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Staircase Arrangements of Pillars with Distinct Heights

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Staircase Arrangements of Pillars with Distinct Heights

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May, 2023

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Abstract

We study the family $\mathcal{A}_n$ of sequences $(a_1, a_2, \ldots, a_n)$ where $0 \leq a_k \leq k$ and nonzero entries are distinct. We show that these sequences are in bijection with the set partitions of $[n+1]$. These sequences have a natural poset structure, and we analyze the maximal chains of this poset. Finally, we explore various subfamilies of $\mathcal{A}_n$, including sequences whose largest entry is k and sequences missing the value k.

Contents

Chapter 1

Introduction

Combinatorics is a branch of mathematics that focuses on counting sets and their various constructions. We focus on the field of enumerative combinatorics to investigate a family of integer sequences and their construction.

In this introductory chapter, we will introduce various combinatorial families, including permutations, set partitions, partially ordered sets, and binary triangles.

In Chapter 2, we will define and describe $\mathcal{A}_n$, our family of sequences highlighted in this project. We will show that $\mathcal{A}_n$ are in bijection with set partitions of $[n+1]$.

In Chapter 3, we will explore the partially ordered set of $\mathcal{A}_n$. This will include our maximal and minimal elements, maximal chains, and coverings of elements in the poset.

In Chapter 4, we look at three different subfamilies of $\mathcal{A}_n$.

1.1 Combinatorial Families

We begin by offering definitions of combinatorial families that are relevant to the main focus of this paper. This will include permutations, set partitions, and binary triangles.

1.1.1 Permutations

Definition 1.1. A permutation is an ordering of n distinct elements such that each element appears once.

Table 1.1 depicts all the permutations of the set $[4] = \{1, 2, 3, 4\}$. There are $n!$ permutations for n elements since we can choose one of n elements for the first spot, $(n-1)$ elements for the second spot, and so on.

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Table 1.1 The $4! = 24$ Permutations of $[4]$

1.1.2 Set Partitions

Definition 1.2. A set partition of the set $S = \{1, 2, 3, \ldots, n\}$ into k parts is a collection of k disjoint subgroups, or blocks, such that the union of these blocks is S.

4 | 3 | 2 | 1	4 | 3 | 21	4 | 31 | 2	41 | 3 | 2	4 | 32 | 1
42 | 3 | 1	43 | 2 | 1	43 | 21	42 | 31	41 | 32
432 | 1	431 | 2	421 | 3	4 | 321	4321

Table 1.2 Set Partitions of $[4]$

Table 1.2 depicts all the set partitions of $[4]$. We write our set partitions in decreasing form, such that we have blocks

$$a_0, a_1, \ldots, a_i | b_0, b_1, \ldots, b_j | \ldots | f_0, f_1, \ldots, f_k$$

where $a_0 > b_0 > \cdots > f_0$ and $a_0 > a_1 > \cdots > a_i$ and likewise for all other blocks. In other words,

- the elements within our blocks are arranged in decreasing order, and
- the blocks are arranged in decreasing order of maximum element.

The nth Bell number B_n counts all the ways to partition a set with n elements. Here are the first few Bell numbers, starting with B_0:

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, \ldots$$

The Bell numbers can be generated from the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

1.1.3 Partially Ordered Sets

Definition 1.3. A partially ordered set, or poset $(S, \preceq)$, is a set S along with a partial ordering $\preceq$ such that

1. $s \preceq s$ for all $s \in S$,
2. if $s \preceq t$ and $t \preceq s$, then $s = t$, and
3. if $r \preceq s$ and $s \preceq t$, then $r \preceq t$.

When $s \preceq t$, we say that s and t are comparable. We say that the set S is *partially* ordered because not every element in S needs to be comparable. There may be elements $s, t \in S$ such that neither $s \preceq t$ nor $t \preceq s$.

1.1.4 Binary Triangles

Definition 1.4. A binary triangle is a left-justified binary array $T(i, j)$ for $1 \leq j \leq i \leq n$.

A binary triangle of size n has n rows and n columns, enumerated bottom to top and left to right respectively. Binary triangles can be visualised as in Figure 1.1 with filled and unfilled squares, corresponding to 1 and 0 respectively.

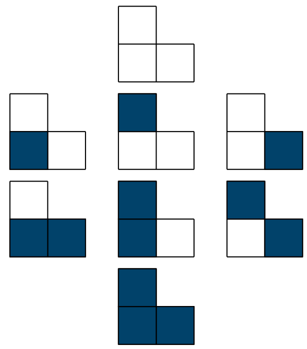


Figure 1.1 Binary Triangles of Size 2

1.2 Bijections

Definition 1.5. A bijection is a function $\varphi : A \to B$ between two sets A an B such that

1. For all $b \in B$, there exists $a \in A$ such that $\varphi(a) = b$, and
2. If for $a_1, a_2 \in A$ we have $\varphi(a_1) = \varphi(a_2)$, then $a_1 = a_2$.

1.3 Approval Ballot Triangles

This project is preceded by research on approval ballot triangles (ABTs). Approval ballot triangles (ABTs) are a family of triangular arrays that is in bijection with totally symmetric self-complimentary plane partitions (TSSCPPs). A TSSCPP of order n is a plane partition in a $2n \times 2n \times 2n$ box with with maximum possible symmetry. Andrews [1] proved that the number of TSSCPPs of order n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!},$$

see OEIS A005130 [8]. See Bressoud [5] for a recounting of the history and mathematical connections of this remarkable formula.

Beveridge and Calaway [4] recently introduced the family of approval ballot triangles. ABTs are a binary encoding of a nest of lattice paths obtained from the fundamental domain of a TSSCPP. This was proven in [4], but equivalent encodings appear in [6] and [9].

Definition 1.6. An *approval ballot triangle (ABT)* of order n is a binary triangular array $A(i,j)$ for $1 \le j \le i \le n-1$ satisfying the following row condition

$$\sum_{k=j}^{k} A(i,k) \le \sum_{k=j}^{i+1} A(i+1,k) \text{ for } 1 \le j \le i \le n-2.$$

Note that an ABT of order n has $n-1$ rows. Intuitively, a binary triangle is an ABT when row $i+1$ ends with at least as many ones as row i. Beveridge and Calaway show that ABTs encode an approval voting process with $n-1$ ballots in which candidate i never trails candidate j whenever $1 \le i < j \le n$. This generalizes many known ballot problems, including the famous Bertrand Ballot Problem for two candidates. See Barton and Mallows [3], Takàcs [10], and Renault [7] for surveys of ballot problems.

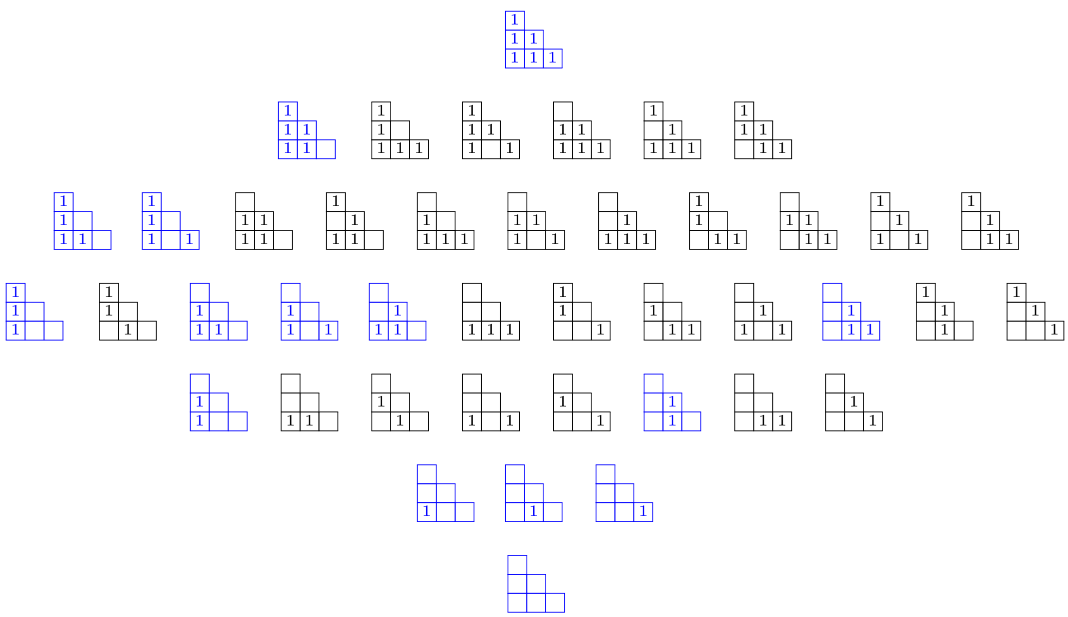


Figure 1.2 The 42 approval ballot triangles of order 4. The zero entries are rendered blank for visual clarity. The blue arrays are the 15 staircase arrangements of pillars with distinct height in a triangular array of size 3.

The 42 ABTs of order 4 are shown in Figure 1.2. Approval ballot triangles have many natural subfamilies that are in bijection with famous combinatorial families, including permutations, set partitions and Catalan numbers.

See [2] for an extensive list of these subfamilies and their structures. The 15 blue triangles are staircase arrangements of pillars with distinct height in a triangular array of size 3.

Chapter 2

Staircase Arrangements of Pillars with Distinct Heights

2.1 Defining Our Main Family

Our primary focus in this paper is on the family $\mathcal{A}_n$.

Definition 2.1. A sequence $a = (a_1, a_2, \ldots, a_n)$ is an element of $\mathcal{A}_n$ if

1. For $1 \leq i \leq n$, we have $0 \leq a_i \leq i$, and
2. For $1 \leq i < j \leq n$, if $a_i > 0$ and $a_j > 0$ then $a_i \neq a_j$.

