's inequality is in the Brascamp-Lieb family, as is the Cauchy-Schwarz inequality. Broadly, Cauchy-Schwarz is a specific case of Hölder's which is a specific case of Brascamp-Lieb. Subsequently, Hölder's inequality plays a crucial role in solving a Brascamp-Lieb inequality and will be directly included in my example.

2.3 Loomis-Whitney Inequality

In technical terms, the Loomis-Whitney inequality is as follows:

Fix a dimension, $d \geq 2$, and consider the projections

$\pi_j : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$

$\pi_j : x = (x_1, ..., x_d) \mapsto \hat{x}_j = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_d)$

For each $1 \leq j \leq d$, let

$g_j : \mathbb{R}^{d-1} \rightarrow [0, +\infty)$

$g_j \in L^{d-1}(\mathbb{R}^{d-1})$

Then the Loomis-Whitney inequality holds

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} g_j(\pi_j(x))dx \leq \prod_{j=1}^{d} ||g_j||_{L^{d-1}(\mathbb{R}^{d-1})}$$

And equivalently

$f_j(x) = g_j(x)^{d-1}$

$\int_{\mathbb{R}^d} \prod_{j=1}^{d} f_j(\pi_j(x))^{\frac{1}{d-1}} dx \leq \prod_{j=1}^{d} \left(\int_{\mathbb{R}^{d-1}} f_j(\hat{x}_j) d\hat{x}_j \right)^{\frac{1}{d-1}}$

This may be the most complicated statement of this chapter, but the overall outcome is one of the simplest: the Loomis-Whitney inequality allows us to estimate the size of a d-dimensional set using the sizes of it's $(d-1)$-dimenional projections.

To see how we arrived at that conclusion, we need to take this statement apart piece by piece. First, we define π_j as a function that maps from d-dimensional space to $(d-1)$-dimensional space by taking a series of x's as input and returning the same series just without the jth entry. To do this we need a value of d that is at minimum 2, as starting with $d = 1$ would put us into 0-dimensional space. Next, we defined a function g_j that maps from $(d-1)$-dimensional space to positive numbers and is contained in the $d-1$ Lebesgue space denoted L^p. Lebesgue spaces will show up a few times throughout this paper and morally just denotes that we have a set of functions whose domain is a defined dimensional space (in this case $d-1$ dimensions) such that the integral of each function to the p power is finite. With functions defined as such, we can then proceed to the full Loomis-Whitney inequality which states that the integral of product of the $(d-1)$-dimensional functions composed with linear projections will be at most the product of the p-norm of the same functions. If we fully expand to include the definition of p-norm, the Loomis-Whitney inequality allows us to make broad statements about what happens when we change the order of the integral and product while moving to one fewer dimension.

As with Hölder's inequality, this is very similar to the Brascamp-Lieb inequality. In fact, the Loomis-Whitney inequality is actually a special case of the Brascamp-Lieb inequality using projections π_j that by definition map onto subspaces of the one fewer dimension.

Chapter 3

The Nitty Gritty

Now that we have seen more specific versions of the Brascamp-Lieb inequality, let us return to the generalized inequality itself. As we briefly discussed previously, the Brascamp-Lieb inequality takes the form

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(B_i x)^{c_i} \leq C \prod_{i=1}^{m} \Big(\int_{\mathbb{R}^{n_i}} f_i \Big)^{c_i} \tag{3.1}$$

where each f_i is a non-negative function, each B_i is a surjective linear map from $\mathbb{R}^n \to \mathbb{R}^{n_i}$, n and m and n_i are positive integers, and $c_i \in [0,1]$. Morally, the inequality allows us to isolate the integrals of the functions and then take their product, instead of taking the integral of the product of functions, which vastly simplifies the calculations required. However this is not the only formulation of the Brascamp-Lieb inequality. In fact, there are two other forms of the inequality used in the literature.

We can equivalently say

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{p_j} \tag{3.2}$$

again requiring n and m to be positive integers, $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ to be surjective linear transformations, each p_j to be a real number in [0, 1], and, for each $j \in 1, ..., m$, $n_j \leq n$.

And, in another equivalent formulation of the inequality, we can say

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j \circ L_j \leq C \prod_{j=1}^{m} ||g_j||_{L^{q_j}(\mathbb{R}^{n_j})} \tag{3.3}$$

with the same constraints as the previous formulation, but additionally requiring $q_j = p_j^{-1}$ for all j.

We now have three different formulations of the same inequality, each with their own slight differences, so let's work through what these differences tell us. On the left side, all three formulations take the product of functions composed of their linear maps–shown by $f_i(B_i x)$ in 3.1, $f_j \circ L_j$ in 3.2, and $g_j \circ L_j$ 3.3. Now, in 3.1 and 3.2, these functions are exponentiated: by c_i in 3.1 and p_j in 3.2. These exponents may raise concern as we have said these three formulations are equivalent, but two include an exponent the third does not. This is where the additional constraint, $q_j = p_j^{-1}$, is incorporated. Essentially, we set $g_j = f_j^{p_j}$, which allowed us to move the exponent and account for its move.

On the right side, we again have some slight differences. In 3.1 and 3.2, they look essentially the same, save a difference of i vs. j and c vs. p. However, the right side of 3.3 looks vastly different: it uses p-norms. This is again due to the fact that the left sides of 3.1 and 3.2 were exponentiated, while that of 3.3 was not. Because 3.3 simply took the functions (not exponentiated functions), it is transformed into exponentiated functions (not just functions). This formulation simply shifted the placement of the exponent. Furthermore, we would be remiss if we did not address the value of p for this p-norm: $L^{q_j}(\mathbb{R}^{n_j})$. This may look very complex, but again just says we have a set of functions contained in n_j dimensional space such that $\int f^{q_j} < \infty$.

With the various formulations and their differences nailed down, we can move to the two conditions necessary for finiteness in the Brascamp-Lieb inequality: the scaling condition and the dimension condition. The scaling condition,

$$\sum_{j=1}^{m} p_j n_j = n,$$

and the dimension condition,

$$\dim(V) \leq \sum_{j=1}^{m} p_j \dim(L_j V),$$

must hold for all subspaces $V \subseteq \mathbb{R}^n$. Now the scaling condition establishes a required relationship for the exponents and the number of dimensions. If we look back to the related inequalities, the scaling condition shows up clearly in Hölder's inequality and restricts the value of p for each p-norm. The dimension condition is more subtle but in essence is a version of the scaling condition that must hold for every possible subspace. Directly checking the dimension condition would be painful, but luckily we can often determine if the inequality holds by other means, such as shown in Chapter 6.

3.1 The Formal Statement

Now that we have seen and worked through a few different formulations of the Brascamp-Lieb inequality, let's take a look at the formal statement. The Brascamp-Lieb inequality is as follows:

Fix natural numbers m and n. For $1 \leq i \leq m$, let $n_i \in \mathbb{N}$ and let $c_i > 0$ such that

$$\sum_{i=1}^{m} c_i n_i = n.$$

Choose non-negative, integrable functions

$$f_i \in L^1(\mathbb{R}^{n_i}; [0, +\infty])$$

and surjective linear maps

$$B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}.$$

Then the following inequality holds:

$$\int_{R^n} \prod_{i=1}^{m} f_i(B_i x)^{c_i} dx \leq D^{-\frac{1}{2}} \prod_{i=1}^{m} \Big(\int_{\mathbb{R}^{n_i}} f_i(y) dy \Big)^{c_i} \tag{3.4}$$

where D is given by

$$D = \inf\{ \frac{\det(\sum_{i=1}^{m} c_i B_i^* A_i B_i)}{\Pi_{i=1}^{m} (\det A_i)^{c_i}} | A_i \text{ is a positive-definite } n_i \times m_i \text{ matrix}\}.$$

Here we see more information about the Brascamp-Lieb constant: a formal definition of it. That constant is incredibly important, so let's focus on it a bit.

3.2 The Brascamp-Lieb Constant in Depth

In 3.4, the Brascamp-Lieb constant is defined to be

$$D = \inf \Big(\frac{\det(\sum_{i=1}^{m} c_i B_i^* A_i B_i)}{\Pi_{i=1}^{m} (\det A_i)^{c_i}} \Big) \tag{3.5}$$

Just as with the inequality itself, the constant has multiple formulations of the same definition. In (6), Elliot Lieb proved the best constant to be

$$\mathrm{BL}(\mathbf{L}, \mathbf{p}) = \sup \left(\frac{\prod_{j=1}^{m} (\det A_j)^{p_j}}{\det(\sum_{j=1}^{m} p_j L_j^* A_j L_j)} \right)^{\frac{1}{2}} = D^{-\frac{1}{2}}. \tag{3.6}$$

In the paper, Lieb showed that the best constant is the same as the best constant when we restrict input functions to centered Gaussians. Essentially this means that in any instance of equality in the Brascamp-Lieb inequality, the input functions are Gaussians. The specifics of this proof are beyond the purview of this paper, but is an interesting avenue of further study for anyone so inclined.

