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Sharp Inequalities of the X-Ray Transform and the Competing Symmetries Argument

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Abstract

We examine the $k = 1$ case of a conjecture by Baernstein and Loss pertaining to the operator norm of the k-plane transform from $L^p(\mathbb{R}^d)$ space to $L^q(\mathcal{M})$ space. Previous work on this problem by Carlen and Loss, as well as by Drouot, has used an iterative technique known as the "competing symmetries argument" to prove this conjecture in the $q = 2$ and $q = d + 1$ cases. We summarize the conjecture and this proof technique, then perform work that strongly suggest that no sufficiently "nice" transformation exists that can be used to apply the competing symmetries argument to other cases of the conjecture.

Contents

Notation

In what follows, d will always be assumed to be the dimension of the Euclidean space $\mathbb{R}^d$ that we are working in. Furthermore, if $x \in \mathbb{R}^d$, then for any $i \in \{1, \ldots, d\}$, we take x_i to mean the ith entry in the vector x, so that $x = (x_1, \ldots, x_d)$ (similar notation applies regardless of what variable name we're using). Similarly, if $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ is a function, then for any $i \in \{1, \ldots, d\}$, we take $\gamma_i : \mathbb{R}^d \to \mathbb{R}$ to be the function representing the ith entry of the output of γ, so that $\gamma(x) = (\gamma_1(x), \ldots, \gamma_d(x))$. Finally, $\mathbb{R}^{>0}$ will be taken to mean $\{x \in \mathbb{R} : x > 0\}$, the set of positive real numbers, while $\mathbb{R}^{\geq 0}$ will be taken to mean $\{x \in \mathbb{R} : x \geq 0\}$, the set of nonnegative real numbers.

Chapter 1

Introduction

In 1997, Albert Baernstein II and Michael Loss published a paper entitled "Some Conjectures about L^p Norms of k-Plane Transforms" [1], in which they discussed the titular k-plane transform $T_{k,d}$, a particular functional operator defined such that if f is a function $\mathbb{R}^d \to \mathbb{R}$, then $T_{k,d}f$ is a function $M_{k,d} \to \mathbb{R}$ (where $M_{k,d}$ represents the set of affine k-planes in $\mathbb{R}^d$). Of course, Baernstein and Loss were not the first people to consider this particular transform (indeed, their paper cites previous work on the topic, including work that proves weaker versions of their own conjectures); however, they do appear to be the first to have seriously treated the specific question of the k-plane transform's operator norm (with respect to the L^p norm of functions on $\mathbb{R}^d$ and the L^q norm of functions on $M_{k,d}$).

Indeed, their paper proposes three conjectures pertaining to the operator norm of the k-plane transform. Of these three, the first one they propose, which explicitly states what Baernstein and Loss believe to be the operator norm of $T_{k,d}$, is the most important one, with the other two (one concerning how the symmetric decreasing rearrangement operation interacts with $T_{k,d}$, and one concerning the operator norm of a similar transformation that acts on functions $\mathbb{R}^{>0} \to \mathbb{R}$) mostly existing to aid in proofs of the first one. Baernstein and Loss then go on to prove the conjecture in the $q = 2$ case, by showing that in that case, it is equivalent to the already-proven sharp Hardy-Littlewood-Sobolev inequality.

Since the publication of this paper, other analysts have devoted themselves to tackling this conjecture. One major advance came in 2014, with the publication of Alexis Drouot's "Sharp Constant for a k-Plane Transform Inequality" [5], which proved the conjecture in the $q = d + 1$ case. And within the context of this thesis, Druout's paper especially notable for the method it used. As noted, the $q = 2$ case of the Baernstein-Loss conjecture is equivalent to the sharp version of the Hardy-Littlewood-Sobolev inequality. Many proofs of this statement exist, among them the "competing symmetries argument," first used (as Baernstein and Loss note) by Carlen and Loss in 1990, in which a transformation D is constructed so that for any function $f : \mathbb{R}^d \to \mathbb{R}$, the sequence $\{(DR)^n f\}_{n=0}^{\infty}$ (where R is the symmetric decreasing rearrangement) possesses certain properties – properties from which the desired inequality can be immediately derived. Drouot's insight was extending this technique to a new case; specifically, he constructed a new transformation J (a slight modification of a transformation originally discovered in a 2011 paper by Christ [4] regarding a closely related topic) such that for any function $f : \mathbb{R}^d \to \mathbb{R}$, the sequence $\{(JR)^n f\}_{n=0}^{\infty}$ possesses certain properties, from which one can derive an inequality that is equivalent to the Baernstein-Loss conjecture in the $q = d + 1$ case.

Drouot's proof led analysts working in the domain to speculate that it was possible to extend the logic of the "competing symmetries argument" even further – to even more values of q, beyond just 2 and $d + 1$. And it was reasonable for them to expect as such. After all, Baernstein and Loss's paper notes that their conjecture is only known to make sense when $1 \leq q \leq d + 1$, that the ways in which k-plane transforms and the symmetric decreasing rearrangement interact are most well-understood when q is an integer, and that the $q = 1$ case of the conjecture is comparatively much easier to prove than the other cases; thus, in some appreciable sense, 2 is the smallest value of q we are interested in, and $d + 1$ is the largest. So, if we can use the competing symmetries argument to prove the Baernstein-Loss conjecture "at the endpoints" of q, with the transformation D being used at one "endpoint" and the transformation J being used at the other, then it makes sense to posit that by finding transformations "between" D and J in some manner, we could apply the competing symmetries argument using those transformations to obtain proofs of the Baernstein-Loss conjecture for values of q between 2 and $d + 1$. Further bolstering this hope was the fact that the transformation D is based off of a sphere, while J is based off of a hemisphere – so perhaps a transformation between D and J would be based off of a three-quarter sphere or something to that effect.

In this thesis, we will address this prospect of extending the competing symmetries argument to other values of q, specifically as it concerns the 1-plane (or X-ray) transform. We will first rigorously define and contextualize the set of affine 1-planes (i.e., lines) in $\mathbb{R}^d$, the X-ray transform, and the Baernstein-Loss conjecture. Then, we will state and prove a general form of the competing symmetries argument – the exact form in which we might hope to slot in a transformation "between" D and J so as to prove the conjecture for other values of q – and discuss its previous success with those D and J transformations. Afterwards, we will state a newly-discovered condition that any transformation that hopes to be slotted into the competing symmetries argument must satisfy; and finally, we will show that that condition is such that no sufficiently "nice" transformation can be slotted into the competing symmetries argument for values of q other than 2 and $d+1$, thus putting a significant damper on future attempts to use this technique to prove the Baernstein-Loss conjecture.

Chapter 2

The Domain of Lines in $\mathbb{R}^d$

As mentioned previously, in [1], Baernstein and Loss discuss functions that act on $M_{k,d}$, the set of affine k-planes in $\mathbb{R}^d$ (such spaces are also discussed in other papers on the topic, of course). Since we will be restricting our attention to the $k = 1$ case of Baernstein and Loss's conjecture (the case of the X-ray transform), we will naturally restrict our attention (as far as affine spaces are concerned) to $M_{1,d}$, the set of 1-dimensional affine subspaces (lines) in $\mathbb{R}^d$ (the domain of functions subjected to the X-ray transform); for simplicity, we will denote this set as simply $\mathcal{M}$.

Definition 1. *For a given Euclidean space $\mathbb{R}^d$, $\mathcal{M}$ represents the set of lines in $\mathbb{R}^d$. In other words, we define $\mathcal{M}$ such that for any given $\ell \subseteq \mathbb{R}^d$, $\ell \in \mathcal{M}$ if and only if we may write it in the form*

$$\ell = \{a + bt : t \in \mathbb{R}\}$$

for some $a \in \mathbb{R}^d$ and some $b \in \mathbb{R}^d \setminus \{0\}$.