Sequences of $\mathcal{A}_n$ can also be represented as binary triangles where a_i is the number of ones in column i, placed from the bottom up. The number of ones in a column must be different for all columns of a. These ones form pillars of distinct heights that we arrange in the staircase template. Figure 2.1 depicts sequences of $\mathcal{A}_n$ for small n alongside their binary triangle representations. For example, the sequence $(1, 0, 2) \in \mathcal{A}_3$ can be visualized as a staircase arrangement with a pillar of height one in the first location and a pillar of height two in the third location.

Throughout this project, it may be advantageous to think of elements of $\mathcal{A}_n$ as both sequences and binary triangles, depending on the context.

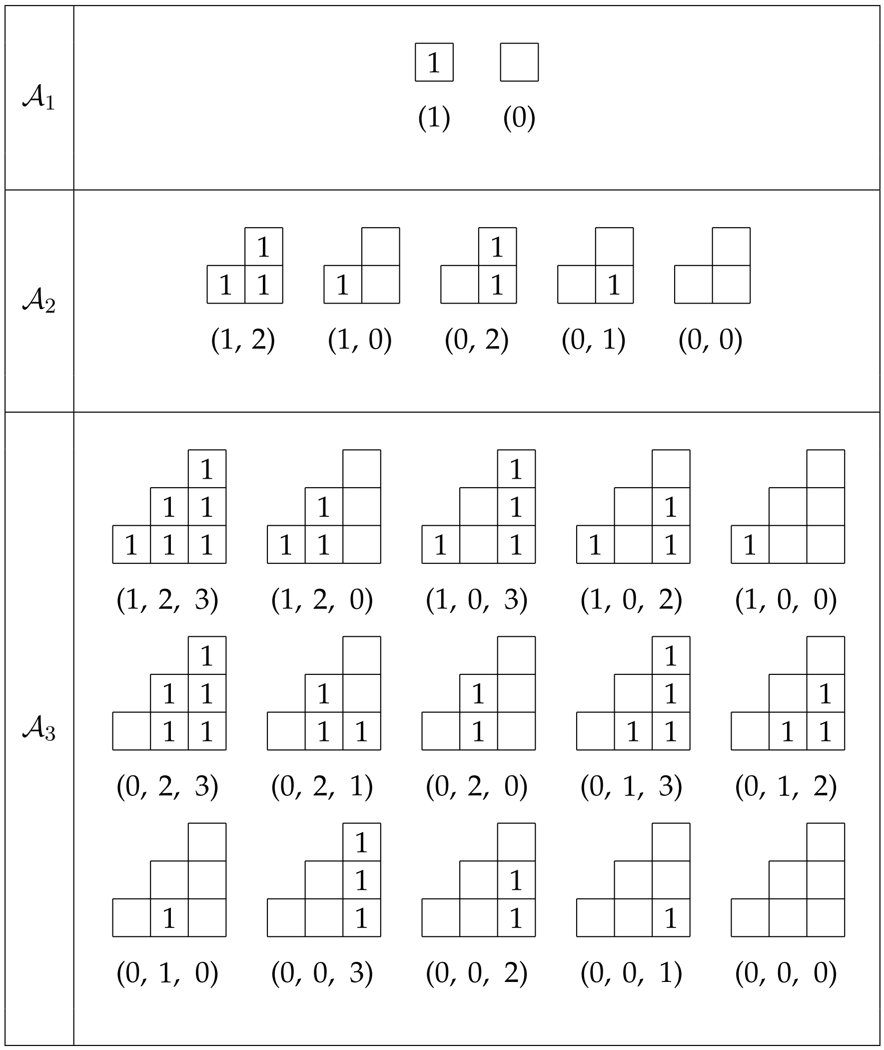


Figure 2.1 Elements of $\mathcal{A}_n$, represented as both sequences and binary triangles.

2.2 Bijection to Set Partitions of $[n+1]$

2.2.1 Sequence to Set Partition

The set $\mathcal{A}_n$ is in bijection with set partitions of $[n+1]$. The mapping from $a \in \mathcal{A}_n$ to a set partition of $[n+1]$ is given by Algorithm 1.

Algorithm 1: Build-A-Partition

$\mathcal{B} \leftarrow \{\}$
$S \leftarrow \{1, 2, \ldots, n, n+1\}$
$a \leftarrow \{a_0 = 0, a_1, \ldots, a_n\}$
while $S \neq \emptyset$ **do**
  $k \leftarrow \max S$
  $B = \emptyset$
  while $k > 0$ **do**
    add k to B
    remove k from S
    $k \leftarrow a_{k-1}$
  end while
  add B to $\mathcal{B}$
end while
return $\mathcal{B}$

As we build the set partition, our main rule is that a block ends if $a_{k-1} = 0$ and we add a_{k-1} to the block if $a_{k-1} > 0$.

We begin our set partitions with the greatest element that is not yet in a block. There are 2 options for the next element in the block.

1. The next entry in the sequence is zero and the block is ended.
2. The next entry in the sequence is a nonzero k and k is added to the block.

Figure 2.2 demonstrates this mapping from a 4-sequence to a set partition of $[5]$. Here is the mapping process in detail.

- Algorithm 1 elements:
  $\mathcal{B} = \{\}$
  $S = \{5, 4, 3, 2, 1\}$
  $B = \{\}$

- Beginning with the sequence $(1, 0, 2, 3)$, we start our first block of the partition with 5, the greatest element of $[5]$.
  $\mathcal{B} = \{\}$
  $S = \{4, 3, 2, 1\}$
  $B = \{5\}$
- We have $a_4 = 3$, so we add 3 into the 5-block.
  $\mathcal{B} = \{\}$
  $S = \{4, 2, 1\}$
  $B = \{5, 3\}$
- Next, $a_2 = 0$, so the block is ended.
  $\mathcal{B} = \{\{5, 3\}\}$
  $S = \{4, 2, 1\}$
  $B = \{\}$
- We start a new block with 4, the largest number in $[n + 1]$ that is not yet in a block.
  $\mathcal{B} = \{\{5, 3\}\}$
  $S = \{2, 1\}$
  $B = \{4\}$
- We have $a_3 = 2$, so we add 2 into the 4-block.
  $\mathcal{B} = \{\{5, 3\}\}$
  $S = \{1\}$
  $B = \{4, 2\}$
- Next, $a_2 = 1$, therefore, we add 1 into the 4-block as well.
  $\mathcal{B} = \{\{5, 3\}\}$
  $S = \{\}$
  $B = \{4, 2, 1\}$
- We have $a_0 = 0$, so the block ends and is added to the set partition
  $\mathcal{B} = \{\{5, 3\}, \{4, 2, 1\}\}$
  $S = \{\}$
  $B = \{4, 2, 1\}$
- Finally, since S is empty, we have obtained our set partition: $53|421$.
  $\mathcal{B} = \{\{5, 3\}, \{4, 2, 1\}\}$

Since this algorithm inspects every a_k in the sequence a, we know that we did not miss any entries during the mapping.

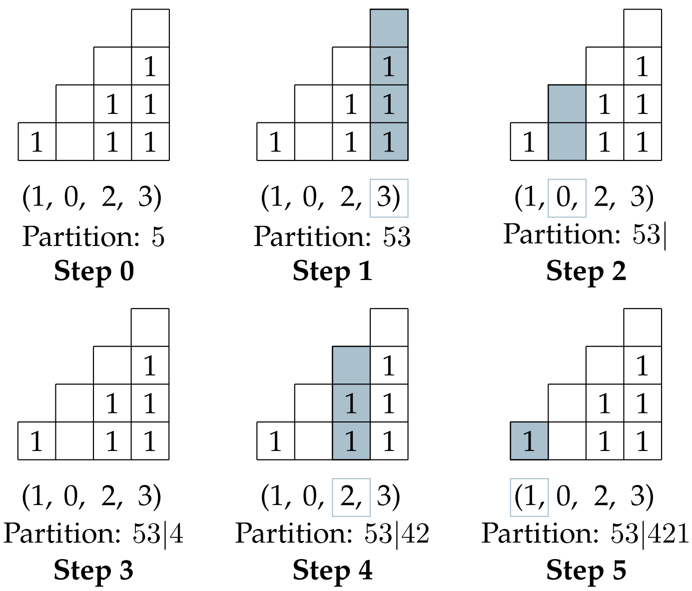


Figure 2.2 Example of Build-A-Partition

2.2.2 Set Partition to Sequence

As we form a sequence a of $\mathcal{A}_n$ from a set partition, we arrange the blocks of the partition in decreasing order and update terms of a according to the entry following it in the block. Figure 2.3 depicts the steps of our Algorithm 2 in the following way:

1. Begin with the set partition $542|31$ and a sequence of four zeros.

2. Consider the first block 542. $b_1 = 5$ and $b_2 = 4$. Therefore, we update $a_{5-1} = a_4$ to 4.

3. Next we move to $b_2 = 4$ and $b_3 = 2$. We update $a_{4-1} = a_3$ to 2.

4. The last element of the block is $b_3 = 2$, so we do nothing and keep $a_{2-1} = a_1 = 0$. From this block, our sequence is $a = (0, 0, 2, 4)$.

5. Consider the next block 31. $b_1 = 3$ and $b_2 = 1$; therefore, we update $a_{3-1} = a_2$ to 1.

6. The last element of this block is $b_2 = 1$, so we do nothing.

Algorithm 2: Build-A-Sequence

$\mathcal{P} \leftarrow \{B_1, B_2, \dots, B_m\}$
$\mathcal{P}$ is a set partition where $B_m = \{b_1, b_2, \dots, b_k\}$ and $b_i > b_i + 1$
$a \leftarrow \{a_0 = 0, a_1 = 0, \dots, a_n = 0\}$
for block B_i in P **do**
  for element b_j in B_i **do**
    if b_{j+1} exists **then**
      $a_{b_j - 1} \leftarrow b_{j+1}$
    end if
  end for
end for
return a

Since we have considered every block and every element of the set partition and made the corresponding updates, our mapping from a set partition of $[5]$ to a sequence of $\mathcal{A}_n$ is complete. $542|31$ maps to $(0, 1, 2, 4)$.