Again, since we have multiple formulations, let us work through the differences to clarify any questions of equivalence. When 3.5 is put into 3.4, it is raised to a power of $-\frac{1}{2}$. Raising a infimum to a negative power is equivalent to raising an supremum to the same positive power. After this change, we are simply left with the same notional incongruities we have seen previously: p for c and L for B.

While this definition is certainly more tangible, it is still incredibly difficult to work with. It requires us to take the product of exponentiated determinants and divide by the determinant of an $m \times m$ matrix, then we must take the supremum of this dividend, and finally square root the supremum. In fact, it took another 15 years to determine when the constant is finite.

3.3 The Rank One Case

The rank one case is a special case of the Brascamp-Lieb inequality where $n_j = 1$, i.e. using functions that map from $\mathbb{R}^n$ to $\mathbb{R}^1$. In this case, each B_j is given by $B_j(x) = \langle x, v_j \rangle$ for a certain vector v_j in $\mathbb{R}^n$. Because we end in one-dimensional space, this simplifies much of the calculations required to find the Brascamp-Lieb constant. To start, let's look at the inequality itself when $n_j = 1$:

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^1} f_j \right)^{p_j} \tag{3.7}$$

This may not look visually simpler–in fact only one thing changes. Nevertheless, knowing that we only need to take the integrals of our input functions over one-dimensional space is vastly more simple. Furthermore, using inequality 3.3, we can transform it to be

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j \circ L_j \leq C \prod_{j=1}^{m} ||g_j||_{q_j} \tag{3.8}$$

Furthermore, let us consider the scaling condition in the rank one case. It becomes

$$\sum_{j=1}^{m} p_j = n$$

which is much simpler than the previous, generalized form. This more closely resembles the restriction on p and q in Hölder's inequality.

Unsurprisingly, deciding that $\mathbb{R}^{n_j}$ will be $\mathbb{R}^1$ simplifies the necessary calculations. However, this is not an insignificant simplification and the required calculations are still extensive. The rank one case allowed mathematicians to expand their understanding of Brascamp-Lieb inequalities and is integral to scholarship on higher rank cases. In chapter 6, we will work through a rank one case for a chosen family of linear maps.

3.4 Types of Brascamp-Lieb Data

The Brascamp-Lieb datum, $(\mathbf{L}, \mathbf{p})$, has been further classified based on the adherence of the subsequent constant, $\mathrm{BL}(\mathbf{L}, \mathbf{p})$, to various conditions. This allows us to further classify points of interest from the family of linear maps and determine answers and ideas about their significance. The three types we will explore are extremizable, simple, and equivalent. These are not exclusive types, a Brascamp-Lieb datum can be multiples types at the same time. Here, I offer some quick introductions, and these types will be explored with a specific datum in chapter 6.

3.4.1 Extremizable

The constant is extremizable if there exists an m-tuple of functions that achieve equality between the two sides of the Brascamp-Lieb inequality with the constant. That is to say, the constant is extremizable if the supremum (or infimum) in the definition of the constant is a maximum (or a minimum). In (6), Lieb proved that if the datum is extremizable, it is always possible to choose the functions to be Gaussians. Commonly in the rank one case, the constant will be extremizable except values where a linear map would be repeated.

3.4.2 Simple

The constant is said to be simple if the dimension condition holds with strict inequality for all possible subspaces. That is,

$$\dim(V) < \sum_{j=1}^{m} p_j \dim(L_j V)$$

for all $V \subset \mathbb{R}^n$. Again in the rank one case, this typically holds for values except those which would repeat a linear map.

Furthermore, a vector space is called critical if, for some $V \subset \mathbb{R}^n$,

$$\dim(V) = \sum_{j=1}^{m} p_j \dim(L_j V)$$

3.4.3 Equivalent

Two Brascamp-Lieb data are said to be equivalent if changing the variables in one of the integrals superficially changes the inequality to the other without changing the validity of the inequality. This is most easily thought of as a situation where a change of variables could make it so that a family of linear maps, $\mathbf{L}$, could be transformed into another family of linear maps, $\tilde{\mathbf{L}}$.

Chapter 4

A Computer Science Example

4.1 "Structural and Computational Aspects of Brascamp-Lieb Inequalities," Avi Wigderson

In (4), Garg, Gurvits, Oliveira, and Wigderson joined the computational, geometric, and analytical aspects of Brascamp-Lieb inequalities. This was achieved by associating each Brascamp-Lieb datum with an operator scaling problem, which both created algorithmic versions of the known structural results of Brascamp-Lieb inequalities and made proofs more concise than previous scholarship. In short, this paper allowed for the previously separate computer science, geometric, and analytical scholarship on Brascamp-Lieb inequalities to meet and showed a method for concisely associating a Brascamp-Lieb datum to operator scaling. This algorithm also simplified computation for the Brascamp-Lieb constant by reducing the number of matrices which need to be optimized.

In (8), Avi Wigderson gives an adapted lecture on the methodology of the paper, most importantly on the connection from Brascamp-Lieb inequalities to operator scaling problems. The connection will be explained later in this section, but first we will focus on the background portion of (8) and the requirements of a geometric Brascamp-Lieb inequality. Finally, we will explore the connection between the geometric and computer science approaches.

Wigderson introduces Brascamp-Lieb inequalities by first giving an overview of the Cauchy-Schwarz inequality, Holder's inequality, and the Loomis-Whitney inequality-all of which we have already seen. Then Wigderson introduces a simple definition of the upper bound for the area of a 2D shape as the area of the smallest square that fully encompasses the shape. This same idea is then translated into 3D with the shape being fully encompassed by a cube. In generality, this is the essence of a Brascamp-Lieb inequality: finding an upper bound for the size (in this case area and volume which have simple area and volume computations) of a function.

To define a Brascamp-Lieb inequality, Wigderson first defines the Brascamp-Lieb datum, (B, p) in $\mathbb{R}^n$ as a list of linear maps or matrices $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ and a list of p's where $p_j \geq 0$ for all j. With this datum, for all $f_j : \mathbb{R}^{n_j} \to \mathbb{R}^+$, it is known

$$\int_{x \in \mathbb{R}^n} \prod_{j=1}^{m} (f_j(B_j x))^{p_j} \leq C \prod_{j=1}^{m} \Big(\int_{x_i \in \mathbb{R}^{n_j}} f_j(x_j) dx_j \Big)^{p_j}$$

where C is the Brascamp-Lieb constant. This is most similar to formulation 3.1 given above, save the notation $f_j(x_j)$ and the inclusion of dx_j. This does not formally show the p-norm, but as we saw previously is equivalent to formulations that do.

Furthermore, a geometric Brascamp-Lieb inequality must meet two conditions (and here remember that the B_j's are commonly matrices):

(1) The Projection Property: For all j, $B_j B_j^* = I_{n_j}$

(2) The Isotropic Property: $\sum p_j B_j^* B_j = I_n$

To find an equivalent geometric Brascamp-Lieb inequality from a general BL inequality, we must perform changes of variables such that these two conditions are met. Broadly, this is done by finding the convergence of optimal values of p_j where both conditions are met. The first condition ensures we are not breaking any foundational elements of matrices, and the second ensures the scaling condition is met in this matrix formulation. To ensure p_j converges, the Brascamp-Lieb constant itself is used as progress measure since the relationship between the constant and the basis is known even though the value of the constant is not known definitively. This is done using an application of ϵ-neighborhoods, which define the set of points within than any arbitrary distance ϵ from the point in question. From this we can determine that $BL(B, p)$ must be at least 1 and at most the largest e^{n^c}, which comes from the Brascamp-Lieb datum itself. Here c is a numerator which will be explored momentarily.

Finally, all of these elements are combined to show how to reduce Brascamp-Lieb inequalities to operator scaling problems. Before diving into that, let us first define operator scaling. Given any completely positive map of matrices $T = (A_1, A_2, ..., A_m)$ where all matrices A_i are equal in size, $T(P) = \sum A_i^* P A_i$ where A_i^* represents the transpose of A_i. This sum will be entirely positive when $P \geq 0$. Here, the analog of the Brascamp-Lieb constant is the capacity of the map operator,

$$\text{cap}(T) = \inf(\frac{\det(T(x))}{\det(x)})$$

where $x > 0$. When $\text{cap}(T)$ is positive, $\text{cap}(T) \geq e^{-n^c}$, which is the analog of the upper bound found using the geometric approach. This analog of the BL constant has fewer matrices to optimize and is computationally simpler, but still encompasses all questions from the original BL setting.

The following algorithm shows us how to apply this idea to a traditional Brascamp-Lieb inequality. To start, we are given $BL(B, p)$ where $\mathbf{B} = \{B_1, B_2, ...B_m\}$ and $p = \{p_1, p_2, ..., p_m\}$. Now, in order to make the connection to capacitance and computer science, we must restrict the p_i's to rational numbers as irrational numbers would not work. This restriction will allow us to re-write the p_i's as $\{\frac{c_1}{d}, \frac{c_2}{d}, ..., \frac{c_m}{d}\}$ where d is a common denominator. Furthermore, on the computer science side, the p_i's need to be rational numbers (computers do not like irrational numbers). At this point, we have a pile of matrices $\mathbf{B}$ which we will use to create a pile of matrices $A_1, A_2, ..., A_{m'}$ (and the difference between m and m' will be addressed later). It is important to note that A has been used as notation in various and important places throughout this paper; here, each A_i is a newly created matrix built from $\mathbf{B}$.