While it is relatively simple to describe what $\mathcal{M}$ is, both intuitively and rigorously, it is harder to describe $\mathcal{M}$ in a way that makes defining operations and performing calculations on it easy and logical. Whereas, for instance, Cartesian coordinates are the canonical way of describing elements (points) of $\mathbb{R}^d$ as numbers or lists of numbers (due to the fact that we can construct a useful, continuous, bijection between points in Euclidean space and such lists), there is no such canonical "best" way of describing $\mathcal{M}$ in terms of $\mathbb{R}$. Instead, there are multiple different natural ways we may choose to parametrize the elements of $\mathcal{M}$ – to parametrize the set of lines in $\mathbb{R}^d$ – each of which will be useful in different circumstances.

2.1 The "Slope-Intercept" Parametrization

The first parametrization of $\mathcal{M}$ we will cover is one that is based off of one of the first ways of describing lines that many students of math learn about: slope-intercept form. To wit, any non-vertical line in $\mathbb{R}^2$ can be uniquely described as the graph of a function of the form $y = mx + b$, where $m \in \mathbb{R}$ represents the line's slope and $b \in \mathbb{R}$ represents the line's y-intercept (i.e., where the line intersects the y-axis $\{(x, 0) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$). In this way, every line in $\mathbb{R}^2$ (or almost every line, rather – again, vertical lines are not counted) can be represented with a pair of numbers in $\mathbb{R}$ (or an element of $\mathbb{R}^2$).

Extending the notion of slope-intercept form to higher dimensions is a bit trickier, though, as it impossible to represent lines in $\mathbb{R}^d$ (with $d > 2$) using just one equation (the graph of the function $z = 2x + 3y - 4$ in $\mathbb{R}^3$, for instance, is a plane, not a line), to say nothing of the fact that lines in higher-dimensional space don't have a single defined slope or intercept. However, if we, effectively, write any line in $\mathbb{R}^d$ as a combination of $d - 1$ different slope-intercept forms, we can extend this notion. To wit, if $m_2, \ldots, m_d, b_2, \ldots, b_d \in \mathbb{R}$, then the equations

$$\begin{aligned} x_2 &= m_2 x_1 + b_2 \\ x_3 &= m_3 x_1 + b_3 \\ &\cdots \\ x_d &= m_d x_1 + b_d, \end{aligned}$$

uniquely describe a line in $\mathbb{R}^d$ (specifically, as that line will consist of all points $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that satisfy those equations). We may equivalently describe this line as

$$\{(1, m_2, \ldots, m_d)t + (0, b_2, \ldots, b_d) : t \in \mathbb{R}\}.$$

In this way, nearly every line in $\mathbb{R}^d$ can be represented with $d - 1$ slopes and $d - 1$ intercepts, and thus with $2(d - 1)$ numbers in total (or one element of $\mathbb{R}^{2(d-1)}$).

Despite its relative simplicity, this parametrization carries with it several disadvantages, such as the fact that it does not induce a particularly elegant measure on $\mathcal{M}$, and the fact that it cannot represent lines on or parallel to the hyperplane $\{(0, x_2, \ldots, x_d) : x_2, \ldots, x_d \in \mathbb{R}\}$. However, the fact that this parametrization forms a perfect bijection between $\mathbb{R}^{2(d-1)}$ and the elements of $\mathcal{M}$ it can represent, as well as the elegant way it interacts with the transformation J, means that it will still be useful to us.

2.2 The “Line Segment” Parametrization

The next parametrization of $\mathcal{M}$ we will cover is also based off of a well-known parametrization of lines in $\mathbb{R}^d$ – or rather, line segments. Specifically, it is known that for any two different points $u, v \in \mathbb{R}^d$, then $(1-t)u + tv$, $0 \leq t \leq 1$ parametrizes the line segment starting at u and ending at v – however, if we let t take on any real number, not just ones in the range $[0, 1]$, then $(1-t)u + tv$ parametrizes the entire line through u and v. To put it another way, if $u, v \in \mathbb{R}^d$ (where $u \neq v$) are two distinct points on a line $\ell \in \mathcal{M}$, then we may write ℓ as

$$\{(1-t)u + tv : t \in \mathbb{R}\}.$$

In this fashion, any line $\ell \in \mathcal{M}$ can be represented by two different points $u, v \in \mathbb{R}^d$.

There are certain factors that make this parametrization less than ideal for working with, chief among them the fact that any line ℓ can be represented infinitely many ways (as every single possible pair of distinct points u and v on the line ℓ will induce a valid parametrization $\{(1-t)u + tv : t \in \mathbb{R}\}$ of ℓ). However, thinking of lines in ℓ in this fashion actually ends up being essential to bridging the gap between the Baernstein-Loss conjecture and the sharp Hardy-Little-Sobolev inequality, and thus to presenting the Baernstein-Loss conjecture in a fashion that allows the competing symmetries argument to be used to prove it.

2.3 The (θ, y) Parametrization

But perhaps the most useful parametrization of $\mathcal{M}$ for our purposes will be one that is derived from a way of describing lines that is somewhat less well known (at least outside of analysis). To wit, let $\theta \in \mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ be any unit vector in $\mathbb{R}^d$, and let $y \in (\text{span}(\{\theta\}))^{\perp} = \theta^{\perp} \subseteq \mathbb{R}^d$ be any vector in $\mathbb{R}^d$ that is orthogonal to θ. Then, the set

$$\{\theta t + y : t \in \mathbb{R}\}$$

is a line in $\mathcal{M}$. Thus, lines in $\mathcal{M}$ can be represented by a unit vector $\theta \in \mathbb{S}^{d-1}$ and an orthogonal vector $y \in \theta^{\perp}$.

As with the previous two ways of representing lines, there is not a strict bijection between pairs $(\theta, y) \in \mathbb{S}^{d-1} \times \theta^{\perp}$ and lines $\ell \in \mathcal{M}$. However, unlike the previous two ways (which were, respectively, only bijective if you ignored a certain subset of $\mathcal{M}$, and which mapped infinitely many pairs $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ to any given line), there is a strict 2-to-1 correspondence between pairs $(\theta, y) \in \mathbb{S}^{d-1} \times \theta^\perp$ and lines $\ell \in \mathcal{M}$. Specifically, if $\ell \in \mathcal{M}$, then there always exists a pair $(\theta, y) \in \mathbb{S}^{d-1} \times \theta^\perp$ such that $\ell = \{\theta t + y : t \in \mathbb{R}\}$; and (θ, y) and $(-\theta, y)$ are the only two pairs that correspond to ℓ in this fashion.

This correspondence means that this (θ, y) parametrization of $\mathcal{M}$ is ideal for defining integrals of real-valued functions over $\mathcal{M}$. After all, $\mathbb{S}^{d-1}$ and $\theta^\perp$ are both spaces that have canonical integrals defined over them; and if we integrate a function over both $\mathbb{S}^{d-1}$ and $\theta^\perp$, we will effectively have integrated it over $\mathcal{M}$ twice over (thanks to the 2-to-1 correspondence between $\mathbb{S}^{d-1} \times \theta^\perp$ and $\mathcal{M}$). The following definition makes this intuition more explicit:

Definition 2. *Let F be a function $\mathcal{M} \to \mathbb{R}$. For any given $\ell \in \mathcal{M}$, if ℓ can be written as $\{\theta t + y : t \in \mathbb{R}\}$ for some $\theta \in \mathbb{S}^{d-1}$ and some $y \in \theta^\perp$, we will take the expression $F(\theta, y)$ to mean $F(\ell)$.*

With this in mind, we will define the integral of F over $\mathcal{M}$ to be equal to

$$\int_{\mathcal{M}} F = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{\theta^\perp} F(\theta, y) \, dy d\theta,$$

where integrals over $\mathbb{S}^{d-1}$ are taken using the standard $d-1$-dimensional spherical measure, and integrals over any given $\theta^\perp$ are taken using the $d-1$-dimensional Lebesgue measure.

For a more detailed discussion of how to treat $\mathcal{M}$ as a measure space over which we may integrate, consult the beginning of [1].