$(0,\ 0,\ 0,\ 0)$	$(0,\ 0,\ 0,\ \boxed{4})$	$(0,\ 0,\ \boxed{2,}\ 4)$
$5\ \ 4\ \ 2 \mid 3\ \ 1$	$\boxed{5\ \ 4\ \ 2} \mid 3\ \ 1$	$\boxed{5\ \ 4\ \ 2} \mid 3\ \ 1$
Step 0	**Step 1**	**Step 2**
$(0,\ \boxed{0,}\ 2,\ 4)$	$(0,\ \boxed{1,}\ 2,\ 4)$	
$\boxed{5\ \ 4\ \ 2} \mid 3\ \ 1$	$5\ \ 4\ \ 2 \mid \boxed{3\ \ 1}$	
Step 3	**Step 4**	

Figure 2.3 Example of Build-A-Sequence

Theorem 2.2. *Sequences of $\mathcal{A}_n$ are in bijection with set partitions of $[n+1]$.*

Proof. Let φ be the mapping from $\mathcal{A}_n$ to set partitions of $[n+1]$ by Algorithm 1. Additionally, let a be a sequence in $\mathcal{A}_n$ and let s be a set partition of $[n+1]$. To show that φ is a bijection, we need to show that it is one-to-one and onto.

First we will show that φ is one-to-one. This means that if $\varphi(a) = \varphi(b)$ for $a, b \in \mathcal{A}_n$ then $a = b$. We will prove the contrapositive. Consider $\varphi(a) = s, \varphi(b) = t$ are different set partitions of $[n+1]$. We can reverse φ with Algorithm 2. Let $\ell \in [n+1]$. ℓ must appear in both s and t. We will notate its position in s as $s_{i,j}$ as ℓ is in the jth position of the ith block of s. ℓ is also in t at the location $t_{k,h}$, where it is in the hth position of the kth block.

Since $s \neq t$, there must be an ℓ such that $s_{i,j+1} \neq t_{k,h+1}$ while $s_{i,j} = t_{k,h} = \ell$. This means that the position of ℓ in s differs from the one in t. In this case, we have $a_\ell = s_{i,j+1}$ and $b_\ell = t_{k,h+1}$. Since we update the ℓth position of the sequence to different entries, a and b must not be the same sequence since $\varphi(a) \neq \varphi(b)$. Therefore, φ is one-to-one.

Now that we have shown φ is one-to-one, we will show that φ is onto. This means that for every b in the set partitions of $[n+1]$, there exists an $a \in \mathcal{A}_n$ such that $\varphi(a) = b$. We know from Algorithm 1 that there exists a b in the set partitions of $\mathcal{A}_n$ so that $\varphi(a) = b$ for all $a \in \mathcal{A}_n$. We also know from Algorithm 2 that there exists an $a \in \mathcal{A}_n$ such that $\varphi(a) = b$ for all b in the set partitions of $[n+1]$. Therefore, φ is an onto mapping.

Because φ is both one-to-one and onto, sequences of $\mathcal{A}_n$ are in bijection with set partitions of $[n+1]$ □

Next, we show that we can determine the number of blocks in the corresponding set partition by counting the zero entries of $a \in \mathcal{A}_n$.

Corollary 2.3. *The number of zeros in $A \in \mathcal{A}_n$ is equivalent to one less than the number of blocks in the corresponding set partition.*

Proof. Let $a \in \mathcal{A}_n$. When mapping to a set partition, a block is ended when $a_i = 0$. But since each element k in the set partitions corresponds to a_{k-1} in the sequence, element 1 will always be the least element in its block but does not have a mapping in the sequence a. Therefore, all blocks except the one containing 1 correspond to an $a_i = 0$ element in the sequence. So if the set partition has j blocks, a has $j-1$ zero entries. □

Chapter 3

Partial Orderings and Chains

3.1 Comparing Size of Elements

For elements $a, b \in \mathcal{A}_n$, we say that $a \preceq b$ if and only if $a_i \leq b_i$ for all $i \in n$. In Figure 3.1, $a \preceq c$ and $b \preceq c$, but neither $a \preceq b$ nor $b \preceq a$ since $a_1 > b_1$ while $a_2 < b_2$. The elements a and b are not comparable.

Using the bijection of Algorithm 1, the poset for $\mathcal{A}_n$ induces a poset on the set partitions of $[n+1]$. We explore the properties of this set partition poset.

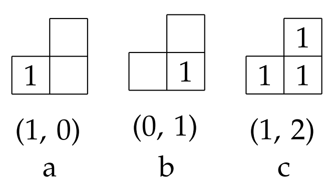
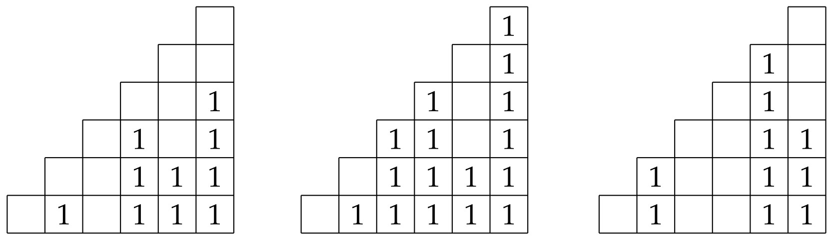


Figure 3.1 Example of Comparable and Non-comparable Elements.

Figure 3.5 and Figure 3.6 show the posets for $\mathcal{A}_3$ and $\mathcal{A}_4$, respectively.

3.1.1 Comparing size of set partitions

We start by setting some notation. We write our set partitions so that each element of a block is in decreasing order and the greatest elements of each block are in decreasing order. In other words, the set partition A with k



(0, 1, 0, 3, 2, 4) Partition: $74|62|531$ A

(0, 1, 3, 4, 2, 5) Partition: $75431|62$ B

(0, 2, 0, 0, 5, 3) Partition: $732|65|4|1$ C

Figure 3.2 Comparable and Non-comparable Elements

blocks is

$$\begin{aligned} A &= A_1\big|A_2\big|\cdots\big|A_k \\ &= a_{1,1}a_{1,2}\cdots a_{1,r_1}\big|a_{2,1}a_{2,2}\cdots a_{2,r_2}\big|\cdots\big|a_{k,1}a_{k,2}\cdots a_{k,r_k} \end{aligned}$$

where

$$a_{1,1} > a_{2,1} > \cdots > a_{k,1}$$

and

$$a_{i,1} > a_{i,2} > \cdots > a_{i,r_i} \quad \text{for} \quad 1 \le i \le k.$$

We let $|A| = k$ denote the number of blocks in the set partition. Finally, for convenience, we will define

$$a_{i,r_{i+1}} = 0 \quad \text{for} \quad 1 \le i \le k.$$

We will proceed with an example of comparable set partitions in this induced ordering and then follow with classifying when $A \prec B$ for set partitions A and B.

Consider Figure 3.2 for an example of comparable and non-comparable elements. $A \succ B$ while A and C are not comparable. In order to compare the sizes of these set partitions, we need to consider all $\ell \in [n]$ and do the following:

1. Locate ℓ in A and B where $a_{i,j} = b_{k,h} = \ell$.
2. Compare $a_{i,j+1}$ and $b_{k,h+1}$.

If $a_{i,j+1} \ge b_{k,h+1}$ for all $\ell \in [n]$, then $A \preceq B$. If $a_{i,j+1} \le b_{k,h+1}$ for all $\ell \in [n]$,

ℓ	(i,j)	(k,h)	$a_{i,j+1}$		$b_{k,h+1}$
7	$(1,1)$	$(1,1)$	4	$<$	5
6	$(2,1)$	$(2,1)$	2	$=$	2
5	$(3,1)$	$(1,2)$	3	$<$	4
4	$(1,2)$	$(1,3)$	0	$<$	3
3	$(3,2)$	$(1,4)$	1	$=$	1
2	$(2,2)$	$(2,2)$	0	$=$	0
1	$(3,3)$	$(1,5)$	0	$=$	0

Table 3.1 Comparing Set Partitions. We find the location of ℓ in the induced set partition for a and b and then evaluate the relationship between the elements following ℓ in the set partitions

then $A \succeq B$. Finally, if there are some ℓ where $a_{i,j+1} < b_{k,h+1}$ and some where $a_{i,j+1} > b_{k,h+1}$, then A and B are not comparable.

We record these relations between A and B from Figure 3.2 in Table 3.1.

1. Let us begin with $\ell = 7$. $a_{1,1} = 7$ and $b_{1,1} = 7$. So we compare $a_{1,2} = 4$ with $b_{1,2} = 5$ and see that for $\ell = 7$, $a < b$.

2. Let us consider $\ell = 3$. $a_{3,2} = 3$ and $b_{1,4} = 3$. When we compare $a_{3,3} = 1$ and $b_{1,5} = 1$, we find that for $\ell = 3$, $a = b$.

After comparing all $\ell \in [n]$, we find that $a_{i,j+1} \leq b_{k,h+1}$ for all $\ell \in [n]$ and since there is at least one ℓ where $a_{i,j+1} < b_{k,h+1}$, we have $A \prec B$.

Lemma 3.1. *Let $a, b \in \mathcal{A}_n$ and let A, B be the corresponding set partitions. If $A \prec B$ then $|A| \geq |B|$.*

Proof. We prove the contrapositive. Recall that a zero in sequence a signals the end of a block in the corresponding set partition A. When set partition A has more blocks than set partition B, the sequence a has more zeros than sequence b. Therefore, if a and b are comparable, then we must have $a \prec b$. $\square$

Let

$$A = A_1 \mid A_2 \mid \cdots \mid A_r$$

where $A_i = a_{i,1}a_{i,2}\cdots a_{i,r_i}$, and let

$$B = B_1 \mid B_2 \mid \cdots \mid B_s$$

where $B_i = b_{i,1}b_{i,2}\cdots b_{i,s_i}$ and $r \leq s$. By the previous lemma, we either have $A \succ B$ or these set partitions are not comparable. This leads us to the following theorem.