To create the primary matrix A from which we will build $A_1, A_2, ..., A_{m'}$, we repeat each B_i the corresponding c_i number of times. That is, A is composed of the matrices $B_1, B_2, ..., B_m$ repeated the corresponding number of times as their numerator when represented as a fraction with common denominator d, only possible because each p_i is rational. This matrix will have n columns coming from the original $\mathbf{B}$ matrices, however it will have nd rows (since it is known $\sum_{i=1}^{m} n_i p_i = n$ and each $p_i = \frac{c_i}{d}$). In general, the matrix A will be as shown below.

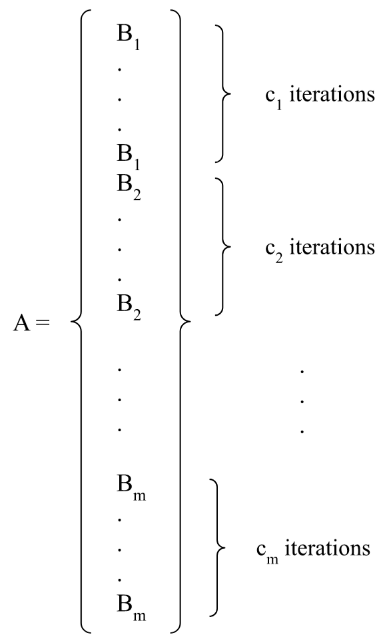


Figure 4.1.1. Matrix Template.

Once the matrix is divided in this manner, we can start to build $A_1, A_2, ..., A'_m$. In essence, each A_i is the A matrix where a specific row is isolated and all other rows are set equal to 0. As such, we are not simply creating m matrices; instead, we are creating the same number of matrices as the number of rows in A, which has been defined as $m' = \sum_{i=1}^{m} c_i$, the sum of all the numerators c defined previously. The general form of each A_i is below and is repeated such that each possible is isolated. Each matrix is again an $n \times nd$ matrix, but isolates only a single iteration of a specific B_i.

$$A_i = \left\{ \begin{array}{c} 0 \\ . \\ . \\ . \\ 0 \\ B_i \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\}$$

Figure 4.1.2 Transformed A_{ij} Matrix.

Now that we have this newly transformed matrix, we can define the operator to be the sum of matrix algebra across all the A_i's. That is to say, the operator,

$$T(x) = \sum_{i=1}^{m'} A_j^* X A_j$$

where X is Wigderson's denotation of the A found in analysis notations of $BL(B, p)$. With this, the final operator will be an $n \times n$ matrix which can be easily related to and solved for the Brascamp-Lieb constant using the relationship

$$BL(B, p) = \frac{1}{\text{cap}(T(B_n, p)}^{1/2}$$

Morally, Wigderson shows that an upper bound for the Brascamp-Lieb constant, $BL(B, p)$, found through a known geometric approach is analog to the capacity of an operator scaling problem using the above template and transformation. The analog is preferable because it requires fewer matrices to be optimized. This created a bridge between capacitance, which is an important notion in computer science, to the Brascamp-Lieb constant.

Chapter 5

A Geometric Example

5.1 Multilinear Kakeya

Kakeya problems focus on the idea of dimension and specifically how the mathematical definition of dimension aligns with a physical, human notion of dimension. This all stems from Kakeya sets, a set of points in Euclidean space that contains a line segment pointing in every direction. There has been much conjecture about how small this set can be: for example, a disk in 2D and a sphere in 3D. However, the connection to analysis and specifically Brascamp-Lieb inequalities relates to the definition of dimension. Specifically, we are asking does a Kakeya set, which includes a line segment pointing in every direction, have n dimensions?

Now you may be thinking this is pretty simple, of course it would have n dimensions. If you put a pen on a table and spin it you create a set with a line segment pointing in every direction that can fill all of 2D space. You could also do the same with a pen in space: rotating the pen would again create a set that contains a line segment pointing in every dimension that covers all of 3D space. For 2D space, the pen example holds some water: the Kakeya set has been proven to fill 2 dimensions. However, the 3D pen example is elusive. The Kakeya set has not been proven to be 3D. This is because the definition of dimension used for fractals like a Kakeya set is more of a measure of complexity than a representation of physical dimension. Even though these sets are 2D and 3D in the linear algebra sense (where the number of dimensions represents the number of coordinates needed to identify a unique point in the space), the fact that the Kakeya set is a fractal introduces fractal dimension and therefore an added level of complexity.

As an example of the dimension difficulties making the 2 dimension proof very complex and preventing the 3 dimensional proof, let us turn to triangles and some basic geometry. First, take Figures 5.1.1 and 5.1.2 below. On the left, Figure 5.1.1 shows a triangle with base b and height h. The area of this triangle will be $\frac{1}{2}bh$. Now, Figure 5.1.2 shows a triangle with base nb and height nh where n is any arbitrary positive scalar. The area of this triangle will be $\frac{1}{2}n^2bh$, that is n^2* area of left triangle. Here, the power of 2 on the scalar is an indicator of 2 dimensional space.

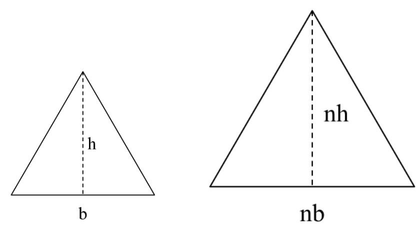


Figure 5.1.1. (left) Triangle with base b and height h.
Figure 5.1.2. (right) Triangle with base nb and height nh (specifically showing $n > 1$).

Now that we have a clear example of dimension, the next logical step is to move to a confusing example of dimension. Here, take the Sierpinski Triangle on the left with base b and height h. We will call the area shaded in blue A. On the right, we have a larger version of the fractal with base $2b$ and height $2h$. If the above pattern of dimension fit, the blue space in this triangle would have area 2^2A. However, we have 3 copies of the smaller triangle in the larger one, meaning the area of the larger triangle is actually $3A$ or $2^{\log 3/\log 2}A$.

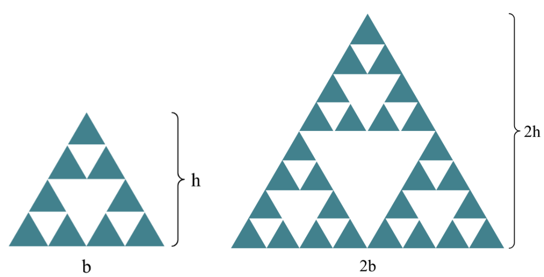


Figure 5.1.3. (left) Sierpinski Triangle with base b and height h.
Figure 5.1.4. (right) Sierpinski Triangle with base $2b$ and height $2h$.

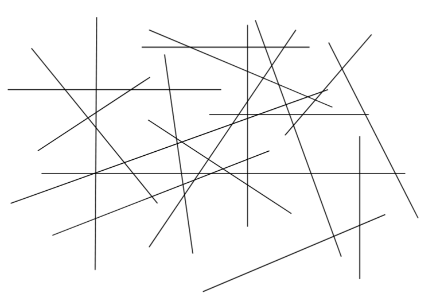
With this example, we find ourselves in $\frac{\log 3}{\log 2}$ dimensional space, i.e. 1.585 dimensional space... which doesn't seem possible. This isn't the typical definition of dimension that you might be thinking of. It is actually a fractal dimension, which is any positive real number, not necessarily a whole number. To formalize this for other fractals, we look to cover the fractal with boxes and count how many boxes are needed on different scales. Put a bit more tangibly, we are essentially projecting a grid over the fractal and counting how many boxes in that grid touch the fractal. When we repeat this with a finer grid and compare the number of boxes from the two iterations, they are related with the fractal dimension.

Now let us return to a Kakeya set but with the covering-with-boxes methodology. Take for instance a Kakeya set that includes the graph of the function

$$f(x) = \begin{cases} 1 & x \in [1,2] \\ \text{undefined} & \text{else} \end{cases}$$

We have a line segment at $y = 1$ from $x = 1$ to $x = 2$. In this instance, the covering-with-boxes methodology will, as the same suggests, cover the line with boxes of a given side length. In essence, this will fatten up the line, draw a rectangle around it, and make its height equal to the side length of the box. With various scalars, we can cover $x \in [1,2]$ for various values of y: for example, with a side length of 0.5, we would cover the region $x \in [1,2]$ and $y \in [0.75, 1.25]$ with two boxes.

This can be done to all items in the Kakeya set, such that for the set below on the left the set on the right would be created (for now, disregard colors).



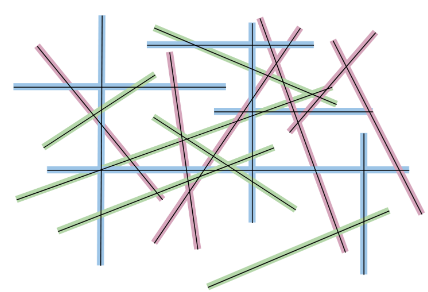


Figure 5.1.3. (left) A (small) subset of a Kakeya Set.