In any case, because we now have a canonical measure and integral on $\mathcal{M}$, we are able to go even further and define an L^p norm for functions $\mathcal{M} \to \mathbb{R}$ – though to distinguish this norm from the more commonly-used L^p norm of functions $\mathbb{R}^d \to \mathbb{R}$ (which will also be discussed heavily in this thesis), we will usually call it the L^q norm of $\mathcal{M}$ (in keeping with standard notation for codomains of functional transformations). More formally:

Definition 3. *Let F be a measurable function $\mathcal{M} \to \mathbb{R}$, and let $q \geq 1$. Then, the L^q norm (or, more precisely, the $L^q(\mathcal{M})$ norm) of F is defined as*

$$\|F\|_{L^q(\mathcal{M})} = \left(\int_{\mathcal{M}} |F|^q \right)^{\frac{1}{q}}.$$

The set of all measurable functions $F : \mathcal{M} \to \mathbb{R}$ (with functions that are equal almost everywhere being treated as equal) such that $\|F\|_{L^q(\mathcal{M})} < \infty$ is known as $L^q(\mathcal{M})$ space, or simply $L^q(\mathcal{M})$.

Chapter 3

The X-Ray Transform and the Baernstein-Loss Conjecture

Now that we have discussed at length the space $\mathcal{M}$ of lines in $\mathbb{R}^d$, as well as the basics of analyzing functions on that space $\mathcal{M}$, we have laid the foundation for the central concepts of this thesis, to which we will turn our attention now.

3.1 The X-Ray Transform $\mathcal{X}$ and Related Concepts

As alluded in this thesis's introduction, Baernstein and Loss's original paper discussed the k-plane transform $T_{k,d}$, an operation that takes functions acting on points in $\mathbb{R}^d$ and turns them into functions acting on k-dimensional affine subspaces of $\mathbb{R}^d$. While that paper proposed conjectures concerning $T_{k,d}$ for all possible $k \in \{1, \ldots, d-1\}$, in this thesis, we will exclusively focus on the $k = 1$ case – which is to say, we will exclusively focus on the transform $T_{1,d}$, which is also known as (and will henceforth almost exclusively be referred to as) the X-ray transform, or $\mathcal{X}$.

The X-ray transform $\mathcal{X}$ is an operation that takes real-valued functions acting on points in $\mathbb{R}^d$ and turns them into real-valued functions acting on lines in $\mathbb{R}^d$ – or, in other words, on elements of $\mathcal{M}$. (The X-ray transform can also be applied to complex-valued functions on $\mathbb{R}^d$ to turn them into complex-valued functions on $\mathcal{M}$, but for simplicity's sake, we will only consider the real case in this thesis.) The way in which it transforms these functions is surprisingly simple: The X-ray transform of a function f at a given line $\ell \in \mathcal{M}$ is just the integral of f along ℓ. More specifically, we may define the X-ray transform as follows:

Definition 4. *Let f be a measurable function $\mathbb{R}^d \to \mathbb{R}$. The X-ray transform $\mathcal{X}$ of f, then, is the function $\mathcal{X}f : \mathcal{M} \to \mathbb{R}$, defined such that for any line $\ell \in \mathcal{M}$, if ℓ can be written as $\{\theta t + y : t \in \mathbb{R}\}$ for some $\theta \in \mathbb{S}^{d-1}$ and some $y \in \theta^{\perp}$, then*

$$\mathcal{X}f(\ell) = \int_{\ell} f = \int_{\mathbb{R}} f(\theta t + y)\, dt.$$

The X-ray transform derives its name from the fact that, in some sense, it functions similarly to X-ray imaging. One can imagine the function $f : \mathbb{R}^d \to \mathbb{R}$ as a body or specimen being analyzed (perhaps one existing in d-dimensional space that has density $f(x)$ at a given point $x \in \mathbb{R}^d$), and the lines ℓ as the paths of the X-rays being generated by the imaging machine. The integral $\int_{\ell} f$, then, is analogous to the amount of mass in the specimen of density f that a given X-ray travelling along line ℓ would pass through; and just as real X-ray machines record how much mass their X-rays encounter by having those X-rays leave impressions on film (with rays travelling through denser parts of the specimen leaving lighter impressions), the X-ray transform records "how much of the function f" a line ℓ passes through via the function $\mathcal{X}f$ (with lines ℓ passing through "larger" parts of the f having greater values of $\mathcal{X}f(\ell)$). And in fact, the connection between the X-ray transform and real-world X-ray imaging is more than just a metaphor, as the X-ray transform is actually used in the study of medical imaging.

We will now define a new functional Φ, which can broadly be described as measuring how much the X-ray transform increases the size of a given function f. (The last chapter, of course, gave us a coherent notion of size for functions $\mathcal{X}f : \mathcal{M} \to \mathbb{R}$.)

Definition 5. *Let $p \geq 1$ and $q \geq 1$ be given. For any function $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$), we define:*

$$\Phi(f) = \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}}.$$

In the broadest of possible terms, the concept that interests us in this thesis is what bounds exist on the possible values of Φ. And as the next theorem shows, if we desire Φ to be bounded at all, we are actually significantly constrained in our choice of norms $L^q(\mathcal{M})$ and $L^p(\mathbb{R}^d)$.

Lemma 3.1. *For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and any positive real number $r > 0$, let $f_r : \mathbb{R}^d \to \mathbb{R}$ be a function defined such that $f_r(x) = f(rx)$. Additionally, let $p \geq 1$ and $q \geq 1$ be given. Then, for all $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$) and all $r > 0$,*

$$\Phi(f_r) = r^{\frac{1-d-q}{q}+\frac{d}{p}}\Phi(f).$$

Proof. First, note that:

$$\|f_r\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f_r|^p\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d} |f(rx)|^p\,dx\right)^{\frac{1}{p}},$$

which, via a change in variables (from x to u, such that $x = r^{-1}u$ and $dx = (r^{-1})^d\,du = r^{-d}\,du$, with the power of d being introduced because x and u are d-dimensional variables), becomes

$$\left(\int_{\mathbb{R}^d} |f(rx)|^p\,dx\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d} |f(u)|^p r^{-d}\,du\right)^{\frac{1}{p}} = \left(r^{-d}\int_{\mathbb{R}^d} |f(u)|^p\,du\right)^{\frac{1}{p}}$$
$$= r^{-\frac{d}{p}}\left(\int_{\mathbb{R}^d} |f(u)|^p\,du\right)^{\frac{1}{p}} = r^{-\frac{d}{p}}\|f\|_{L^p(\mathbb{R}^d)}.$$

Thus, $\|f_r\|_{L^p(\mathbb{R}^d)} = r^{-\frac{d}{p}}\|f\|_{L^p(\mathbb{R}^d)}$. (Note that this statement also implies that f_r must still be in $L^p(\mathbb{R}^d)$ – and as $r \neq 0$, $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$ as well.)

Next, note that, for any $\theta \in \mathbb{S}^{d-1}$ and $y \in \theta^\perp$:

$$\mathcal{X}f_r(\theta, y) = \int_{\mathbb{R}} f_r(\theta t + y)\,dt = \int_{\mathbb{R}} f(r(\theta t + y))\,dt = \int_{\mathbb{R}} f(\theta(rt) + ry)\,dt,$$

which, via a change in variables (from t to u, such that $t = r^{-1}u$ and $dt = r^{-1}\,du$, becomes

$$\int_{\mathbb{R}} f(\theta(rt) + ry)\,dt = \int_{\mathbb{R}} f(\theta u + ry)r^{-1}\,du$$
$$= r^{-1}\int_{\mathbb{R}} f(\theta u + ry)\,du = r^{-1}\mathcal{X}f(\theta, ry).$$

Thus, $\mathcal{X}f_r(\theta, y) = r^{-1}\mathcal{X}f(\theta, ry)$.