Theorem 3.2. *Let A, B be set partitions of $[n+1]$. $A = A_1 \mid A_2 \mid \cdots \mid A_r$ where each $A_i = a_{i,1}a_{i,2}\cdots a_{i,r_i}$, and $B = B_1 \mid B_2 \mid \cdots \mid B_s$ where each $B_i = b_{i,1}b_{i,2}\cdots b_{i,s_i}$. $A \succeq B$ if and only if $a_{i,j+1} \geq b_{k,h+1}$ for all $\ell \in [n+1]$ where $\ell = a_{i,j} = b_{k,h}$.*

Proof. We prove the forward direction first. From Theorem 2.2, we know that set partitions of $[n+1]$ are in bijection with $\mathcal{A}_n$ sequences. Therefore, let $a, b \in \mathcal{A}_n$ be the induced sequences from the set partitions A, B respectively. Given that $A \succeq B$, we also have that $a \succeq b$ and $a_i \geq b_i$ for all $i \in [n]$. Let $\ell \in [n+1]$. From Algorithm 1, we know there exists i, j, k, h such that $a_{i,j} = b_{k,h} = \ell$. Because $a \succeq b$, the mapping gives that $a_{i,j+1} \geq b_{k,h}$ for all $\ell \in [n+1]$. This proves the forward direction.

Now assume that $a_{i,j+1} \geq b_{k,h+1}$ for all $\ell \in [n+1]$ where $\ell = a_{i,j} = b_{k,h}$. From Algorithm 2, we know that $a_{\ell-1} \geq b_{\ell-1}$ for all $\ell \in [n+1]$. Therefore $a \succeq b$ and $A \succeq B$ by the bijection. This proves the backward direction. □

3.2 Covers of Elements

Definition 3.3. For elements $a, b \in \mathcal{A}_n$, we can say that a covers b if $a \succ b$ and there is no $c \in \mathcal{A}_n$ such that $a \succ c \succ b$.

Let J_i be the set of elements j such that $j \notin a$ and $a_i < j \leq i$. In other words, these are the numbers that could be in a_i to increase a. Beginning with an element $a \in \mathcal{A}_n$, its covers are constructed by doing one of the following

1. Increasing a_k to $\min(J_k)$ when $a_k > 0$, or

2. Setting $a_k = \min(J_k)$ when $a_k = 0$.

Note that there may be instances in which either one of the options may not be possible. We will look at a few examples of covers to clarify this point. In Figure 3.3 we begin with the sequence $a = (0, 0, 2)$, the sequences above are the possible covers of a. Note that $\{1, 3\} \notin a$. We can obtain covers of a by doing one of the following:

1. For $a_1 = 0$, $J_1 = \{1\}$, therefore, we can set $a_1 = 1$, achieving the cover $a = (1, 0, 2)$. This is an example of the second case where $a_i = 0$.

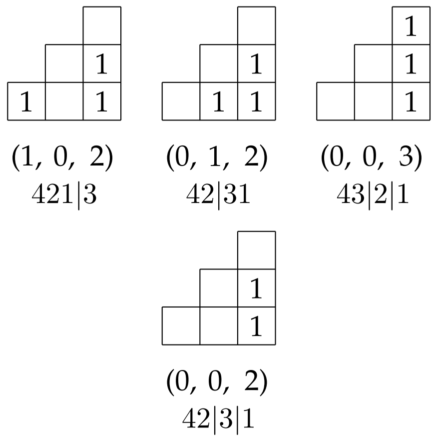


Figure 3.3 Covers of 002

2. For $a_2 = 0$, we again have $J_2 = \{1\}$, therefore, we can set $a_2 = 1$ and achieve the cover $(0, 1, 2)$. This is also an example of the second case where $a_i = 0$.

3. Finally, for $a_3 = 2$, we have $J_3 = \{3\}$. We can set $a_3 = 3$ and obtain the cover $(0, 0, 3)$. This is an example of the first case where $a_i > 0$.

For an example of when it is not possible to increase an entry a_i to achieve a cover of a, consider the sequence $(0, 1, 2)$. In this example, $a_1 = 0$ and $J_1 = \emptyset$, so there are no ways to achieve a cover by increasing a_1. Similarly, we can not increase $a_2 = 1$ since $J_2 = \emptyset$. Therefore, the only cover for a is $(0, 1, 3)$ where we increase $a_3 = 2$ to $\min J_3 = \min\{3\} = 3$.

Lemma 3.4. *An $\mathcal{A}_n$ sequence a has at most one covering from increasing a given $a_i \in a$.*

Proof. Let $a \in \mathcal{A}_n$. Consider achieving a cover with an arbitrary $a_i \in a$. We know the set J_i is the set of all j such that $a_i < j \leq k$ and $j \notin a$. J_i is either the empty set of has at least one element.

Suppose J_i is empty. In this case, a has no achievable coverings from a_i. Now suppose j_i is not empty. There is at least one element j that a_i can be increased to. Since coverings are achievable from $\min(J_i)$, there is only one option for increasing a_i. Therefore, a has at most one covering achieved by increasing $a_i \in a$. $\square$

Theorem 3.5. *The number of coverings a' for a sequence $a \in \mathcal{A}_n$ is equivalent to the number of nonempty J_i for all $1 \le i \le n$ where $J_i = \{j : a_i < j \le i$ and $j \notin a\}$.*

Proof. From Lemma 3.4, we know that there is at most one way to achieve a cover from any $a_i \in a$. Additionally, covers are only achieved when $J_i \neq \emptyset$. Therefore, the total number of covers for a given sequence a will be the number of entries a_i that are able to increase, which is the number of nonempty J_i. □

3.2.1 Coverings with Set Partitions

From the mapping of $\mathcal{A}_n$ sequences to set partitions of $[n+1]$, we have a natural ordering of these set partitions and can analyze the process of coverings for set partitions. Recall that we write the set partition A with k blocks as

$$\begin{aligned} A &= A_1 \big| A_2 \big| \cdots \big| A_k \\ &= a_{1,1}a_{1,2}\cdots a_{1,r_1}\big| a_{2,1}a_{2,2}\cdots a_{2,r_2}\big| \cdots \big| a_{k,1}a_{k,2}\cdots a_{k,r_k} \end{aligned}$$

where

$$a_{1,1} > a_{2,1} > \cdots > a_{k,1}$$

and

$$a_{i,1} > a_{i,2} > \cdots > a_{i,r_i} \quad \text{for} \quad 1 \le i \le k.$$

As mentioned earlier, a covering a' of a sequence $a \in \mathcal{A}_n$ can be achieved by one of the following methods:

1. The cover changes $a_k = 0$ to $a'_k > 0$ where $a'_k \notin a$, or
2. The cover changes $a_k > 0$ to $a'_k > a_k$ where $a'_k \notin a$.

In the first case, two blocks of our set partition merge together. Consider the partition $42|3|1$ in Figure 3.3. We can do either of the following merges:

1. Block 42 merges with block 1 because $0 < 1 < 2$ to achieve the top left partition $421|3$, or
2. Block 3 merges with block 1 to achieve the top middle partition $42|31$.

Note that we can not merge 42 and 3 together because $2 < 3$. Doing so results in the partition $423|1$, which we then reorder to be decreasing as

$432|1$. This partition shows that we have increased a_3 in the sequence to 3 as well as increased a_2 to 2.

In the second case, we swap elements of blocks depending on their relation to others. Consider the set partition $42|31$ in Figure 3.3. We can swap the block 31 in for element 2 in block 42 since $3 > 2$. This results in the partition $431|2$.

Generally, merges and swaps are permitted if the following conditions are met.

1. Block A_i can merge with block A_j when $a_{j,1} = \min(a_{k,1} : a_{k,1} < a_{i,r_i})$. In other words, this is when the greatest element in block A_j, or $a_{j,1}$ is the minimum of all the blocks whose greatest element $a_{k,1}$ is less than the least element of block A_i, which is a_{i,r_i}.

2. Let $a_{i,\ell} \in A_i$. Elements $a_{i,p} \in A_i$ such that $p > \ell$ can swap with block A_j when $a_{j,1} = \min(a_{k,1} | a_{i,\ell} > a_{k,1} > a_{i,\ell+1})$.

3.3 The Partially Ordered Set

Figure 3.4 depicts the partially ordered set $\mathcal{A}_3$ by connecting elements $a, b \in \mathcal{A}_n$ when a covers b.

3.3.1 Layers of the poset

Big elements

The maximum element of $\mathcal{A}_n$ is $(1, 2, 3, \ldots, n)$. These are the binary triangles where all entries are 1. The corresponding set partition is the unique partition with exactly one block.

This maximal element covers n elements of the $\mathcal{A}_n$ family. These are the triangles that are completely full except for one empty column. The corresponding set partitions have two blocks: one has $n - 1$ elements and the other has one element.

There are exactly $n \cdot (n - 1)$ elements in the third highest layer of the poset. This is because, starting with the maximum sequence, we pick one of the n entries to make zero and then pick on of the remaining $n - 1$ entries to reduce. This row is actually $\mathcal{A}_{n,2}$, the subfamily explored in Chapter 4 of $\mathcal{A}_n$ elements in which the maximum element is 2.

Counting the size of layers below the third highest layer is an open problem.

Small elements

The minimum element in $\mathcal{A}_n$ is the sequence of all zeroes. When given as set partitions, these are partitions in which all $n+1$ elements are in separate blocks.

There are n elements in the second layer from the bottom. These are the sequences of all zeroes except for one $a_i = 1$.

The third layer has $n \cdot (n-1)$ elements. These sequences come from first choosing one of the n entries to make a 1. Then we choose from $n-1$ places to place a 2. Note that it is possible that we select the same a_i to place 1 and then place 2 if $i > 1$. This is fine since it is equivalent to placing 1 in a_i and then increasing a_i by 1.

Counting the size of layers above the third layer is an open problem.