Figure 5.1.4. (right) The same subset of a Kakeya Set using cover-with-boxes methodology.

In 2D space we create rectangles; in 3D space, tubes. In either dimension, we want to find the area or volume contained in the rectangles or tubes. However, we have created a new problem with the cover-with-boxes method: if we calculated the area of each individual rectangle (regardless of color) in the above subset of a Kakeya set and summed the areas, some places would be counted twice and the final sum would be inaccurate. We will come back to this in a minute.

So far this chapter, we have talked about Kakeya sets and fractal dimension, but have neglected a crucial part of the chapter's name: multilinear. Multilinear simply means that we are taking a function of multiple functions which are linear in each variable, therefore making a multilinear function. This is a slightly complex because both our input and output will be functions. Ultimately, a multilinear function could look something like $T(af_1 + bg_1, af_2 + bg_2, ...)$ where f and g are functions used as input variables. Using properties of linearity, we could expand this function to be

$$aT(f_1, f_2, ...) + bT(g_1, g_2, ...).$$

Returning to Kakeya, to find the area contained within a multilinear Kakeya set, we would expect to see something similar to

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j$$

That is to say, finding the area of a multilinear Kakeya set requires the integral of a product. Larry Guth explores multilinear Kakeya inequalities in depth in (5). Specifically Guth shows that multilinear Kakeya inequalities reduce to a nearly axis-parallel case. This reduction vastly simplifies the mathematics, but is still incredibly complex. Therefore, we will be focusing on a further simplified case: the axis-parallel case.

For the axis-parallel case–a small subset of which are shaded in blue in the above example–the inequality comes directly from Loomis-Whitney (Section 2.3). Remember that Loomis-Whitney allows us to estimate the size of a set in d dimensions from $(d-1)$ dimensional projections, meaning that in the context of a 2 dimensional Kakeya set we can look at the projections in 1 dimension. Conceptually the application of Loomis-Whitney makes sense because a line parallel to an axis cannot give useful information in all of it's dimensions. For instance, in 2D space the line $x = 2$ will only meaningfully provide information in the y dimension.

In the axis-parallel case, we have a collection of line segments all parallel to an axis. In the above example (again the segments shaded in blue) we start to see the beginning of a grid. In fact, the axis-parallel case will make a grid. Using the cover with boxes methodology and Loomis-Whitney logic described above, if we project a sufficiently fine grid over the axis-parallel case, all boxes in the grid will contain part of the set. Therefore, the axis-parallel case is a conceptually relatively intuitive (although mathematically rigorous to prove) example of multilinear Kakeya inequalities.

Just as with the geometric example previously, now that we have a relatively conceptually easy example of multilinear Kakeya inequalities, it is only natural to turn to a complex example. The axis-parallel case is a wonderful dream, applicable in very very few cases. The nearly axis-parallel case expands the possibilities, but the majority of Kakeya sets will not include a nearly axis-parallel case and certainly not a true axis-parallel case. This is where Brascamp-Lieb inequalities and the area/volume sum issue raised a few paragraphs back come into play. The Brascamp-Lieb inequality allows us to isolate the spaces contained in multiple tubes and put an upper bound on the area or volume (or any other metric of size) of the fattened line segments.

It would take several pages to fully explain the application of Brascamp-Lieb inequalities in this setting. As such, we will go through the main alterations needed to use one, but will not be going through the math directly. In general, moving from the axis-parallel case to a general case requires understanding how the Brascamp-Lieb constant depends on the linear maps, which is explored for a specific family in Chapter 6. For more information on the math specifically, please refer to (5) and (9).

The most important difference in multilinear Kakeya applications is that, while previously we had seen Brascamp-Lieb inequalities with the general form

$$\int \prod_i f_i \leq C \prod_i \int f_i$$

to find a bound for the area of these Kakeya sets we must transform the input of the Brascamp-Lieb inequality to generally be of the form

$$\int \prod_i \Big(\sum_j f_{ij} \Big)$$

Where j is the set of tubes that point in a similar direction. The sum over j is performed so that the spaces included within multiple tubes is not counted multiple times.

Now you may be asking, what does 'the set of tubes that point in a similar direction' mean? Let us again return to figure 5.1.4. Each j would be a general direction of the line segment, here shown by color. Blue is axis parallel segments, pink is generally vertical segments, and green is generally horizontal segments. In a more complex example there would be more categories with formally definitions, but the general idea is the same. We categorize the segments so we can sum over each general direction while accounting for the areas included in multiple tubes.

Now, the changes we have made to the Brascamp-Lieb inequality may look big; including a sum when we are already taking the integral of a product is complex. However, in reality we are only repeating the work of a Brascamp-Lieb inequality multiple times and accounting for that repetition.

Chapter 6

My Example

Now that we have seen uses of the Brascamp-Lieb inequality in both computer science and fractal geometry, we will turn to an analysis example. Where the previous sections have had a more zoomed-out focus looking at broad applications and important innovations, this section will be specifically working through finding the best constant for the Brascamp-Lieb inequality for a chosen family of linear maps, which will require us to use other inequalities in the Brascamp-Lieb family. Furthermore, while previous sections have summarized the work of others, this example is the main deliverable from this project and was chosen as a progression of Professor Flock's previous examples. Here, we will dive into the following family of linear maps:

$$B_1(x) = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; B_2(x) = x \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; B_3(x) = x \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$B_4(x) = x \cdot \begin{bmatrix} 1 \\ -a \end{bmatrix} \; B_5(x) = x \cdot \begin{bmatrix} b \\ 1 \end{bmatrix}$$

The general overview of this process is to start by composing functions from these linear maps. To compose a function from a linear map, you can think of the top number in the map as a coefficient on x and the bottom number as a coefficient on y. Then these two terms are added to create the function input. Therefore, $B_1(x)$ would be the function $f(x)$, $B_2(x)$ $g(y)$, $B_3(x)$ $h(x-y)$, $B_4(x)$ $k(x-ay)$, and $B_5(x)$ $l(bx+y)$. Instead of keeping the B_i notation, we will switch to f, g, h, k, and l to avoid possible mistakes. Now, we can put these functions into the left side of the Brascamp-Lieb inequality (using equation 3.3) like so:

$$\int \int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dydx.$$

From here, we will split the functions into 2 groups of three functions each (with one function repeated in both groups) using the Cauchy-Schwarz inequality, which will yield

$$\int \int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dydx \leq \\ (\int \int |f(x)g(y)l(bx+y)^{\frac{1}{2}}|^2 dydx)^{\frac{1}{2}} \\ (\int \int |h(x-y)k(x-ay)l(bx+y)^{\frac{1}{2}}|^2 dydx)^{\frac{1}{2}}.$$

However, this is only one possible way to split our functions. We will have to use the same methodology for all possible function splits and plot the calculated constants for each. As we move through these possible splits, the third and final function in each group of three will be the split function and will therefore be raised to the $\frac{1}{2}$ power.

Now that we have our functions split into two groups of three, again with one function repeated in both groups, we will have to perform changes of variables in order to be able to use Young's Convolution inequality. Through the changes of variables, we will try to remove a and b when possible, but in some cases will not be able to. For these cases, we will apply a, b, or some combination of the two to the entire function input unilaterally. This will put us into a separate function space, where a and b act as constants on the entire function input, but will allow us to use Young's Convolution inequality to find the relationship in terms of p-norms, still with a and b acting as constants on the entire function input. After using Young's Convolution, we will again perform changes of variables to move the a's and b's from inside the functions to outside the functions, returning us to our original function space. Clearly, we will need to start by introducing Young's Convolution inequality.

6.1 Young's Convolution Inequality

As you can probably guess from the name, Young's convolution inequality hinges on convolution. Convolution is a mathematical operation that takes two functions as input and produces a third function which shows one way the shape of a function can be modified by another function. It can be thought of as a graphical representation of the integral of the product of the two functions where one function is reversed and shifted, evaluated for all possible values of shift. Since this is a little abstract to understand simply from reading on a page, please refer to the following image for a visual representation.

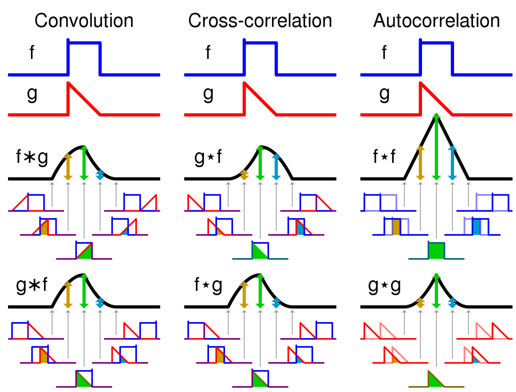


Figure 6.1.1. "Comparison convoultion correlation" by Cmglee on Wikipedia is licensed under CC BY-SA 3.0.