And with this in mind, note that:

$$\|\mathcal{X}f_r\|_{L^q(\mathcal{M})} = \left(\int_{\mathcal{M}} |\mathcal{X}f_r|^q\right)^{\frac{1}{q}} = \left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} |\mathcal{X}f_r(\theta, y)|^q\,dyd\theta\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} |r^{-1}\mathcal{X}f(\theta,ry)|^q \, dyd\theta\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} |r^{-1}|^q|\mathcal{X}f(\theta,ry)|^q \, dyd\theta\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} r^{-q}|\mathcal{X}f(\theta,ry)|^q \, dyd\theta\right)^{\frac{1}{q}},$$

which, via a change in variables (from y to u, such that $y = r^{-1}u$ and $dy = (r^{-1})^{d-1}\,du = r^{1-d}\,du$, with the power of $d-1$ being introduced because y and u are $d-1$-dimensional variables), becomes

$$\left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} r^{-q}|\mathcal{X}f(\theta,ry)|^q \, dyd\theta\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} r^{-q}|\mathcal{X}f(\theta,u)|^q r^{1-d} \, dud\theta\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{2}r^{1-d-q}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} |\mathcal{X}f(\theta,u)|^q \, dud\theta\right)^{\frac{1}{q}}$$
$$= r^{\frac{1-d-q}{q}}\left(\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^\perp} |\mathcal{X}f(\theta,u)|^q \, dud\theta\right)^{\frac{1}{q}} = r^{\frac{1-d-q}{q}}\|\mathcal{X}f\|_{L^q(\mathcal{M})}.$$

Thus, $\|\mathcal{X}f_r\|_{L^q(\mathcal{M})} = r^{\frac{1-d-q}{q}}\|\mathcal{X}f\|_{L^q(\mathcal{M})}$.

Finally, with all of the preceding statements in mind, we may write that:

$$\Phi(f_r) = \frac{\|\mathcal{X}f_r\|_{L^q(\mathcal{M})}}{\|f_r\|_{L^p(\mathbb{R}^d)}} = \frac{r^{\frac{1-d-q}{q}}\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{r^{-\frac{d}{p}}\|f\|_{L^p(\mathbb{R}^d)}}$$
$$= r^{\frac{1-d-q}{q}+\frac{d}{p}}\frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}} = r^{\frac{1-d-q}{q}+\frac{d}{p}}\Phi(f),$$

as desired. □

Theorem 3.2. *If $p \neq \frac{dq}{d+q-1}$ (and $p,q \geq 1$), then $\Phi(f)$ is unbounded, which is to say that for any arbitrarily large positive real number $M > 0$, there exists a function $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$) such that $\Phi(f) \geq M$.*

Proof. Let $M > 0$ be any positive real number.

First, note that:

$$\begin{aligned} &\frac{1-d-q}{q}+\frac{d}{p}=0 \\ \Rightarrow &p(1-d-q)+dq=0 \\ \Rightarrow &p(1-d-q)=-dq \\ \Rightarrow &p=\frac{-dq}{1-d-q} \\ \Rightarrow &p=\frac{dq}{d+q-1}. \end{aligned}$$

Thus, if $p \neq \frac{dq}{d+q-1}$, then $\frac{1-d-q}{q}+\frac{d}{p} \neq 0$.

With this in mind, consider any function $f \in L^p(\mathbb{R}^d)$ with nonzero $L^p(\mathbb{R}^d)$ norm. If $\Phi(f) \geq M$, then the proof is complete. Otherwise, let $r = (\frac{M}{\Phi(f)})^{(\frac{1-d-q}{q}+\frac{d}{p})^{-1}}$ (note that the reciprocal $(\frac{1-d-q}{q}+\frac{d}{p})^{-1}$ is well-defined, as $p \neq \frac{dq}{d+q-1}$, so $\frac{1-d-q}{q}+\frac{d}{p} \neq 0$), and consider the function f_r. By Lemma 3.1, $f_r \in L^p(\mathbb{R}^d)$, $\|f_r\|_{L^p(\mathbb{R}^d)} \neq 0$, and:

$$\begin{aligned} \Phi(f_r) = r^{\frac{1-d-q}{q}+\frac{d}{p}}\Phi(f) &= \left(\left(\frac{M}{\Phi(f)}\right)^{(\frac{1-d-q}{q}+\frac{d}{p})^{-1}}\right)^{\frac{1-d-q}{q}+\frac{d}{p}}\Phi(f) \\ &= \frac{M}{\Phi(f)}\Phi(f) = M \geq M, \end{aligned}$$

in which case the proof is also complete. □

As a result of Theorem 3.2, our efforts from here on out will be exclusively focused on the case where $p = \frac{dq}{d+q-1}$, as it is only when that equation is satisfied that Φ can possibly be bounded. In fact, in 1984, Michael Christ proved the converse of Theorem 3.2 – that $\Phi(f)$, the ratio between the norm of $\mathcal{X}f$ and the norm of f, is always bounded when $p = \frac{dq}{d+q-1}$ and $q \leq d+1$ [3]. And in doing so, Christ established that when $p = \frac{dq}{d+q-1}$ and $q \in [1, d+1]$, $\mathcal{X}$ may be treated not just as something that turns functions on $\mathbb{R}^d$ into functions on $\mathcal{M}$, but as an operator that maps elements of one normed vector space to another – as an operator $\mathcal{X} : L^p(\mathbb{R}^d) \to L^q(\mathcal{M})$.

In fact, because

$$\mathcal{X}(\lambda_1 f+\lambda_2 g)(\ell) = \int_\ell (\lambda_1 f+\lambda_2 g) = \lambda_1 \int_\ell f + \lambda_2 \int_\ell g = \lambda_1 \mathcal{X}f(\ell) + \lambda_2 \mathcal{X}g(\ell)$$

for all $f, g \in L^p(\mathbb{R}^d)$, all $\lambda_1, \lambda_2 \in \mathbb{R}$, and all $\ell \in \mathcal{M}$, we can conclude that $\mathcal{X}$ is specifically a linear operator; and because the ratio Φ between $\|\mathcal{X}f\|_{L^q(\mathcal{M})}$ and $\|f\|_{L^p(\mathbb{R}^d)}$, we can conclude that $\mathcal{X}$ is furthermore specifically a bounded linear operator. So, for the rest of this thesis, we will consider $\mathcal{X}$ as a bounded linear operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathcal{M})$.

3.2 The Baernstein-Loss Conjecture: Statement and Proof for $q = 1$

Now that we are thinking of $\mathcal{X}$ as a bounded linear operator from one normed vector space to another, we may consider its operator norm – the "size" of the X-ray transform itself.

Definition 6. *Let V and W be two normed vector spaces, and let $T : V \to W$ be a bounded linear operator. The operator norm of T, then, is*

$$\|T\| = \sup_{\substack{x \in V \\ \|x\|_V \neq 0}} \frac{\|Tx\|_W}{\|x\|_V}.$$

In our case, the operator norm of the X-ray transform will be equal to

$$\|\mathcal{X}\| = \sup_{\substack{f \in L^p(\mathbb{R}^d) \\ \|f\|_{L^p(\mathbb{R}^d)} \neq 0}} \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}} = \sup_{\substack{f \in L^p(\mathbb{R}^d) \\ \|f\|_{L^p(\mathbb{R}^d)} \neq 0}} \Phi(f).$$

It is this very operator norm – this very supremum of Φ – that is the central focus of the conjectures laid out in [1]. In fact, at this point, we are ready to directly state the primary conjecture from [1] – the very conjecture whose proofs (both achieved and hypothetical) that the rest of this thesis will be dedicated to analyzing.