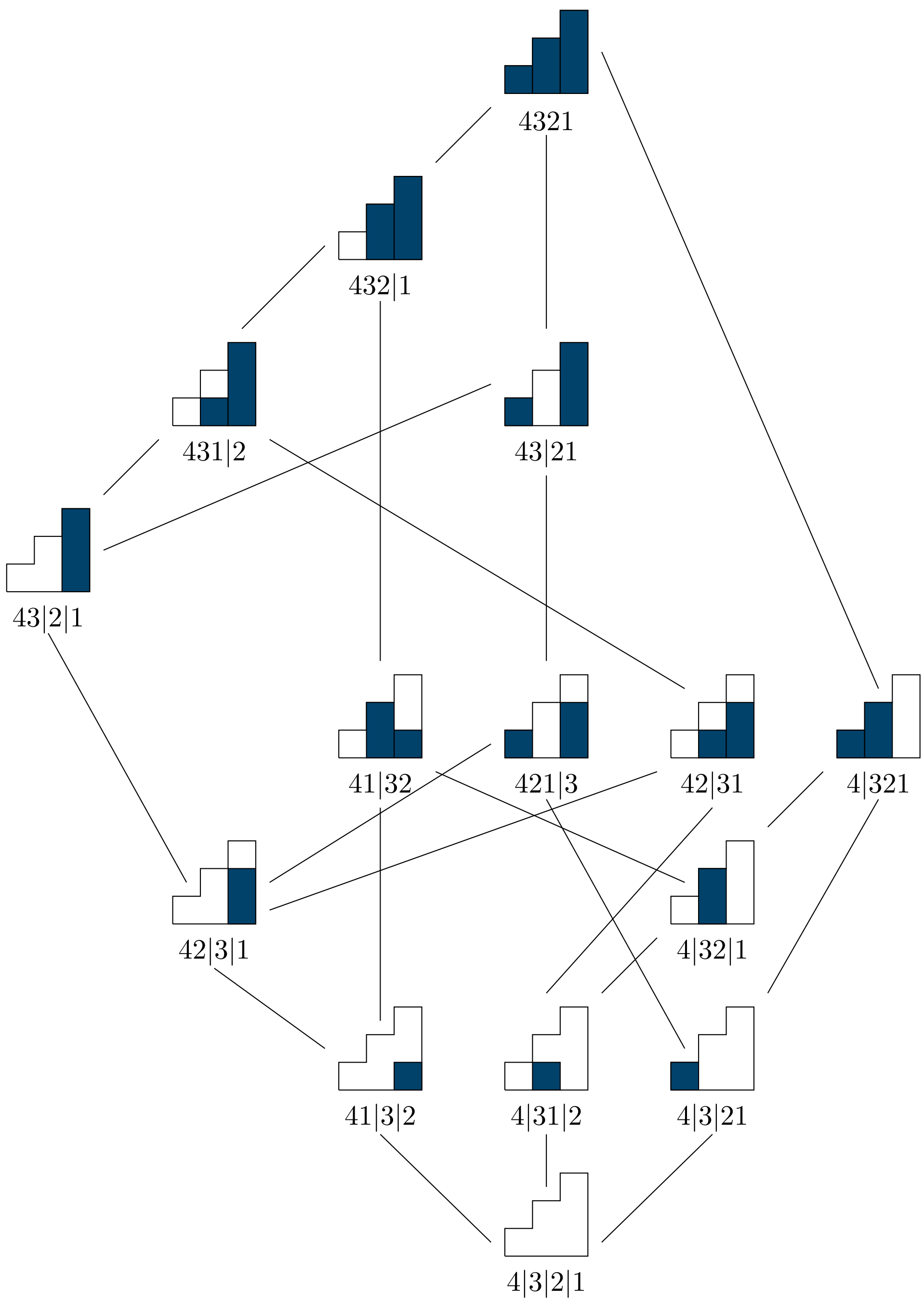


Figure 3.4 Partially Ordered Set of $\mathcal{A}_3$. Elements are given as both binary triangles where the filled in blocks represent ones and their induced set partition from our previously defined mapping.

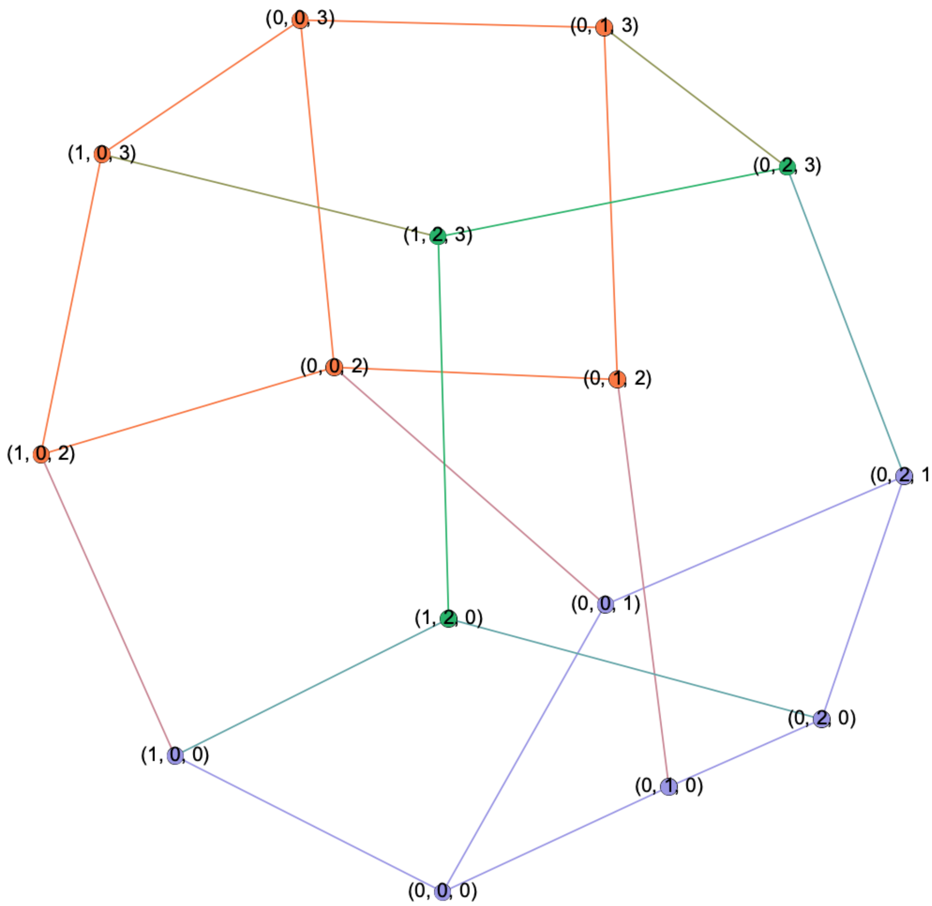


Figure 3.5 The poset for $\mathcal{A}_3$.

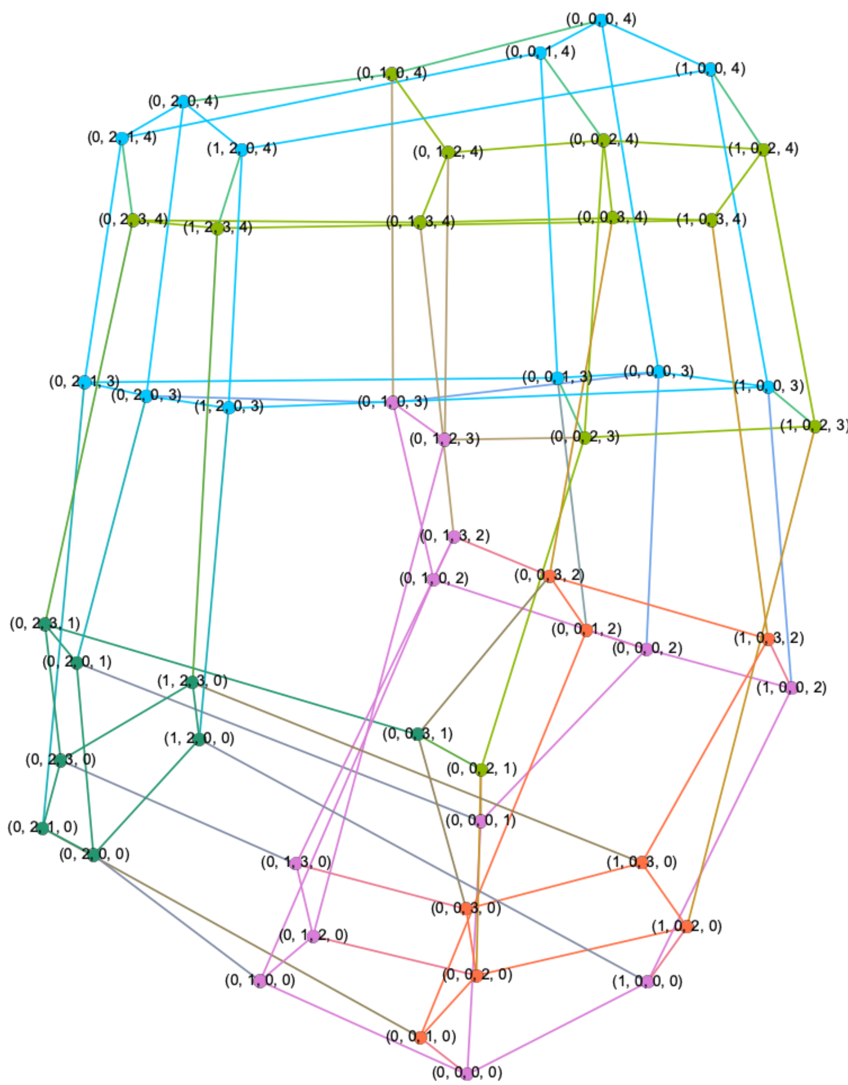


Figure 3.6 The poset for $\mathcal{A}_4$.