Now that we have a shared understanding of convolution, let us turn to the formal statement of the inequality using $*$ to denote convolution:

Suppose f is in $L^p(\mathbb{R}^d)$ and g is in $L^q(\mathbb{R}^d)$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$ with $1 \leq p,q,r \leq \infty$, then

$$||f*g||_r \leq \Big(\int_{\mathbb{R}^d}|f|^p\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^d}|g|^q\Big)^{\frac{1}{q}} = ||f||_p||g||_q$$

Through duality and an application of Hölder's inequality, the two function version implies a three function version and vice versa. Therefore, we can say, if $p,q,r \geq 1$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)h(x-y)dxdy \leq A\Big(\int_{\mathbb{R}^d}|f|^p\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^d}|g|^q\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^d}|h|^r\Big)^{\frac{1}{r}}$$

where $A = A_pA_qA_r$ and, given $1-\frac{1}{p}=\frac{1}{p'}$, $A_p = \frac{p^{1/p}}{p'^{1/p'}}$.

In essence, Young's convolution inequality allows us to go from the integral of the product of functions to the product of the functions' p-norms, given one function takes a difference of the two variables as input, and is such another example of a Brascamp-Lieb inequality. This is used in the following example after our five linear maps are split into two groups of three (again with one function included in both) using the Cauchy-Schwarz inequality. After using a change of variables transformation, we will use Young's convolution to find the relationship in terms of p-norms and calculate the constant.

6.2 Calculating the Brascamp-Lieb Constant

The family of linear maps used in this example,

$$B_1(x) = x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; B_2(x) = x \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; B_3(x) = x \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$B_4(x) = x \cdot \begin{bmatrix} 1 \\ -a \end{bmatrix} \; B_5(x) = x \cdot \begin{bmatrix} b \\ 1 \end{bmatrix}$$

has $n = 2$, $m = 5$, $n_j = 1$ for $j \in 1, 2, 3, 4, 5$, and (using the Brascamp-Lieb formulation in 3.3) $q_j = \frac{5}{2}$ in compliance with the scaling condition as $n = 2$ and we have 5 functions of equal weight.

When directly put into the left side of the Brascamp-Lieb inequality, this family of linear maps yields

$$\int \int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dxdy \tag{6.1}$$

In order to work with this and calculate the Brascamp-Lieb constant, we will have to use Cauchy-Schwarz to split the functions into 2 groups, a changes of variables to superficially change the functions, and Young's convolution to find the relationship in term of p-norms. Since we split our 5 total functions in 2 groups, we will have many different combinations to work with. We will fully work through two of these combinations to show the process, and the remaining combinations will be listed with their respective changes of variables and constants at the end of this section.

Furthermore, although we start with constant values of q_j, and therefore constant values of p_j, the Cauchy-Schwarz split changes some of the values. Since we have 5 functions, we have to repeat one function–raised to the $\frac{1}{2}$ power–on both sides of the split. For continuity the repeated function is always listed third, meaning it always has the p_3 value for its p-norm. Therefore, we know that while p_1 and p_2 remain unchanged at $\frac{5}{4}$, p_3 becomes $\frac{5}{2}$. Nevertheless, knowing these set values of each p_j, the value of A_p

is constant and calculated to be

$$\begin{aligned} A_p &= A_{p_1} A_{p_2} A_{p_3} \\ &= \Big(\frac{(\frac{5}{4})^{\frac{4}{5}}}{5^{\frac{1}{5}}} \Big)^{\frac{1}{2}} \Big(\frac{(\frac{5}{4})^{\frac{4}{5}}}{5^{\frac{1}{5}}} \Big)^{\frac{1}{2}} \Big(\frac{(\frac{5}{2})^{\frac{2}{5}}}{(\frac{5}{3})^{\frac{3}{5}}} \Big)^{\frac{1}{2}} \\ &= \frac{3^{\frac{3}{10}} 5^{\frac{1}{2}}}{2^{\frac{9}{5}}} \approx 0.893 \end{aligned}$$

6.2.1 One Example Combination

The first combination is comparatively straight forward and chosen as the resulting constant is relatively simple. Using Cauchy-Schwarz,

$$\begin{aligned} &\int \int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dxdy \\ &\leq \Big(\int \int |k(x-ay)h(x-y)f(x)^{\frac{1}{2}}|^2 dxdy \Big)^{\frac{1}{2}} \\ &\qquad \Big(\int \int |l(bx+y)g(y)f(x)^{\frac{1}{2}}|^2 dxdy \Big)^{\frac{1}{2}} \end{aligned}$$

Now that we have chosen our split, we must use changes of variables to transform each side before moving on to Young's Convolution. We need each side to be in the form $f(x)g(y)h(x-y)$. The $\frac{1}{2}$ power on the third function does not need to be changed as it will be reflected in the value of p later. Furthermore, due to the nature of these functions we will have to use a scaling trick in our changes of variables. Essentially, we will not be able to remove all instances of a and b, but we can transform the functions so that a, b, and/or some combination of the two are unilaterally applied to the entire function input, not just one variable. This trick is shown below

$$\begin{aligned} &k(x-ay)h(x-y)f(x)dxdy \\ &\qquad \text{Let } z = x-y \\ &\qquad dx = dz \\ &= k(z+y-ay)h(z)f(z+y)dzdy \\ &\qquad \text{Let } aw = z+y-ay \\ &\qquad dy = |\frac{a}{1-a}|dw \\ &= k(aw)h(z)f(\frac{-a}{1-a}(z-w))|\frac{a}{1-a}|dzdw \end{aligned}$$

and

$$\begin{aligned}
&l(bx+y)g(y)f(x)dxdy \\
&\qquad \text{Let } -z = bx \\
&\qquad dx = |\frac{1}{b}|dz \\
&= l(-(z-y))g(y)f(\frac{-1}{b}(z))|\frac{1}{b}|dzdy
\end{aligned}$$

When we put this into our previous inequality (and changing all remaining y's to be w's for continuity), we have

$$\leq |\frac{a}{1-a}||\frac{1}{b}|(\int\int |k(aw)h(z)f(\frac{-a}{1-a}(z-w))^{\frac{1}{2}}|^2 dzdw)^{\frac{1}{2}}$$
$$(\int\int |l(-(z-w))g(w)f(\frac{-1}{b}(z))^{\frac{1}{2}}|^2 dzdw)^{\frac{1}{2}}.$$

We have used Cauchy-Schwarz and a change of variables to transform our inequality, enabling us to now use Young's convolution inequality. This yields

$$\leq \left(A_p|\frac{a}{1-a}|||k(aw)||_{p_1}||h||_{p_2}||f(\frac{-a}{1-a}(z-w))||_{p_3}\right)$$
$$\left(A_p|\frac{1}{b}|||l(-(z-w))||_{p_1}||g||_{p_2}||f(\frac{-1}{b}(z))||_{p_3}\right)$$

again with $p_1 = p_2 = \frac{5}{4}$ and $p_3 = \frac{5}{2}$. Now, the change of variables trick to apply a, b, or some combination of the two to the entire function is clearly seen. We can now use another change of variables to remove the constant from the inside of the p-norm, but still keep its effect, using the definition of p-norm.

$$\begin{aligned}
||k(aw)||_{p_1} &= (\int k(aw)^{p_1} dw)^{\frac{1}{p_1}} \\
aw = y \quad dw &= |\frac{1}{a}|dy \\
&= (\int k(y)^{p_1}|\frac{1}{a}|dy)^{\frac{1}{p_1}} \\
&= |\frac{1}{a}|^{\frac{1}{p_1}}(\int k(y)^{p_1}dy)^{\frac{1}{p_1}} = |\frac{1}{a}|^{\frac{1}{p_1}}||k||_{p_1}.
\end{aligned}$$

We then do the same for all other p-norms such that the input for each function is just the variable. For the functions with input $x-y$ or $z-w$, the function input is treated as one unit. This results in

$$\leq A_p^2|\frac{a}{1-a}||\frac{1}{b}||\frac{1}{a}|^{\frac{1}{p_1}}|\frac{1-a}{-a}|^{\frac{1}{p_3}}|b|^{\frac{1}{p_3}}||k||_{p_1}||h||_{p_2}||f||_{p_3}||l||_{p_1}||g||_{p_2}||f||_{p_3}$$

, which can be simplified to

$$\leq A_p^2|\frac{1}{a}|^{\frac{1}{5}}|\frac{1}{1-a}|^{\frac{3}{5}}|\frac{1}{b}|^{\frac{3}{5}}|k||_{p_1}||h||_{p_2}||f||_{p_3}||l||_{p_1}||g||_{p_2}||f||_{p_3}$$

.