Conjecture 1. *Let $d > 1$ be a given integer, let $q \in [1, d+1]$ be given, and let $p = \frac{dq}{d+q-1}$. Furthermore, for any positive real $a, b > 0$, let us define the function $f_0 : \mathbb{R}^d \to \mathbb{R}$ such that for any $x \in \mathbb{R}$,*

$$f_0(x) = (a + b|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}. \tag{3.1}$$

Then, for all $a, b > 0$, the operator norm of the X-ray transform is equal to

$$\|\mathcal{X}\| = \sup_{\substack{f \in L^p(\mathbb{R}^d) \\ \|f\|_{L^p(\mathbb{R}^d)} \neq 0}} \Phi(f) = \Phi(f_0).$$

It should be noted the conjecture (which we will often call the "Baernstein-Loss conjecture") as stated here is not exactly identical to the conjecture as stated in Baerstein and Loss's original paper. First, as discussed previously, the original conjecture considered the operator norm $\|T_{k,d}\|$ of the k-plane transform for all possible $k \in \{1, \ldots, d-1\}$, whereas this rendering of the conjecture is restricted to the $k = 1$ case; and second, Baernstein and Loss did not use the terminology of the functional Φ. However, the conjecture as stated here is otherwise equivalent to the one from the original paper.

It should also be noted that because the function (or rather, the functions) f_0 is, according to the conjecture, the function that maximizes – in other words, extremizes – the value of Φ, we often refer to it as the "conjectured extremizer".

We will now prove the simplest case of the Baernstein-Loss conjecture: the $q = 1$ case.

Lemma 3.3. *For any function $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$),*

$$\Phi(f) \leq \Phi(|f|).$$

Proof. First, note that, for any $f \in L^p(\mathbb{R}^d)$:

$$\begin{aligned} |f| &= |(|f|)| \\ |f|^p &= |(|f|)|^p \\ \int_{\mathbb{R}^n} |f|^p &= \int_{\mathbb{R}^n} |(|f|)|^p \\ \left(\int_{\mathbb{R}^n} |f|^p\right)^{1/p} &= \left(\int_{\mathbb{R}^n} |(|f|)|^p\right)^{1/p} \\ \|f\|_{L^p(\mathbb{R}^d)} &= \||f|\|_{L^p(\mathbb{R}^d)} \end{aligned}$$

Next, note that, for any $f \in L^p(\mathbb{R}^d)$, we may use $\mathcal{X}$'s linearity, as well as the monotonicity of integrals (and thus the monotonicity of $\mathcal{X}$) to write that:

$$\begin{gathered} -|f| \leq f \leq |f| \\ \forall \ell \in \mathcal{M}, \int_\ell -|f| \leq \int_\ell f \leq \int_\ell |f| \\ \mathcal{X}(-|f|) \leq \mathcal{X}f \leq \mathcal{X}(|f|) \\ -\mathcal{X}(|f|) \leq \mathcal{X}f \leq \mathcal{X}(|f|) \\ |\mathcal{X}f| \leq |\mathcal{X}(|f|)| \\ |\mathcal{X}f|^q \leq |\mathcal{X}(|f|)|^q \end{gathered}$$

$$\int_{\mathcal{M}} |\mathcal{X}f|^q \leq \int_{\mathcal{M}} |\mathcal{X}(|f|)|^q$$
$$\left(\int_{\mathcal{M}} |\mathcal{X}f|^q\right)^{1/q} \leq \left(\int_{\mathcal{M}} |\mathcal{X}(|f|)|^q\right)^{1/q}$$
$$\|\mathcal{X}f\|_{L^q(\mathcal{M})} \leq \|\mathcal{X}(|f|)\|_{L^q(\mathcal{M})}$$

Since $\|\mathcal{X}f\|_{L^q(\mathcal{M})} \leq \|\mathcal{X}(|f|)\|_{L^q(\mathcal{M})}$ and $\|f\|_{L^p(\mathbb{R}^d)} = \||f|\|_{L^p(\mathbb{R}^d)}$ for all $f \in L^p(\mathbb{R}^d)$, and as all values involved in these statements are nonnegative (by virtue of being norms), we can conclude that for all $f \in L^p(\mathbb{R}^d)$ where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$,

$$\frac{\|\mathcal{X}(f)\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}} \leq \frac{\|\mathcal{X}(|f|)\|_{L^q(\mathcal{M})}}{\||f|\|_{L^p(\mathbb{R}^d)}},$$

and thus that

$$\Phi(f) \leq \Phi(|f|),$$

as desired. □

Theorem 3.4. *Conjecture 1 holds when $q = 1$.*

Proof. For any dimension d, if $q = 1$, then we have that $p = \frac{dq}{d+q-1} = \frac{d(1)}{d+1-1} = \frac{d}{d} = 1$.

Now, consider any nonnegative function $f \geq 0$ in $L^p(\mathbb{R}^d)$ where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$. We may write that:

$$\Phi(f) = \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}} = \frac{\|\mathcal{X}f\|_{L^1(\mathcal{M})}}{\|f\|_{L^1(\mathbb{R}^d)}} = \frac{\left(\int_{\mathcal{M}} |\mathcal{X}f|^1\right)^{\frac{1}{1}}}{\left(\int_{\mathbb{R}^d} |f|^1\right)^1} = \frac{\int_{\mathcal{M}} |\mathcal{X}f|}{\int_{\mathbb{R}^d} |f|}$$
$$= \frac{\frac{1}{2}\int_{\mathbb{S}^{d-1}} \int_{\theta^\perp} |\mathcal{X}f(\theta, y)|\, dy d\theta}{\int_{\mathbb{R}^d} |f(x)| dx}.$$

Now, note that as $f \geq 0$, not only does $|f| = f$, but for all $\ell \in \mathcal{M}$, $\int_\ell f \geq 0$, meaning that $\mathcal{X}f \geq 0$, and thus that $|\mathcal{X}f| = \mathcal{X}f$. So, from here, we may write that:

$$\frac{\int_{\mathcal{M}} |\mathcal{X}f|}{\int_{\mathbb{R}^d} |f|} = \frac{\int_{\mathcal{M}} \mathcal{X}f}{\int_{\mathbb{R}^d} f} = \frac{\frac{1}{2}\int_{\mathbb{S}^{d-1}} \int_{\theta^\perp} \mathcal{X}f(\theta, y)\, dy d\theta}{\int_{\mathbb{R}^d} f(x)\, dx}$$
$$= \frac{\frac{1}{2}\int_{\mathbb{S}^{d-1}} \int_{\theta^\perp} \int_{\mathbb{R}} f(\theta t + y)\, dt dy d\theta}{\int_{\mathbb{R}^d} f(x)\, dx},$$

which, via a change in variables (from $t \in \mathbb{R}$ and $y \in \theta^{\perp}$ to $x \in \mathbb{R}^d$, such that $x = \theta t + y$ and $dx = dtdy$), becomes

$$\frac{\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\theta^{\perp}}\int_{\mathbb{R}} f(\theta t + y)\,dtdyd\theta}{\int_{\mathbb{R}^d} f(x)\,dx} = \frac{\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d} f(x)\,dxd\theta}{\int_{\mathbb{R}^d} f(x)\,dx} = \frac{\frac{1}{2}\int_{\mathbb{R}^d} f(x)\,dx\int_{\mathbb{S}^{d-1}} d\theta}{\int_{\mathbb{R}^d} f(x)\,dx}$$
$$= \frac{1}{2}\int_{\mathbb{S}^{d-1}} d\theta = \frac{1}{2}\left(\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\right) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

Thus, $\Phi(f) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ for all $f \geq 0$ in $L^p(\mathbb{R}^d)$ with nonzero norm.

Note that $f_0(x) = (a + b|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$ is a nonnegative function; after all, $a, b > 0$ by definition and $|x|^2$ must be nonnegative for all x, meaning $a + b|x|^2$ must be nonnegative as well, which in turn means that $(a + b|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$ must be nonnegative as well. And of course, for any $f \in L^p(\mathbb{R}^d)$, $|f|$ is a nonnegative function. This means that, based on Lemma 3.3 and what we just proved, we may write that for all $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$),

$$\Phi(f) \leq \Phi(|f|) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = \Phi(f_0).$$

Therefore, $\Phi(f_0) = \max_{\substack{f \in L^p(\mathbb{R}^d) \\ \|f\|_{L^p(\mathbb{R}^d)} \neq 0}} \Phi(f) = \sup_{\substack{f \in L^p(\mathbb{R}^d) \\ \|f\|_{L^p(\mathbb{R}^d)} \neq 0}} \Phi(f)$, meaning that Conjecture 1 holds, as desired. □

Unfortunately, to prove this statement for other values of $q \in [1, d+1]$, more complicated proofs will be required.