3.4 Chains

Now that we have formed a hierarchy of elements in the family $\mathcal{A}_n$, we can explore the relationships between them. In this section, we will look at chains in the poset.

Definition 3.6. A chain is a set of elements $\{c_0, c_1, \ldots, c_i\}$ such that we have $c_0 \prec c_1 \prec \cdots \prec c_i$ where each element is a covers the one before it.

3.4.1 Maximal Chains

Definition 3.7. A maximal chain is not a subset of any other chain.

A maximal chain of $\mathcal{A}_n$ must start at the unique minimum element $(0, 0, \ldots, 0)$ and end at the unique maximum element $(1, 2, 3, \ldots, n)$. Note that there may be multiple maximal chains and they may be different lengths. Figures 3.7 and 3.8 depict two different maximal chains for $\mathcal{A}_3$.

Theorem 3.8. *The length of the longest maximal chain for $\mathcal{A}_n$ is*

$$1 + \sum_{k=1}^{n} k = 1 + \binom{n+1}{2}.$$

Proof. Beginning with our minimum element $\overline{a}$, we increase a_n by 1 until $a_n = n$. This will take n steps. Then we increase a_{n-1} by 1 until $a_{n-1} = n-1$. This will take $n-1$ steps. We continue this pattern until we reach the maximum element $\hat{a}$. This takes $n + (n-1) + (n-2) + \cdots + 2 + 1$ steps, which is the number of elements used minus one. Therefore, the chain consists of $1 + \sum_{k=1}^{n} k$ elements.

It is clear that this is the maximum number of possible steps in a chain: we must increase each column by at least 1 at each step. □

Figure 3.7 depicts a maximal chain of longest length for $\mathcal{A}_3$. For $n \geq 3$, there are multiple longest maximal chains. The number of maximal chains for $1 \leq n \leq 6$ is

$$1, 1, 2, 12, 286, 33592, \ldots$$

These are the *strict sense ballot numbers*, see OEIS A003121 [8], and the nth strict sense ballot number is

$$\binom{n+1}{2}! \frac{\prod_{k=1}^{n-1} k!}{\prod_{k=1}^{n} (2k-1)!},$$

We will prove that the collection of longest maximal chains of $\mathcal{A}_n$ are in bijection with strict sense ballots with n candidates.

Definition 3.9. Consider an election with n candidates in which candidate k receives exactly k votes. An ordering of these $\binom{n+1}{2}$ votes is a *strict-sense ballot* (SSB) when candidate k always leads candidate $k-1$ for $2 \le k \le n$ during the vote count.

For example, the 12 SSBs for four candidates are

$$\begin{array}{cccc} 4444333221 & 4444332321 & 4443433221 & 4443432321 \\ 4443343221 & 4443342321 & 4443324321 & 4434433221 \\ 4434432321 & 4434343221 & 4434342321 & 4434324321. \end{array}$$

Each of these SSBs encodes a longest maximal chain of $\mathcal{A}_n$. The values indicate the entry that we increment at each step of the maximal chain. For example, the SSB

$$4434342321$$

corresponds to the maximal chain

$$\begin{aligned} &(0,0,0,0) \to (0,0,0,1) \to (0,0,0,2) \to (0,0,1,2) \\ &\to (0,0,1,3) \to (0,0,2,3) \to (0,0,2,4) \to (0,1,2,4) \\ &\to (0,1,3,4) \to (0,2,3,4) \to (1,2,3,4). \end{aligned}$$

We now show that this mapping is a bijection.

Theorem 3.10. *The longest maximal chains for $\mathcal{A}_n$ are in bijection with strict sense ballots of order n.*

Proof. Let $\mathcal{L}_n$ denote the collection of longest maximal chains of $\mathcal{A}_n$, and let $\mathcal{B}_n$ to denote the collection of strict sense ballots for n candidates. We construct a bijective mapping $f : \mathcal{L}_n \to \mathcal{B}_n$.

Let $a = (a^0, a^1, \ldots, a^{n(n+1)/2})$ be a maximal chain of $\mathcal{A}_n$. We construct strict sense ballot $b = f(a)$ where $b = (b_1, b_2, \ldots, b_{n(n+1)/2})$ by setting b_k to be the unique index whose entry is incremented by 1 when we move from a_{k-1} to a_k.

We claim that $b \in \mathcal{B}_n$. In the maximal chain a, the value of entry k is incremented k times, so the number k appears exactly k times in b. Next, we show that for $1 \le m \le n(n+1)/2$, each partial sequence $b^m = (b_1, b_2, \ldots, b_m)$ contains at least as many k's and $(k-1)$'s. If not, then then consider the smallest m such that $b_m = k-1$ and the number of $(k-1)$'s in b^m equals the number of k's in b^m. In the corresponding maximal chain, the element a^m

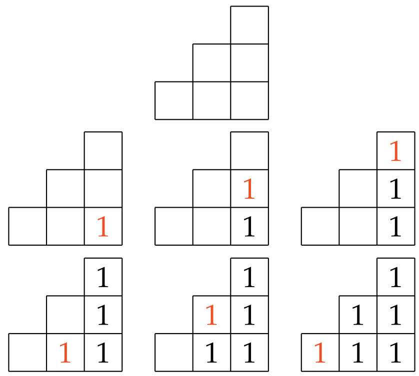


Figure 3.7 A Longest Maximal Chain for $\mathcal{A}_3$. Reading from top to bottom and left to right, we show how ones are added to the binary triangle one at a time.

would have $a^m_{k-1} = a^m_k$, contradicting the fact that all the nonzero entries of a^m are distinct.

Next, we prove that this mapping is a bijection by creating a mapping $g : \mathcal{B}_n \to \mathcal{L}_n$ that is the inverse of f. Given SSB $b = (b_1, \ldots, b_{n(n+1)/2})$, we define $g(b) = a = (a^0, a^1, \ldots, a^{n(n+1)/2})$ where a^0 is the all-zero sequence of length n, and for $1 \leq m \leq n(n+1)/2$, we set a^m to be the sequence $(a_1, a_{,} \ldots, a_n)$ where a_k is the number of times that k appears in the subsequence $b^m = (b_1, \ldots, b_m)$. It is straight-forward to confirm that $g \circ f$ is the identity map on $\mathcal{L}_n$ and that $f \circ g$ is the identity map on $\mathcal{B}_n$. □

Theorem 3.11. *There is a unique shortest maximal chain, which has length $n+1$.*

Proof. Let $a \in \mathcal{A}_n$ be the minimum sequence of all zero entries. We describe our candidate maximal chain C of minimal length. First we will change a_1 from 0 to 1.

Next we move to the second column where we can changer a_2 from 0 to 2, since $a_1 = 1$. Continuing this way, we can change a_k directly from 0 to k by induction for $1 \leq k \leq n$. This takes n steps to move from $(0, 0, \ldots, 0)$ to $(1, 2, 3, \ldots, n)$.

This is the smallest possible length for a maximal chain. We must change each entry at least once, and this change reaches the maximum entry in exactly n steps.

Furthermore, this is the unique maximal chain of minimum length. Considering any other maximal chain C', consider the first step where it

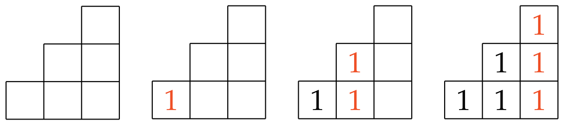


Figure 3.8 Shortest Maximal Chain for $\mathcal{A}_3$. Reading from left to right, we show how entire pillars of ones are added to the binary triangle.

differs from chain C. Let k be the index of the entry that change at this step. The value goes from 0 to 1 the value at index $k-1$ is zero. $\square$

Chapter 4

Subfamilies of Staircase Arrangements

We now investigate three subfamilies of staircase arrangements $\mathcal{A}_n$. These families are:

- arrangements with maximum element k,
- arrangements with maximum element at most k,
- arrangements missing element k.

The first two families appear to be new combinatorial sequences: they do not appear in the OEIS. The last family does appear in the OEIS, and we give a new interpretation for this sequence.

4.1 Maximum Element Exactly k

Let $\mathcal{A}_{n,k} \subset \mathcal{A}_n$ be the sequences in which the maximum element is exactly k. Let $A(n,k) = |\mathcal{A}_{n,k}|$ be the size of this set. The values for $A(n,k)$ for small n are shown in Table 4.1. These are staircase arrangements of pillars with distinct heights where the pillar of height k appears in the arrangement and is the tallest. Since $\mathcal{A}_n$ are in bijection with set partitions of $[n+1]$, we can also describe $\mathcal{A}_{n,k}$ as set partitions of $[n+1]$. $\mathcal{A}_{n,k}$ are set partitions $A = a_{1,1}, a_{1,2}, \ldots, a_{1,r_1}|a_{2,1}, a_{2,2}, \ldots, a_{2,r_2}| \ldots |a_{b,1}, a_{b,2}, \ldots, a_{b,r_b}$ where A has b blocks such that the following are true:

- there exists i for each $j > k$ such that $a_{i,1} = j$, and

- $a_{i,1} \neq k$ for all $i \in [b]$

In other words, integers greater than k must begin a block and element k must not start a block.

In this section, we prove the recursive relationship for sets $\mathcal{A}_{n,k}$ and provide a general formula for $A(n,k)$. We are a bit excited that this triangular sequence does not appear in the OEIS, but we still found an explicit formula for these numbers.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1	1						
$n=2$	1	2	2					
$n=3$	1	3	6	5				
$n=4$	1	4	12	20	15			
$n=5$	1	5	20	51	74	52		
$n=6$	1	6	30	104	231	302	203	
$n=7$	1	7	42	185	564	1116	1348	877

Table 4.1 Values for $A(n,k)$

Lemma 4.1. *There are $(n-k+1)$ ways to place element k in an $\mathcal{A}_n$ sequence.*

Proof. Recall that $0 \leq a_i \leq i$ for $1 \leq i \leq n$. Therefore the first $(k-1)$ entries cannot take on value k. So there are $n-(k-1) = n-k+1$ possible placements of k. □

Theorem 4.2. *The sets $\mathcal{A}_{n,k}$ have the following recursive relationship*:

$$A(n,0) = 1,$$
$$A(n,k) = (n-k+1)\sum_{i=0}^{k-1} A(n-1,i) \quad \textit{for} \quad 1 \leq k \leq n.$$

Proof. Clearly $A(n,0) = 1$ because the only sequence in $\mathcal{A}_{n,0}$ is the all-zero sequence. So consider $1 \leq k \leq n$. The left hand side $A(n,k)$ is the number of $\mathcal{A}_n$ sequences where the maximum entry is exactly k.