Here are the calculations put concisely in sequence:

$$\begin{aligned}
&\int\int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dxdy \\
&\quad\leq \Big(\int\int |k(x-ay)h(x-y)f(x)^{\frac{1}{2}}|^2 dxdy\Big)^{\frac{1}{2}} \\
&\qquad\quad \Big(\int\int |l(bx+y)g(y)f(x)^{\frac{1}{2}}|^2 dxdy\Big)^{\frac{1}{2}} \\
&\quad\leq |\frac{a}{1-a}||\frac{1}{b}|\Big(\int\int |k(aw)h(z)f(\frac{-a}{1-a}(z-w))^{\frac{1}{2}}|^2 dzdw\Big)^{\frac{1}{2}} \\
&\qquad\quad \Big(\int\int |l(-(z-w))g(w)f(\frac{-1}{b}(z))^{\frac{1}{2}}|^2 dzdw\Big)^{\frac{1}{2}} \\
&\quad\leq A_p^2|\frac{a}{1-a}||\frac{1}{b}|||k(aw)||_{p_1}||h||_{p_2}||f(\frac{-a}{1-a}(z-w))||_{p_3} \\
&\qquad\quad ||l(-(z-w))||_{p_1}||g||_{p_2}||f(\frac{-1}{b}(z))||_{p_3} \\
&\quad\leq A_p^2|\frac{a}{1-a}||\frac{1}{b}||\frac{1}{a}|^{\frac{1}{p_1}}|\frac{1-a}{-a}|^{\frac{1}{p_3}}|b|^{\frac{1}{p_3}} \\
&\qquad\quad ||k||_{p_1}||h||_{p_2}||f||_{p_3}||l||_{p_1}||g||_{p_2}||f||_{p_3} \\
&\quad\leq A_p^2|\frac{1}{a}|^{\frac{1}{5}}|\frac{1}{1-a}|^{\frac{3}{5}}|\frac{1}{b}|^{\frac{3}{5}}||k||_{p_1}||h||_{p_2}||f||_{p_3}||l||_{p_1}||g||_{p_2}||f||_{p_3}
\end{aligned}$$

At long last, we have calculated one possibility for the Brascamp-Lieb constant, $A_p^2|\frac{1}{a}|^{\frac{1}{5}}|\frac{1}{1-a}|^{\frac{3}{5}}|\frac{1}{b}|^{\frac{3}{5}}$. In order to fully determine the constant, we have to calculate this for all other combinations and plot the minimum of these calculated constants. Let us explore one more combination before moving on to this plot.

6.2.2 A Second Example Combination

The previous example combination was chosen for its relative simplicity, and this combination is chosen for the opposite reason: this is the most complex calculation. However, there is no difference in the methodology. We will still split using Cauchy-Schwarz, transform with a change of variables, use Young's convolution, again use changes of variables, and finally simplify a combination of a's and b's.

Using Cauchy-Schwarz, we know

$$\begin{aligned} &\int\int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dxdy \\ &\quad \leq \Big(\int\int |k(x-ay)l(bx+y)h(x-y)^{\frac{1}{2}}|^2 dxdy\Big)^{\frac{1}{2}} \\ &\qquad \Big(\int\int |f(x)g(y)h(x-y)^{\frac{1}{2}}|^2 dxdy\Big)^{\frac{1}{2}} \end{aligned}$$

However, we are now faced with simultaneously a very difficult change of variables–as k, l, and h are together–and no need for a change of variables–as f, g, and h are already together. For k, l, and h, we will have to use multiple changes of variables, and, to visually simplify this process, we will define $\alpha = \frac{a-1}{b+1}$.

$$
\begin{aligned}
&k(x-ay)l(bx+y)h(x-y)dxdy\\
&\qquad \text{Let } z = x-y\\
&\qquad dx = dz\\
&= k(z+y-ay)l(bz+by+y)h(z)dzdy\\
&\qquad \text{Let } w = by+y\\
&\qquad dy = |\frac{1}{b+1}|dw\\
&= k(z+\frac{w}{b+1}-\frac{aw}{b+1})l(bz+w)h(z)dzdw|\frac{1}{b+1}|\\
&= k(z-\alpha w)l(bz+w)h(z)dzdw|\frac{1}{b+1}|\\
&\qquad \text{Let } m = bz+w\\
&\qquad dw = dm\\
&= k(z-\alpha m-\alpha bz)l(m)h(z)dzdm|\frac{1}{b+1}|\\
&\qquad \text{Let } n = z-\alpha bz\\
&\qquad dz = |\frac{1}{1-\alpha b}|dn\\
&= k(n-\alpha m)l(m)h(\frac{1}{1-\alpha b}n)dndm|\frac{1}{b+1}||\frac{1}{1-\alpha b}|\\
&\qquad \text{Let } s = \alpha m\\
&\qquad dm = |\frac{1}{\alpha}|ds\\
&= k(n-s)l(\frac{1}{\alpha}s)h(\frac{1}{1-\alpha b}n)dnds|\frac{1}{b+1}||\frac{1}{1-\alpha b}||\frac{1}{\alpha}|
\end{aligned}
$$

Which, substituting in $x = n$ and $y = s$, results in

$$k(x-y)l(\frac{1}{\alpha}y)h(\frac{1}{1-\alpha b}x)dxdy|\frac{1}{b+1}||\frac{1}{1-\alpha b}||\frac{1}{\alpha}|.$$

Now, we can use Young's convolution to find

$$
\begin{aligned}
\leq \Big(A_p|\frac{1}{b+1}||\frac{1}{1-\alpha b}||\frac{1}{\alpha}|||k||_{p_1}||l(\frac{1}{\alpha})y||_{p_2}||h(\frac{1}{1-\alpha b})x||_{p_3}\Big)\\
(A_p||f||_{p_1}||g||_{p_2}||h||_{p_3}).
\end{aligned}
$$

Again perform the change of variables within each p-norm to yield

$$\leq A_p^2 \left|\frac{1}{b+1}\right|\left|\frac{1}{1-\alpha b}\right|\left|\frac{1}{\alpha}\right||\alpha|^{\frac{1}{p_2}}|1-\alpha b|^{\frac{1}{p_3}}$$
$$||k||_{p_1}||l||_{p_2}||h||_{p_3}||f||_{p_1}||g||_{p_2}||h||_{p_3},$$

which can be simplified to

$$\leq A_p^2 \left|\frac{1}{b+1}\right|\left|\frac{1}{\alpha}\right|^{\frac{1}{5}}\left|\frac{1}{1-\alpha b}\right|^{\frac{1}{5}}||k||_{p_1}||l||_{p_2}||h||_{p_3}||f||_{p_1}||g||_{p_2}||h||_{p_3}.$$

Let's again look at these transformations concisely:

$$\int\int f(x)g(y)h(x-y)k(x-ay)l(bx+y)dxdy$$
$$\leq \left(\int\int |k(x-ay)l(bx+y)h(x-y)^{\frac{1}{2}}|^2 dxdy\right)^{\frac{1}{2}}$$
$$\left(\int\int |f(x)g(y)h(x-y)^{\frac{1}{2}}|^2 dxdy\right)^{\frac{1}{2}}$$
$$\leq \left|\frac{1}{b+1}\right|\left|\frac{1}{1+\alpha b}\right|\left|\frac{1}{\alpha}\right|\left(\int\int |k(x-y)l(\frac{1}{\alpha}y)h(\frac{1}{1+\alpha b}x)^{\frac{1}{2}}|^2 dxdy\right)^{\frac{1}{2}}$$
$$\left(\int\int |f(x)g(y)h(x-y)^{\frac{1}{2}}|^2 dxdy\right)^{\frac{1}{2}}$$
$$\leq \left|\frac{1}{b+1}\right|\left|\frac{1}{1+\alpha b}\right|\left|\frac{1}{\alpha}\right|(A_p||k||_{p_1}||l(\frac{1}{\alpha})y||_{p_2}||h(\frac{1}{1+\alpha b})x||_{p_3})$$
$$(A_p||f||_{p_1}||g||_{p_2}||h||_{p_3})$$
$$\leq A_p^2\left|\frac{1}{b+1}\right|\left|\frac{1}{1+\alpha b}\right|\left|\frac{1}{\alpha}\right||\alpha|^{\frac{1}{p_2}}|1+\alpha b|^{\frac{1}{p_3}}$$
$$||k||_{p_1}||l||_{p_2}||h||_{p_3}||f||_{p_1}||g||_{p_2}||h||_{p_3}$$
$$\leq A_p^2\left|\frac{1}{b+1}\right|\left|\frac{1}{\alpha}\right|^{\frac{1}{5}}\left|\frac{1}{1-\alpha b}\right|^{\frac{1}{5}}||k||_{p_1}||l||_{p_2}||h||_{p_3}||f||_{p_1}||g||_{p_2}||h||_{p_3}.$$

Again, we end with another possible value of the Brascamp-Lieb constant: $A_p^2|\frac{1}{b+1}||\frac{1}{\alpha}|^{\frac{1}{5}}|\frac{1}{1-\alpha b}|^{\frac{1}{5}}$. Now that we have worked through two possible combinations at varying levels of difficulty, we can move on.

6.2.3 All Function Combinations

The following table shows different splits of our 5 functions and their respective constants. The minimum of all of these constants is graphed in the next section.