Chapter 4

The Competing Symmetries Argument

The competing symmetries argument is a proof structure that was discovered by Eric Carlen and Michael Loss and described by them in a 1988 paper [2]. This paper explained the competing symmetries argument in its broadest possible terms (stating the weakest possible conditions that the two titular symmetries must satisfy for the argument to work), and then gave several examples of the argument being applied to prove well-known analysis inequalities, such as the Hardy-Littlewood-Sobolev inequality and Nelson's hypercontractivity inequality.

For our purposes, however, while we will state the competing symmetries argument in a somewhat general form – we will not assume any particular value of p, q, or d, and we will leave one of the two titular "competing symmetries" in the statement completely unspecified (only talking about what properties it must have for the proof to work) – we will only be discussing the argument as it applies to the question of the Baernstein-Loss conjecture. As a result, for our purposes, one of the competing symmetries in question will be assumed to always be the symmetric decreasing rearrangement, and the functional we are trying to find the extremizers for will be assumed to always be Φ.

4.1 Some Important Lemmas

Before we prove anything directly relating to the competing symmetries argument, we must first introduce several lemmas. Although we will be stating these lemmas without proof, and although these lemmas are quite miscellaneous in terms of their subject matter, they will all eventually prove essential to the competing symmetry argument.

The first such lemma we will discuss is known as Helly's selection principle.

Lemma 4.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $\mathbb{R}^d \to \mathbb{R}$ that are all monotonic and are uniformly bounded (i.e., there exists a real number $M \geq 0$ such that for all $i \in \mathbb{N}$ and all $x \in \mathbb{R}^d$, $|f_i(x)| \leq M$). Then, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of that sequence (with $\{n_k\}_{k=1}^{\infty}$ being a strictly increasing sequence of natural numbers) that converges pointwise to some symmetric decreasing function f.*

This principle is, in effect, an extension of the Bolzano-Weierstrass Theorem to functions (and in fact, the Bolzano-Weierstrass Theorem is used to prove it).

The next such lemma we will discuss, like Helly's selection principle, allows us to take a sequence of functions satisfying certain conditions and find a pointwise convergent subsequence.

Lemma 4.2. *If a sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges to a function f under the $L^p(\mathbb{R}^d)$ norm, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of that sequence (with $\{n_k\}_{k=1}^{\infty}$ being a strictly increasing sequence of natural numbers) that converges pointwise to f.*

A proof of this statement may be found in Section 2.7 of [6].

The third such lemma we will discuss is an identity originally discovered by Blaschke in 1935 – or, more specifically, a corollary to Blaschke's identity demonstrated by [1] (restricted, of course, to the k-plane transform case where $k = 1$).

Lemma 4.3. *Let q be an integer greater than or equal to 2. Then there exists a positive constant $C \in \mathbb{R}^{>0}$ such that, for all nonnegative functions $f : \mathbb{R}^d \to \mathbb{R}$,*

$$\int_{\mathbb{R}^d} (\mathcal{X} f)^q = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) f(y) |x-y|^{q-1-d} \left(\int_{\mathbb{R}} f((1-t)x + ty) dt \right)^{q-2} dx dy,$$

and thus, as an immediate consequence,

$$\|\mathcal{X} f\|_{L^q(\mathcal{M})} = \left(C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) f(y) |x-y|^{q-1-d} \left(\int_{\mathbb{R}} f((1-t)x + ty) dt \right)^{q-2} dx dy \right)^{\frac{1}{q}}.$$

Note that, as nonnegative real numbers' equality is preserved under multiplication by a positive constant and under taking powers, Blaschke's identity implies that for any two nonnegative functions $f, g : \mathbb{R}^d \to \mathbb{R}$, $\|\mathcal{X}f\|_{L^q(\mathcal{M})} = \|\mathcal{X}g\|_{L^q(\mathcal{M})}$ if and only if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)f(y)|x-y|^{q-1-d}\left(\int_{\mathbb{R}} f((1-t)x+ty)dt\right)^{q-2} dxdy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x)g(y)|x-y|^{q-1-d}\left(\int_{\mathbb{R}} g((1-t)x+ty)dt\right)^{q-2} dxdy.$$

The final lemma we will discuss here is much more directly relevant to the Baernstein-Loss conjecture itself; effectively, it states that in order to show that every function of the form described in equation (3.1) is an extremizer of Φ, it suffices to show that just one function f_0 of that form is an extremizer of Φ.

Lemma 4.4. *Let $a_1, b_1, a_2, b_2 > 0$ all be positive real numbers, and let p, q, and d be parameters satisfying the conditions in the Baernstein-Loss conjecture. Let us define the functions $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ such that $f_1(x) = (a_1 + b_1|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$ and $f_2(x) = (a_2 + b_2|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$. Then*

$$\Phi(f_1) = \Phi(f_2).$$

While we will not go into a fully detailed proof of this statement, unlike the previous two lemmas, a brief (but complete) summary of the reasoning behind it will be within the scope of this thesis. To wit, any function $f_1(x) = (a_1 + b_1|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$ can be transformed into to any function $f_2(x) = (a_2 + b_2|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$ using exclusively scalar multiplication (i.e., operations of the form $f \mapsto kf$ for some $k \in \mathbb{R}$) and dilation (i.e., operations of the form $f \mapsto f_r$ for some $r > 0$, with f_r defined as in Lemma 3.1). By the linearity of $\mathcal{X}$ and the multiplicativity of the $L^p(\mathbb{R}^d)$ and $L^q(\mathcal{M})$ norms, scalar multiplication preserves Φ (since it increases $\|\mathcal{X}f\|_{L^q(\mathcal{M})}$ and $\|f\|_{L^p(\mathbb{R}^d)}$ by the same factor); and by Lemma 3.1 and Theorem 3.2, dilation also preserves Φ. Thus, Φ must be preserved between $f_1(x) = (a_1 + b_1|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$ and $f_2(x) = (a_2 + b_2|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$, as stated in the lemma.

4.2 The Symmetric Decreasing Rearrangement R

Let us begin our discussion of the competing symmetries argument proper by defining one of the symmetries we will use: the symmetric decreasing rearrangement R.

Definition 7. *For a given set $A \subseteq \mathbb{R}^d$ of finite measure, the symmetric rearrangement of A is A^*, the open ball in $\mathbb{R}^d$, centered at the origin, whose radius is such that its Lebesgue measure is the same as A's. Specifically, letting m represent the Lebesgue measure, for any given measurable $A \subseteq \mathbb{R}^d$ where $m(A) < \infty$,*

$$A^* = \left\{ x \in \mathbb{R}^d : |x| < \left(\frac{m(A)}{V_d} \right)^{\frac{1}{d}} \right\},$$

where $V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$ is the volume or measure of a d-dimensional unit sphere.

Note that the symmetric rearrangement of an ball (whether open or closed) centered at the origin is simply the open ball of that radius centered at the origin. After all, the measure of a ball of radius r centered at the origin is $r^d V_d$, meaning that that ball's symmetric rearrangement is $\{x \in \mathbb{R}^d : |x| < (\frac{r^d V_d}{V_d})^{\frac{1}{d}}\} = \{x \in \mathbb{R}^d : |x| < r\}$ – the open ball of radius r centered at the origin.

Definition 8. *For a given measurable function $f : \mathbb{R}^d \to \mathbb{R}$ that vanishes at infinity (i.e., whose absolute value's level sets all have finite measure), the symmetric decreasing rearrangement of that function is $Rf : \mathbb{R}^d \to \mathbb{R}$, the function on $\mathbb{R}^d$ whose level sets are equal to the symmetric rearrangements of the level sets of $|f|$. Specifically, for any measurable $f : \mathbb{R}^d \to \mathbb{R}$,*

$$Rf(x) = \int_0^\infty \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) dt$$

for any $x \in \mathbb{R}^d$.