Let $\mathcal{A}_{n,k,\ell}$ be the sequences in $\mathcal{A}_n$ whose largest element is k and whose second largest element is $0 \leq \ell \leq k-1$. These sets are disjoint and we have

$$\mathcal{A}_{n,k} = \bigcup_{\ell=0}^{k-1} \mathcal{A}_{n,k,\ell}.$$

We claim that
$$|\mathcal{A}_{n,k,\ell}| = (n-k+1)A(n-1,i)$$
Indeed, let $a = (a_1, a_2, \ldots, a_n) \in \mathcal{A}_{n,k,\ell}$. There are $n-k+1$ options for the location of k in this sequence. Removing this value from a results in a sequence of length $n-1$ whose largest value is ℓ. By Lemma 4.1, removing k from the sequence is a $(n-k+1)$-to-1 mapping from $\mathcal{A}_{n,k,\ell}$ to $A(n-1,i)$, and the claim is proven.

Finally, we have

$$\begin{aligned} A(n,k) = |\mathcal{A}_{n,k}| &= \left|\bigcup_{\ell=0}^{k-1} \mathcal{A}_{n,k,\ell}\right| \\ = \sum_{\ell=0}^{k-1} |\mathcal{A}_{n,k,\ell}| &= \sum_{\ell=0}^{k-1} (n-k+1)A(n-1,i) \\ = (n-k+1)\sum_{i=0}^{k-1} A(n-1,i). \end{aligned}$$

□

Using the recurrence relation of Theorem 4.2, we obtain the following formulas for $A(n,k)$ for $0 \leq k \leq 4$.

$A(n,0) = 1,$

$A(n,1) = n,$

$A(n,2) = (n-1)n,$

$A(n,3) = (n-2)^2(n-1) + (n-2)n,$

$A(n,4) = (n-3)^3(n-2) + (n-3)^2(n-1) + (n-3)(n-2)(n-1) + (n-3)n.$

Next, we give a general formula for $A(n,k)$. We prove two lemmas along the way.

Lemma 4.3. *The number of ways to make an $\mathcal{A}_n$ sequence with i nonzero entries*

$1 \leq c_1 < c_2 < \cdots < c_i = k$ is

$$\prod_{j=1}^{i}(n - c_j + 1 - (i - j)).$$

Proof. We want to make a sequence $a \in \mathcal{A}_n$ with i nonzero entries $1 \leq c_1 < c_2 < \cdots < c_i = k$.

We place elements these from largest to smallest. First, we place element $c_i = k$. From Lemma 4.1, there are $n - c_i + 1$ possible locations for c_i.

Next, we place c_{i-1}. Since c_i has been placed, we can treat the available locations as a sequence of length $(n-1)$. There are $(n-1) - c_{i-1} + 1$ possible locations for c_{i-1}.

Continuing in this pattern, when placing c_j, we have aleady placed $c_{j+1}, \ldots, c_i$. There are $(n - (i - j)$ empty spots in our sequence since $i - j$ elements have been placed. Therefore, from Lemma 4.1, there are $((n - (i - j) - c_j + 1)$ possible locations for c_j.

Because we follow a pattern of placing some $c_j \in c$ and then placing c_{j-1}, we need to multiply all these numbers together to obtain the total number of sequences whose nonzero elements are $c_1, \ldots, c_i$. This results in

$$\prod_{j=1}^{i}(n - c_j + 1 - (i - j))$$

sequences. $\square$

Lemma 4.4. *The number of $\mathcal{A}_n$ sequences with i nonzero entries and greatest element equal to k is*

$$\sum_{r \in R(i,k)} \prod_{j=1}^{i}(n - r_j + 1 - (i - j))$$

where $R(i,k) = \{r_1, r_2, \ldots, r_i : 1 \leq r_1 < r_2 < \cdots < r_i = k\}$.

Proof. Observe that

$$R(i,k) = \{r_1, r_2, \ldots, r_i | 1 \leq r_1 < r_2 < \cdots < r_i = k\}$$

is the set of all sequences of length i where entries are nonzero, distinct, and increasing to k. Since these cases are distinct, we will add up the possible ways to form $\mathcal{A}_n$ sequences from Lemma 4.3. This is

$$\sum_{r \in R(i,k)} \prod_{j=1}^{i} (n - r_j + 1 - (i - j)).$$

□

Theorem 4.5. *The number of $\mathcal{A}_n$ sequences with greatest element equal to k, $A(n,k)$ is given by the following*

$$A(n,k) = \sum_{i=1}^{k} \sum_{r \in R(i,k)} \prod_{j=1}^{i} (n - r_j + 1 - (i - j))$$

where $R(i,k) = \{r_1, r_2, \ldots, r_i | 1 \leq r_1 < r_2 < \cdots < r_i = k\}$.

Proof. From Lemma 4.4, we know the number of ways to construct all $\mathcal{A}_n$ sequences that have i nonzero entries and maximum element k is

$$\sum_{r \in R(i,k)} \prod_{j=1}^{i} (n - r_j + 1 - (i - j))$$

where $R(i,k) = \{r_1, r_2, \ldots, r_i | 1 \leq r_1 < r_2 < \cdots < r_i = k\}$.

The number of nonzero entries can range from 1 (where the only nonzero entry is k) or all the way to k (where all values $1, 2, \ldots, k$ are in the sequence). We add all these possibilities together to achieve the total number of sequences with maximum element k:

$$A(n,k) = \sum_{i=1}^{k} \sum_{r \in R(i,k)} \prod_{j=1}^{i} (n - r_j + 1 - (i - j)).$$

□

4.2 Maximum Element at Most k

Let $B(n,k)$ be the number of $\mathcal{A}_n$ sequences in which the maximum element is at most k. These are the staircase arrangements of pillars with distinct heights where no pillars taller than k are in the arrangement. The values for

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1	2						
$n=2$	1	3	5					
$n=3$	1	4	10	15				
$n=4$	1	5	17	37	52			
$n=5$	1	6	26	77	151	203		
$n=6$	1	7	37	141	372	674	877	
$n=7$	1	8	50	235	799	1915	3263	4140

Table 4.2 Values of $B(n,k)$

$B(n,k)$ for $1 \leq k \leq n \leq 7$ are shown in Table 4.2. This triangular sequence does not appear in the OEIS.

Just as elements of $\mathcal{A}_{n,k}$ can be described as set partitions of $[n+1]$, so too can elements of $\mathcal{B}_{n,k}$. These are set partitions of $[n+1]$ elements $A = a_{1,1}, a_{1,2}, \ldots, a_{1,r_1} | a_{2,1}, a_{2,2}, \ldots, a_{2,r_2} | \ldots | a_{b,1}, a_{b,2}, \ldots, a_{b,r_b}$ where A has b blocks such that the following are true:

- there exists i for each $j > k$ such that $a_{i,1} = j$.

In other words, integers greater than k must begin their own blocks. $\mathcal{B}_{n,k}$ differs from $\mathcal{A}_{n,k}$ since k need not be in the sequence. This means that k could be the largest element in a block for a set partition from $\mathcal{B}_{n,k}$.

Theorem 4.6. *Elements of $B(n,k)$ have the following recurrence relation*:

$$B(n,k) = B(n,k-1) + (n-k+1)B(n-1,k-1).$$

Proof. On the left, $B(n,k)$ is the number of $\mathcal{A}_n$ sequences that have no elements greater than k. We can divide these into two different cases. Either an element of $B(n,k)$ has k in it, or it does not. The number of $\mathcal{A}_n$ sequences that do not have element k or greater is the number of $\mathcal{A}_n$ sequences that have maximum element at most $(k-1)$. Therefore, our first case is $B(n,k-1)$.

Alternatively, consider the family of sequences in $B(n,k)$ that include k. We can construct these sequences by beginning with those that make up $B(n-1,k-1)$ and adding element k into the sequence. There are $(n-k+1)$ ways to place element k in a sequence, so this means there are $(n-k+1) \cdot B(n-1,k-1)$ sequences that have maximum element equal to k.

We add these two possibilities together since they are separate cases and achieve that

$$B(n,k) = B(n,k-1) + (n-k+1)B(n-1,k-1).$$

$\square$

Lemma 4.7. *We have the following relationship between $\mathcal{A}_n$ sequences with maximum element exactly k and $\mathcal{A}_n$ sequences with maximum element at most k:*

$$A(n,k) = (n-k+1) \cdot B(n-1,k-1)$$

Proof. Let $\mathcal{B}_{n,k} \subset \mathcal{A}_n$ be sequences in which no elements are greater than k.

On the left side of the relationship, $A(n,k)$ is the number of $\mathcal{A}_n$ sequences in which the maximum entry is exactly k. We claim

$$A(n,k) = (n-k+1) \cdot |\mathcal{B}_{n-1,k-1}|$$

Let $a \in \mathcal{B}_{n-1,k-1}$. Clearly $a \notin \mathcal{A}_{n,k}$ since there is no element k. We will place k in a, for which Lemma 4.1 tells us there are $n-k+1$ possible positions.