Cauchy-Schwarz Split	Change of Variables: Left Side	Change of Variables: Right Side	Constant	Simplified Constant
$\int kfg * \int lhg$	$w = ay \quad dy = \|\frac{1}{a}\|dw$	$z = x - y \quad dx = dz$ $w = by + y$ $dy = \|\frac{1}{b+1}\|dw$	$A_p^2\|a\|^{\frac{1}{p_3}}\|\frac{1}{a}\|\|\frac{1}{b}\|^{\frac{1}{p_1}}$ $\|\frac{b+1}{b}\|^{\frac{1}{p_3}}\|\frac{b}{b+1}\|$	$A_p^2\|\frac{1}{a}\|^{\frac{3}{5}}\|\frac{1}{b}\|^{\frac{1}{5}}\|\frac{1}{b+1}\|^{\frac{3}{5}}$
$\int kfh * \int lgh$	$z = x - y \quad dx = dz$ $aw = z + y - ay$ $dy = \|\frac{a}{1-a}\|dw$	$z = x - y \quad dx = dz$ $w = by + y$ $dy = \|\frac{1}{b+1}\|dw$	$A_p^2\|1 - a\|^{\frac{1}{p_2}}\|a\|^{\frac{1}{p_3}}$ $\|\frac{1}{1-a}\|\|\frac{1}{a}\|\|b+1\|^{\frac{1}{p_2}}$ $\|b\|^{\frac{1}{p_3}}\|\frac{1}{b}\|\|\frac{1}{b+1}\|$	$A_p^2\|\frac{1}{1-a}\|^{\frac{1}{5}}\|\frac{1}{a}\|^{\frac{3}{5}}$ $\|\frac{1}{b+1}\|^{\frac{1}{5}}\|\frac{1}{b}\|^{\frac{3}{5}}$
$\int kfl * \int ghl$	$w = y + bx \quad dy = dw$ $az = x - abx$ $dx = \|\frac{a}{1-ab}\|dz$	$z = x - y \quad dx = dz$ $w = by + y$ $dy = \|\frac{1}{b+1}\|dw$	$A_p^2\|\frac{1}{ab+1}\|\|\frac{ab+1}{a}\|^{\frac{1}{p_1}}$ $\|\frac{1}{a}\|^{\frac{1}{p_2}}\|\frac{1}{b}\|\|\frac{1}{b+1}\|$ $\|b+1\|^{\frac{1}{p_1}}\|b\|^{\frac{1}{p_2}}$	$A_p^2\|\frac{1}{a}\|^{\frac{3}{5}}\|\frac{1}{ab+1}\|^{\frac{1}{5}}$ $\|\frac{1}{b}\|^{\frac{1}{5}}\|\frac{1}{b+1}\|^{\frac{1}{5}}$
$\int kgf * \int lhf$	$w = ay \quad dy = \|\frac{1}{a}\|dw$	$w = x - y \quad dy = dw$ $bz = bx + x$ $dx = \|\frac{b}{b+1}\|dz$	$A_p^2\|\frac{1}{a}\|^{\frac{1}{p_1}}\|\frac{1-a}{a}\|^{\frac{1}{p_3}}\|\frac{a}{1-a}\|$ $\|b+1\|^{\frac{1}{p_3}}\|\frac{1}{b+1}\|$	$A_p^2\|\frac{1}{a}\|^{\frac{1}{5}}\|\frac{1}{1-a}\|^{\frac{3}{5}}\|\frac{1}{b+1}\|^{\frac{3}{5}}$
$\int kgh * \int lfh$	$z = x - y \quad dx = dz$ $w = ay - y$ $dy = \|\frac{1}{a-1}\|dw$	$w = x - y \quad dy = dw$ $bz = bx + x$ $dx = \|\frac{b}{b+1}\|dz$	$A_p^2\|a - 1\|^{\frac{1}{p_2}}\|\frac{1}{1-a}\|$ $\|b+1\|^{\frac{1}{p_2}}\|\frac{1}{b+1}\|$	$A_p^2\|\frac{1}{a-1}\|^{\frac{1}{5}}\|\frac{1}{b+1}\|^{\frac{1}{5}}$
$\int kgl * \int fhl$	$z = x - y \quad dx = dz$ $-bw = bay + y$ $dy = \|\frac{b}{ba+1}\|dw$	$w = x - y \quad dy = dw$ $bz = bx + x$ $dx = \|\frac{b}{b+1}\|dz$	$A_p^2\|\frac{b}{ba+1}\|\|\frac{ba+1}{b}\|^{\frac{1}{p_1}}$ $\|\frac{1}{b}\|\frac{1}{p_3}\|\frac{1}{b+1}\|\|b+1\|\frac{1}{p_1}$	$A_p^2\|\frac{1}{b}\|^{\frac{1}{5}}\|\frac{1}{ba+1}\|^{\frac{1}{5}}$ $\|\frac{1}{b+1}\|^{\frac{1}{5}}$

Cauchy-Schwarz	Change of Variables			Simplified
Split	Left Side	Right Side	Constant	Constant
$\int khf * \int lgf$	$z = x - y \quad dx = dz$ $aw = z + y - ay \quad dy = \lvert\frac{a}{1-a}\rvert dw$	$z = -bx$ $dx = \lvert\frac{1}{b}\rvert dz$	$A_p^2\lvert\frac{1}{a}\rvert^{\frac{1}{p_1}}\lvert\frac{1-a}{a}\rvert^{\frac{1}{p_3}}$ $\lvert\frac{a}{1-a}\rvert\lvert b\rvert^{\frac{1}{p_3}}\lvert\frac{1}{b}\rvert$	$A_p^2\lvert\frac{1}{a}\rvert^{\frac{1}{5}}\lvert\frac{1}{1-a}\rvert^{\frac{3}{5}}\lvert\frac{1}{b}\rvert^{\frac{3}{5}}$
$\int khg * \int lfg$	$z = x - y \quad dx = dz$ $w = ay - y \quad dy = \lvert\frac{1}{a-1}\rvert dw$	$z = -bx$ $dx = \lvert\frac{1}{b}\rvert dz$	$A_p^2\lvert a-1\rvert^{\frac{1}{p_3}}\lvert\frac{1}{1-a}\rvert$ $\lvert b\rvert^{\frac{1}{p_2}}\lvert\frac{1}{b}\rvert$	$A_p^2\lvert\frac{1}{1-a}\rvert^{\frac{3}{5}}\lvert\frac{1}{b}\rvert^{\frac{1}{5}}$
$\int khl * \int fgl$	$z = x - y \quad dx = dz$ $w = by + y \quad dy = \lvert\frac{1}{b+1}\rvert dw$ $m = bz + w \quad dw = dm$ $n = z - \alpha bz \quad dz = \lvert\frac{1}{1-\alpha b}\rvert dn$ $s = \alpha m \quad dm = \lvert\frac{1}{\alpha}\rvert ds$	$z = -bx$ $dx = \lvert\frac{1}{b}\rvert dz$	$A_p^2\lvert\frac{1}{b+1}\rvert\lvert\frac{1}{1-\alpha b}\rvert\lvert\frac{1}{\alpha}\rvert$ $\lvert 1-\alpha b\rvert^{\frac{1}{p_1}}\lvert\alpha\rvert^{\frac{1}{p_3}}$ $\lvert\frac{1}{b}\rvert\lvert b\rvert^{\frac{1}{p_1}}$	$A_p^2\lvert\frac{1}{b+1}\rvert\lvert\frac{1}{1-\alpha b}\rvert^{\frac{1}{5}}$ $\lvert\frac{1}{\alpha}\rvert^{\frac{3}{5}}\lvert\frac{1}{b}\rvert^{\frac{1}{5}}$
$\int klf * \int ghf$	$w = y + bx \quad dy = dw$ $az = x - abx \quad dx = \lvert\frac{a}{1-ab}\rvert dz$	N/A	$A_p^2\lvert\frac{1}{a}\rvert^{\frac{1}{p_1}}\lvert\frac{ab+1}{a}\rvert^{\frac{1}{p_3}}$ $\lvert\frac{a}{ab+1}\rvert$	$A_p^2\lvert\frac{1}{a}\rvert^{\frac{1}{5}}\lvert\frac{1}{ab+1}\rvert^{\frac{1}{5}}$
$\int klg * \int fhg$	$z = x - y \quad dx = dz$ $-bw = bay + y \quad dy = \lvert\frac{b}{ba+1}\rvert dw$	N/A	$A_p^2\lvert\frac{1}{b}\rvert^{\frac{1}{p_2}}\lvert\frac{ba+1}{b}\rvert^{\frac{1}{p_3}}$ $\lvert\frac{b}{ba+1}\rvert$	$A_p^2\lvert\frac{1}{b}\rvert^{\frac{1}{5}}\lvert\frac{1}{ba+1}\rvert^{\frac{3}{5}}$
$\int klh * \int fgh$	$z = x - y \quad dx = dz$ $w = by + y \quad dy = \lvert\frac{1}{b+1}\rvert dw$ $m = bz + w \quad dw = dm$ $n = z - \alpha bz \quad dz = \lvert\frac{1}{1-\alpha b}\rvert dn$ $s = \alpha m \quad dm = \lvert\frac{1}{\alpha}\rvert ds$	N/A	$A_p^2\lvert\frac{1}{b+1}\rvert\lvert\frac{1}{1-\alpha b}\rvert\lvert\frac{1}{\alpha}\rvert$ $\lvert 1-\alpha b\rvert^{\frac{1}{p_1}}\lvert\alpha\rvert^{\frac{1}{p_2}}$	$A_p^2\lvert\frac{1}{b+1}\rvert\lvert\frac{1}{\alpha}\rvert^{\frac{1}{5}}$ $\lvert\frac{1}{1-\alpha b}\rvert^{\frac{1}{5}}$