Intuitively, the symmetric decreasing rearrangement operator "shifts around" a function's "mass" (i.e., rearranges it) towards the center of $\mathbb{R}^d$ (i.e., so as to make the function radially symmetric and decreasing).

The symmetric decreasing rearrangement of a function possesses many noteworthy properties that make it ideal to be used in the competing symmetries argument, several of which are presented below (with proofs for many of them).

Theorem 4.5. *Let $f : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ be nonnegative measurable functions that vanish at infinity. Then, the following statements are true:*

(1) The function Rf is radially symmetric, which is to say that for any $x, y \in \mathbb{R}^d$, if $|x| = |y|$, then $Rf(x) = Rf(y)$.

(2) *The function Rf is nonincreasing (with respect to the norm $|x|$ of points x in $\mathbb{R}^d$), which is to say that for any $x, y \in \mathbb{R}^d$, if $|x| \leq |y|$, then $Rf(x) \geq Rf(y)$.*

(3) *The function f is (equal almost everywhere to) a radially symmetric and nonincreasing function if and only if $Rf = f$ (almost everywhere).*

(4) *If $f \leq g$, then $Rf \leq Rg$. (In other words, R is order-preserving.)*

(5) *For all $C \in \mathbb{R}^{\geq 0}$, $R(Cf) = CRf$.*

(6) *Let $p > 1$. If $f, g \in L^p(\mathbb{R}^d)$, $\|Rf - Rg\|_{L^p(\mathbb{R}^d)} \leq \|f - g\|_{L^p(\mathbb{R}^d)}$ (i.e., R is nonexpansive). Furthermore, if f is radially symmetric and strictly decreasing (with respect to the norm $|x|$ of points x in $\mathbb{R}^d$), then $\|Rf - Rg\|_{L^p(\mathbb{R}^d)} = \|f - g\|_{L^p(\mathbb{R}^d)}$ if and only if $Rg = g$.*

(7) *If $f \in L^p(\mathbb{R}^d)$, then $\|Rf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$ and $\|\mathcal{X}Rf\|_{L^q(\mathcal{M})} \geq \|\mathcal{X}Rf\|_{L^q(\mathcal{M})}$. Consequently, if $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$, then $\Phi(Rf) \geq \Phi(f)$.*

Proof. Let us begin by proving statement (1). Say that $|x| = |y|$. Now, for any $t \in \mathbb{R}^{>0}$, consider the set $\{u \in \mathbb{R}^d : |f(u)| > t\}^*$. By the definition of the symmetric rearrangement, we must be able to write this set as $\{u \in \mathbb{R}^d : |u| < r\}$ for some $r \geq 0$. If $|x| = |y| < r$, then x and y are both in the set, meaning that $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) = \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y) = 1$. Otherwise, if $|x| = |y| \geq r$, then neither x nor y is in the set, meaning that $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) = \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y) = 0$. In either case, $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x)$ must equal $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y)$ for all $t \in \mathbb{R}^{>0}$, meaning that, under our assumption that $|x| = |y|$, $Rf(x) = \int_0^\infty \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x)dt$ must equal $Rf(y) = \int_0^\infty \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y)dt$, as desired.

Next, let us prove statement (2). Say that $|x| \leq |y|$. Now, for any $t \in \mathbb{R}^{>0}$, consider the set $\{u \in \mathbb{R}^d : |f(u)| > t\}^*$. By the definition of the symmetric rearrangement, we must be able to write this set as $\{u \in \mathbb{R}^d : |u| < r\}$ for some $r \geq 0$. If $|x| \leq |y| < r$, then x and y are both in the set, meaning that $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) = 1$ and $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y) = 1$. If $r \leq |x| \leq |y|$, then neither x nor y is in the set, meaning that $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) = 0$ and $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y) = 0$. The only remaining case is that $|x| < r \leq |y|$, in which case x is in the set but y is not, meaning that $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) = 1$ and $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y) = 0$. In all of these cases, $\mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x) \geq \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y)$ for all $t \in \mathbb{R}^{>0}$, meaning that, under our assumption that $|x| \leq |y|$, $Rf(x) = \int_0^\infty \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(x)dt$ must be greater than or equal to $Rf(y) = \int_0^\infty \mathbb{1}_{\{u \in \mathbb{R}^d : |f(u)| > t\}^*}(y)dt$, as desired.

Now, let us prove statement (3). First, let f be equal almost everywhere to g, a radially symmetric and (radially) nonincreasing function. For any $t \in \mathbb{R}^{>0}$, consider the level set $\{u \in \mathbb{R}^d : |f(u)| > t\}$. Because $f = g$ almost everywhere, $\{u \in \mathbb{R}^d : |f(u)| > t\}$ can differ from $\{u \in \mathbb{R}^d : |g(u)| > t\}$ only by a set of measure zero, which in turn means that $\{u \in \mathbb{R}^d : |f(u)| > t\}$ must have the same measure as $\{u \in \mathbb{R}^d : |g(u)| > t\}$, which in turn means that $\{u \in \mathbb{R}^d : |f(u)| > t\}^* = \{u \in \mathbb{R}^d : |g(u)| > t\}^*$ (since a set's symmetric rearrangement only depends on its measure). So, let us consider g. Because g is nonincreasing with respect to the radius, $\{u \in \mathbb{R}^d : |g(u)| > t\}$ must be describable as the union of several closed balls (of varying radii) in $\mathbb{R}^d$ centered at the origin. To see why, consider any point $x \in \{u \in \mathbb{R}^d : |g(u)| > t\}$ (if the level set is empty, it is vacuously such a union), which is to say, any point x such that $|g(x)| > t$. For any y in the closed ball of radius r centered at the origin (i.e., any $y \in \{u \in \mathbb{R}^d : |u| \leq r\}$), then $y \in \{u \in \mathbb{R}^d : |g(u)| > t\}$ – after all, since $|y| \leq r = |x|$, the fact that g is nonincreasing means that $g(y) \geq g(x) > t$, making y a member of the level set. Consequently, $\{u \in \mathbb{R}^d : |g(u)| > t\}$ can be written as

$$\bigcup_{x \in \{u \in \mathbb{R}^d : |g(u)| > t\}} \{u \in \mathbb{R}^d : |u| = |x|\},$$

which is indeed a union of several closed balls centered at the origin.

Now, note that the union of a collection of closed balls centered at the origin must either be a closed ball centered at the origin, an open ball centered at the origin, or the entirety of $\mathbb{R}^d$; so, the level set $\{u \in \mathbb{R}^d : |g(u)| > t\}$ must be of one of these forms. Because f vanishes at infinity, meaning that g vanishes at infinity, g's level sets $\{u \in \mathbb{R}^d : |g(u)| > t\}$ must have finite measure; consequently, $\{u \in \mathbb{R}^d : |g(u)| > t\}$ cannot equal $\mathbb{R}^d$, meaning that it must be equal to either a closed or open ball centered at the origin. If $\{u \in \mathbb{R}^d : |g(u)| > t\}$ is an open ball centered at the origin, we can conclude that $\{u \in \mathbb{R}^d : |g(u)| > t\} = \{u \in \mathbb{R}^d : |g(u)| > t\}^*$, and thus that $\mathbb{1}_{\{u \in \mathbb{R}^d : |g(u)| > t\}}(x) = \mathbb{1}_{\{u \in \mathbb{R}^d : |g(u)| > t\}^*}(x)$ for all $x \in \mathbb{R}^d$. Otherwise, if $\{u \in \mathbb{R}^d : |g(u)| > t\}$ is a closed ball centered at the origin, the sets $\{u \in \mathbb{R}^d : |g(u)| > t\}$ and $\{u \in \mathbb{R}^d : |g(u)| > t\}^*$ will differ only by a set of Lebesgue measure zero (specifically, the boundary sphere whose radius is the same as $\{u \in \mathbb{R}^d : |g(u)| > t\}$ and $\{u \in \mathbb{R}^d : |g(u)| > t\}^*$'s), meaning that $\mathbb{1}_{\{u \in \mathbb{R}^d : |g(u)| > t\}}(x) = \mathbb{1}_{\{u \in \mathbb{R}^d : |g(u)| > t\}^*}(x)$ for almost all $x \in \mathbb{R}^d$. In either case, we may write that, for almost every $x \in \mathbb{R}^d$,

$$
\begin{aligned}
Rf(x) &= \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>t\}^*}(x)dt \\
&= \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}^*}(x)dt = \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}}(x)dt \\
&= \int_0^{|g(x)|} dt = [t]_{t=0}^{|g(x)|} = |g(x)| - 0 = |g(x)| = |f(x)| = f(x).
\end{aligned}
$$

(The last step arises due to f's nonnegativity.) Thus, when f is radially symmetric and nonincreasing, $Rf = f$ almost everywhere, as desired.