Therefore,

$$\begin{aligned} A(n,k) &= (n-k+1) \cdot |\mathcal{B}_{n-1,k-1}| \\ &= (n-k+1) \cdot B(n-1,k-1) \end{aligned}$$

$\square$

Theorem 4.8. *$B(n,k)$ can be defined entirely by the values $B(n-1,j)$ in row $n-1$ via the following recurrence relationship*

$$B(n,k) = 1 + \sum_{i=1}^{k} (n-i+1) \cdot B(n-1,i-1).$$

Proof. Clearly

$$B(n,k) = \sum_{i=0}^{k} A(n,i) \quad \text{for} \quad 1 \le k \le n.$$

and so we have

$$B(n,k) = A(n,0) + \sum_{i=1}^{k} A(n,i)$$
$$B(n,k) = 1 + \sum_{i=1}^{k} (n-i+1) \cdot B(n-1,i-1)$$

because $A(n,0) = 1$ and $A(n,i) = (n-i+1) \cdot B(n-1,i-1)$ by Lemma 4.7. □

4.3 Missing Element k

Let $\mathcal{C}_{n,k} \subset \mathcal{A}_n$ be the sequences in which the element k does not appear. These are staircase arrangements of pillars with distinct heights where the pillar of height k does not appear. When given as the induced set partitions, these are set partitions of $[n+1]$ elements $A = a_{1,1}, a_{1,2}, \ldots, a_{1,r_1} | a_{2,1}, a_{2,2}, \ldots, a_{2,r_2} | \ldots | a_{b,1}, a_{b,2}, \ldots, a_{b,r_b}$ where A has b blocks such that the following are true:

- there exists i such that $a_{i,1} = k$.

In other words, k must begin its own block/be the greatest element in its block.

We denote the number of these sequences as $C(n,k) = |\mathcal{C}_{n,k}|$. The values of $C(n,k)$ for small n are given in Table 4.3. This triangular array is known as Aitken's array, which is sequence A011971 in [8].

In this section, we prove the recursive relation for the sets $\mathcal{C}_{n,k}$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=1$	1	1	2						
$n=2$	1	2	3	5					
$n=3$	1	5	7	10	15				
$n=4$	1	15	20	27	37	52			
$n=5$	1	52	67	87	114	151	203		
$n=6$	1	203	255	322	409	523	674	877	
$n=7$	1	877	1080	1335	1657	2066	2589	3263	4140

Table 4.3 Values of $C(n,k)$

Lemma 4.9. *$|C_{n,k}| = |\mathcal{A}_n|$ for $k > n$*

Proof. Elements of $\mathcal{A}_n$ are characterized by $0 \le a_i \le i$. So there are no sequences with an element greater than n. Therefore the number of sequence that contain $k > n$ is always zero, which means that $|\mathcal{C}_{n,k}| = |\mathcal{A}_n|$ when $k > n$. □

Theorem 4.10. *The sets $\mathcal{C}_{n,k}$ have the following recursive relationship*:

$$
\begin{aligned}
&C(n, 0) = 1,\\
&C(1, 1) = 1,\\
&C(n, k) = C(n, k-1) + C(n-1, k-1) \quad \textit{for} \quad 2 \le k \le n+1.
\end{aligned}
$$

Proof. We have $C(n, 0) = 1$. Since every element is distinct and nonzero, we have

$$\mathcal{C}_{n,0} = \{(1, 2, 3, \ldots, n)\}.$$

It is also clear that $C(1, 1) = 1$ because $\mathcal{C}_{1,1} = \{(0)\}$.

Considering the recurrence relation, the left side is the number of $\mathcal{A}_n$ sequences that do not contain element k. On the right side, we have the number of $\mathcal{A}_n$ sequences that do not contain element $k-1$ and the number of $\mathcal{A}_{n-1}$ sequences that do not contain element $k-1$.

We partition $\mathcal{C}_{n,k}$ into three subsets.

- Let $\mathcal{C}'_{n,k}$ be the $\mathcal{A}_n$ sequences that do not contain k and also do not contain $k-1$.

- Let $\mathcal{C}''_{n,k}$ be the $\mathcal{A}_n$ sequences that do not contain k and have $a_i = k-1$ for some $i \ge k$.

- Finally, let $\mathcal{C}'''_{n,k}$ be the $\mathcal{A}_n$ sequences that do not contain k and where $a_{k-1} = k-1$.

These are disjoint sets, so

$$C(n, k) = |\mathcal{C}'_{n,k}| + |\mathcal{C}''_{n,k}| + |\mathcal{C}'''_{n,k}|.$$

Let us partition $\mathcal{C}_{n,k-1}$ into two sets. Let $X \subset \mathcal{C}_{n,k-1}$ be sequences that contain neither k nor $k-1$. Let Y be the sequence that do not contain $k-1$ but do contain k. Clearly $\mathcal{C}'_{n,k} = X$, by definition.

Next, we create a bijective mapping from $\mathcal{C}''_{n,k}$ to Y. Given $a = (a_1, a_2, ..., a_n) \in \mathcal{C}''_{n,k}$, we map a to $a' \in Y$ by increasing $a_i = k-1$ to k: this increase is possible because $i \geq k$. This mapping is clearly a bijection.

Taken together, we have

$$|\mathcal{C}'_{n,k}| + |\mathcal{C}''_{n,k}| = C(n, k-1).$$

Finally, we will show that

$$|\mathcal{C}'''_{n,k}| = C(n-1, k-1)$$

by creating a bijection from $\mathcal{C}'''_{n,k}$ to $\mathcal{C}_{n-1,k-1}$.

Let $a \in \mathcal{C}'''_{n,k}$. We will turn a into $a' \in \mathcal{C}_{n-1,k-1}$ by deleting entry a_{k-1} and then decrementing every entry of a that is larger than k. More explicitly, this mapping is

1. Set $a'_i = a_i$ for $1 \leq k-2$.
2. For $k \leq i \leq n$, set $a'_{i-1} = a_i$ if $a_i < k-1$ and set $a'_{i-1} = a_i - 1$ if $a_i > k$.

The result is a sequence $a' \in \mathcal{C}_{n-1,k-1}$.

We show that this is a bijection by giving the reverse mapping. Given $b \in \mathcal{C}_{n-1,k-1}$, we map to $b' \in \mathcal{C}'''_{n,k}$ as follows:

1. Set $b'_i = b_i$ for $1 \leq k-2$.
2. Set $b'_{k-1} = k-1$
3. For $k-1 \leq i \leq n-1$, set $a'_{i-1} = a_i$ if $a_i < k-1$ and set $a'_{i-1} = a_i + 1$ if $a_i \geq k$.

The result is indeed a sequence b' with $b'_{k-1} = k-1$ and missing k, and it is clear that this mapping is the inverse of the process above.

Finally, we conclude

$$\begin{aligned} C(n,k) &= |\mathcal{C}_{n,k}| \\ &= |\mathcal{C}'_{n,k}| + |\mathcal{C}''_{n,k}| + |\mathcal{C}'''_{n,k}| \\ &= |\mathcal{C}_{n,k-1}| + |\mathcal{C}_{n-1,k-1}| \\ &= C(n,k-1) + C(n-1,k-1). \end{aligned}$$

$\square$

In this section, we will provide and prove the explicit equation for $C(n,k)$. As previously mentioned, the values for $C(n,k)$ where $1 \leq k \leq n$ are equivalent to Aitken's array given by OEIS A011971, which gives us the following formula.

Theorem 4.11. *The number of $\mathcal{A}_n$ sequences without k, or $C(n,k)$, when $1 \leq k \leq n$ are given by the following*

$$C(n,k) = \sum_{i=0}^{k-1} \binom{k-1}{i} B_{n-i}$$

where B_m is the mth Bell number.

Proof. Recall that $C(n,k) = |\mathcal{C}_{n,k}|$. Also note that since our sequences do not contain element k, k must be the largest element in its block of the induced set partition. Let us construct all the values of $\mathcal{C}_{n,k}$ by forming their corresponding set partitions of $[n+1]$ where k is the largest in its block.

First, we pick the number of elements that will be in a block with k. This can range from 0 where k is in its own block all the way to $k-1$ where all the integers $[k-1]$ are in a block with k. This gives us $\binom{k-1}{i}$ where i is the number of elements we pick to be in a block with k.

Once we have placed k and the i other elements into a block, we have $n-i$ elements left to place into blocks. We know that these are counted by the Bell numbers. Therefore, we multiply $\binom{k-1}{i}$ with B_{n-i}.

Since the number of elements that can be in a block with k range from 0 to $k-1$, we sum $\binom{k-1}{i}B_{n-i}$ from $i=0$ to $i=k-1$.

So we have our definition

$$C(n,k) = \sum_{i=0}^{k-1} \binom{k-1}{i} B_{n-i}.$$

□

The description for Aitken's array is as follows:

> $a(n,k)$ is the number of equivalence relations on $\{0,...,n\}$ such that k is not equivalent to n, $k+1$ is not equivalent to n, ..., $n-1$ is not equivalent to n.

In this definition, "equivalent" means that the integers are in different blocks. This leads us to the following open question: what is the mapping between

(a) set partitions where $k+1$ is not in the same block as $n+1$ and (b) set partitions where $k+1$ is the largest element in its block?

Chapter 5

Conclusion

Throughout this research project, we explored various features of the sequences in $\mathcal{A}_n$. We began by showing that they are in bijection with set partitions of $[n+1]$. We then gave a natural method of ordering the elements to induce a partially ordered set. With these orderings and covers, we analyzed how covers are achieved through the bijective set partitions. We also considered the lengths of longest and shortest maximal chains of $\mathcal{A}_n$ as well as chains between nearly identical elements. Finally, we explored various subfamilies of $\mathcal{A}_n$, including those with greatest element exactly k, all elements at most k and sequences in which there is no element k. We proved recursive relationships for all of these and also provided an explicit definition for $A(n, k)$.

Next steps in this research would include the following:

- Count the number of maximal chains for $\mathcal{A}_n$. We were able to count the number of longest maximal chains, but not the total number of maximal chains.

- Further examine the length of the longest and shortest chains between comparable elements.

- Count the number of elements at least k steps away from the maximal element of $\mathcal{A}_n$. We were able to count the number of elements exactly 2 steps away from the maximal element but not more than that.

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