Cauchy-Schwarz Split	Change of Variables: Left Side	Change of Variables: Right Side	Constant	Simplified Constant
$\int fgk * \int hlk$	$w = ay \quad dy = \lvert\frac{1}{a}\rvert dw$	$z = x - y \quad dx = dz$ $w = by + y \quad dy = \lvert\frac{1}{b+1}\rvert dw$ $m = bz + w \quad dw = dm$ $n = z - \alpha bz \quad dz = \lvert\frac{1}{1-\alpha b}\rvert dn$ $s = \alpha m \quad dm = \lvert\frac{1}{\alpha}\rvert ds$	$A_p\lvert a\rvert^{\frac{1}{p_2}}\lvert\frac{1}{a}\rvert\lvert\frac{1}{b+1}\rvert$ $\lvert\frac{1}{1-\alpha b}\rvert\lvert\frac{1}{\alpha}\rvert$ $\lvert 1-\alpha b\rvert^{\frac{1}{p_1}}\lvert\alpha\rvert^{\frac{1}{p_2}}$	$A_p^2\lvert\frac{1}{a}\rvert^{\frac{1}{5}}\lvert\frac{1}{b+1}\rvert$ $\lvert\frac{1}{1-\alpha b}\rvert^{\frac{1}{5}}\lvert\frac{1}{\alpha}\rvert^{\frac{1}{5}}$
$\int fhk * \int glk$	$z = x - y \quad dx = dz$ $aw = z + y - ay$ $dy = \lvert\frac{a}{1-a}\rvert dw$	$z = x - y \quad dx = dz$ $-bw = bay + y$ $dy = \lvert\frac{b}{ba+1}\rvert dw$	$A_p^2\lvert\frac{a}{1-a}\rvert\lvert\frac{1-a}{1}\rvert^{\frac{1}{p_1}}$ $\lvert\frac{1}{a}\rvert^{\frac{1}{p_3}}\lvert\frac{1}{ba+1}\rvert\lvert b\rvert$ $\lvert\frac{ba+1}{b}\rvert^{\frac{1}{p_1}}\lvert\frac{1}{b}\rvert^{\frac{1}{p_2}}$	$A_p^2\lvert\frac{1}{a}\rvert^{\frac{1}{5}}\lvert\frac{1}{1-a}\rvert^{\frac{1}{5}}$ $\lvert\frac{1}{ba+1}\rvert^{\frac{1}{5}}\lvert\frac{1}{b}\rvert^{\frac{3}{5}}$
$\int ghk * \int flk$	$z = x - y \quad dx = dz$ $w = ay - y$ $dy = \lvert\frac{1}{a-1}\rvert dw$	$w = y + bx \quad dy = dw$ $az = x - abx$ $dx = \lvert\frac{a}{1-ab}\rvert dz$	$A_p^2\lvert a-1\rvert^{\frac{1}{p_1}}\lvert\frac{1}{1-a}\rvert$ $\lvert\frac{1}{a}\rvert^{\frac{1}{p_3}}\lvert\frac{1-ab}{a}\rvert^{\frac{1}{p_1}}\lvert\frac{a}{1-ab}\rvert$	$A_p^2\lvert\frac{1}{a-1}\rvert^{\frac{1}{5}}\lvert\frac{1}{a}\rvert^{\frac{1}{5}}$ $\lvert\frac{1}{1-ab}\rvert^{\frac{1}{5}}$

6.3 The Brascamp-Lieb Constant Visually

Visually, the minimum of the constants listed above takes the shape

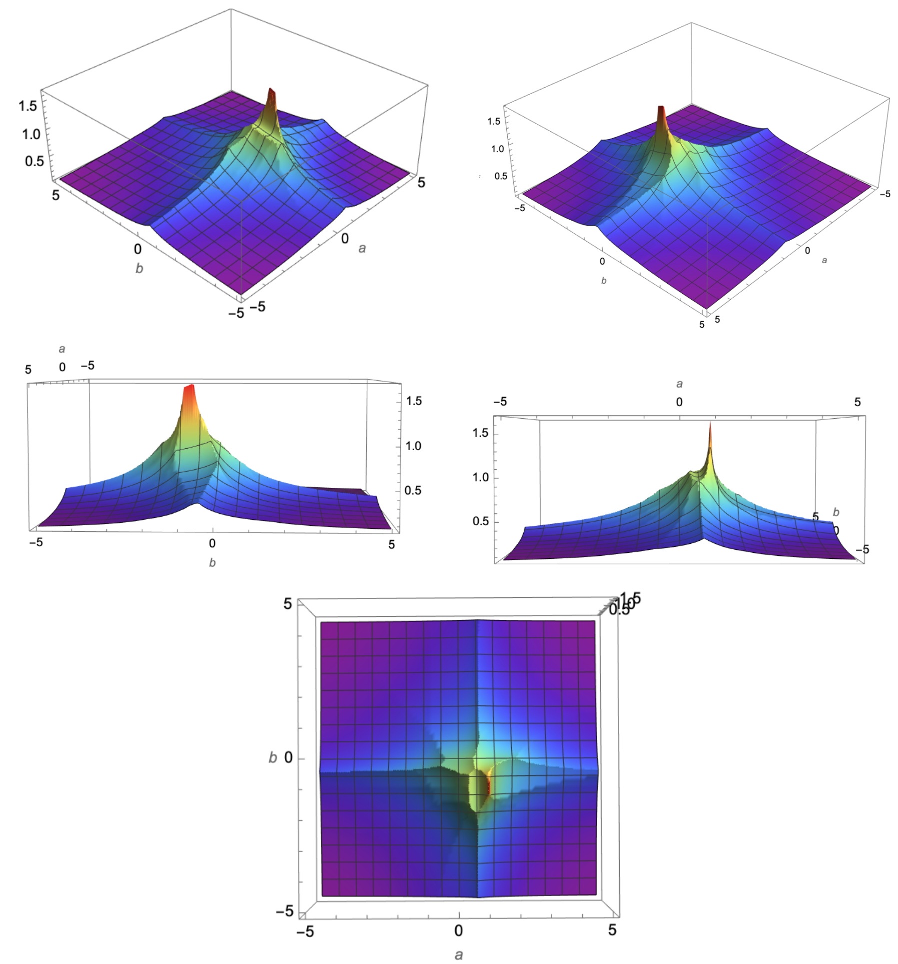


Figure 6.4.1. Various Angles of the Graphed Minimum Constant.

This graph has several interesting features. Firstly, a spike where $-1 \leq b \leq -0.5$ along $a = 1$. Thus it is not differentiable at (a, b) pairs $(1, -1)$ and $(1, -0.5)$. This is because when $a = 1$, B_4 would be equal to B_3, causing the spike. Additionally, we see ridged going up to this spike approximately along $a = 0.5$ and $b = -0.5$. These ridges appear because, with these values, B_4 and B_5 would be identical. The remaining smaller ridges also appear at values of a and b where linear maps would be identical.

6.4 Types of Brascamp-Lieb Data

Returning to the types of Brascamp-Lieb Data discussed in Chapter 3 Section 4, we can determine where the datum graphed above is each type. This datum is extremizable except when $a = 0, 1$ and $b = 0, -1, \frac{-1}{a}$. It is also simple for the same range. There are numerous possible equivalent Brascamp-Lieb data that could be created with changes of variables.

Chapter 7

Conclusion

So, we've been introduced to the Brascamp-Lieb inequality–both broad strokes and details–as well as other inequalities in the Brascamp-Lieb family. We explored an application in computer science as well as fractal geometry. We even calculated the constant for a chosen family of linear maps. The only thing left is to talk about the inequality's significance.

The information in this paper does not translate well to real-world uses. The easiest real-world application would be likening the span of the Kakeya set to moving a pen in space or moving a large log around a forest. Even these examples are lackluster and specific. So does the Brascamp-Lieb inequality have real world applications? Perhaps not, at least not ones that are direct and tangible. The concepts and inequalities covered in this paper tough the laws of math and statistics. However, lacking direct and tangible applications does not make the inequality unimportant.

If you can indulge my Classics minor for a second, I would like to bring us to the earliest studies of mathematics: a branch of philosophy. Pythagoras is yes famous for the Pythagorean Theorem, but also is known for establishing a functionally monastic school where members lived a largely ascetic life, shared their possessions, and had communal meals. Pythagoras was viewed as a divine figure, sent by the gods for the benefit of humankind. While we may be several millennia from Pythagoras (and I will admit, I have never viewed Brascamp or Lieb as god-sends), we can still view mathematics as more than just complex problems begetting more complex problems. Mathematics is an exploration of the limits of human thought, and Brascamp-Lieb inequalities a piece of this expanse.

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