Next, let f be a function such that $Rf = f$ almost everywhere. Since Rf is the symmetric decreasing rearrangement of a function, by parts (1) and (2) of this theorem, we may conclude that Rf is radially symmetric and nonincreasing. Thus, as $Rf = f$ almost everywhere, f is equal (almost everywhere) to a radially symmetric and nonincreasing function, as desired. The implication holds in both directions, and part (3) of this theorem is true, as desired.

Let us continue by proving statement (4). Say that $f \le g$, which is to say that $f(x) \le g(x)$ for all $x \in \mathbb{R}^d$, and consider any $t \in \mathbb{R}^{>0}$. Because $f(x) \le g(x)$ (and f and g are both nonnegative, so $|f| = f$ and $|g| = g$), for any $u \in \mathbb{R}^d$, $|f(u)| > t$ implies that $|g(u)| > t$. Consequently, the level set $\{u \in \mathbb{R}^d : |f(u)| > t\}$ must be a subset of the level set $\{u \in \mathbb{R}^d : |g(u)| > t\}$, which in turn means that the measure of the level set $\{u \in \mathbb{R}^d : |f(u)| > t\}$ must be less than or equal to that of $\{u \in \mathbb{R}^d : |g(u)| > t\}$. And as open balls' measures scale monotonically with their radii, this means that the radius of the ball $\{u \in \mathbb{R}^d : |f(u)| > t\}^*$ (whose measure is the same as that of $\{u \in \mathbb{R}^d : |f(u)| > t\}$) must be less than or equal to the radius of the ball $\{u \in \mathbb{R}^d : |g(u)| > t\}^*$ (whose measure is the same as that of $\{u \in \mathbb{R}^d : |g(u)| > t\}$) – which, as both balls are centered at the origin, means that $\{u \in \mathbb{R}^d : |f(u)| > t\}^* \subseteq \{u \in \mathbb{R}^d : |g(u)| > t\}^*$. The fact that $\{u \in \mathbb{R}^d : |f(u)| > t\}^* \subseteq \{u \in \mathbb{R}^d : |g(u)| > t\}^*$ means that $\mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>t\}^*}(x) \le \mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}^*}(x)$ for all x. After all, if $x \in \{u \in \mathbb{R}^d : |f(u)| > t\}^*$, it must also be in $\{u \in \mathbb{R}^d : |g(u)| > t\}^*$, so $\mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>t\}^*}(x) = 1 \le 1 = \mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}^*}(x)$ and the inequality is satisfied; and otherwise, if $x \notin \{u \in \mathbb{R}^d : |f(u)| > t\}^*$, then $\mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>t\}^*}(x) = 0$, meaning that whether x is in $\{u \in \mathbb{R}^d : |g(u)| > t\}$ (and thus whether $\mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}^*}(x)$ is equal to 0 or 1), the inequality must be satisfied. This logic works for all $t \in R^+$, so, from here, the monotonicity of integrals allows us to conclude that

$$\begin{aligned} &\mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>t\}^*}(x) \leq \mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}^*}(x) \\ \Rightarrow &\int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>t\}^*}(x)dt \leq \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|g(u)|>t\}^*}(x)dt \\ &\Rightarrow Rf(x) \leq Rg(x) \end{aligned}$$

for all $x \in \mathbb{R}^d$. Thus, $Rf \leq Rg$, as desired.

Now, let us prove statement (5). Consider any $C \in \mathbb{R}^{\geq 0}$ and any nonnegative measurable $f : \mathbb{R}^d \to \mathbb{R}$ that vanishes at infinity. By definition:

$$\begin{aligned} R(Cf) &= \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|Cf(u)|>t\}^*}(x)dt = \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>\frac{t}{|C|}\}^*}(x)dt \\ &= \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>\frac{t}{C}\}^*}(x)dt \end{aligned}$$

(with the last change from $|C|$ to C being able to occur because $C \geq 0$). Changing variables from t to $y = \frac{t}{C}$ (so that $dy = \frac{1}{C}dt \Rightarrow dt = Cdy$), we get that:

$$R(Cf) = \int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>\frac{t}{C}\}^*}(x)dt = C\int_0^\infty \mathbb{1}_{\{u\in\mathbb{R}^d:|f(u)|>y\}^*}(x)dy = CRf,$$

as desired.

□

While proofs of parts (6) and (7) of the preceding theorem are theoretically tractable for the level of mathematics that this thesis is putting forward, a fully rigorous treatment of those two proofs would be so involved and would require so many additional lemmas that it becomes entirely outside of this paper's scope. Specifically, a proof of (6) would require demonstration of certain additional properties of R, such as ones involving integrals of functions' products compared to their rearrangements' products and how R interacts with nondecreasing continuous functions $\mathbb{R}^{>0} \to \mathbb{R}^{>0}$; from there, the proof would apply those properties while exploiting the fact that the function $t \mapsto |t|^p$ is a convex function with a strictly increasing derivative. Meanwhile, a proof of (7) would first involve showing $\|f\|_{L^p(\mathbb{R}^d)} = \|Rf\|_{L^p(\mathbb{R}^d)}$ (which would be done via some of those same properties used to show (6)), then showing that $\|\mathcal{X}f\|_{L^q(\mathcal{M})} \leq \|\mathcal{X}Rf\|_{L^q(\mathcal{M})}$ by using Blaschke's identity alongside Riesz's rearrangement inequality, a theorem which states that for any nonnegative $f, g, h : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Rf(x)Rg(x-y)Rh(y)dxdy.$$

For more details regarding these proofs, consult [6].

In any case, we may also note at this point that as an immediate corollary of statement (7) of Theorem 4.5, $f \in L^p(\mathbb{R}^d) \Rightarrow Rf \in L^p(\mathbb{R}^d)$ (and $\|f\|_{L^p(\mathbb{R}^d)} = 0$ if and only if $\|Rf\|_{L^p(\mathbb{R}^d)} = 0$). Thus, we may consider R as an operator $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ (and $\Phi(Rf)$ is well-defined for all $f \in L^p(\mathbb{R}^d)$ where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$).

4.3 Stating and Proving the Competing Symmetries Argument

And now, we are ready to state the competing symmetries argument proper. The competing symmetries argument can actually be thought of as having two parts: one in which we establish a fact about functional convergence (the part of the argument that actually involves the two symmetries competing), and one in which we use that fact to find extremizers of functionals (such as Φ). We give these two parts, and their proofs, below.

Theorem 4.6. *Let $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ be a functional transformation (a symmetry) satisfying the following properties:*

(1) *For all $f \in L^p(\mathbb{R}^d)$, $\|Tf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$. In other words, T is norm-preserving.*

(2) *For all $f, g \in L^p(\mathbb{R}^d)$, if $f \leq g$, then $Tf \leq Tg$. In other words, T is order-preserving.*

(3) *For all $f, g \in L^p(\mathbb{R}^d)$, $\|Tf - Tg\|_{L^p(\mathbb{R}^d)} \leq \|f - g\|_{L^p(\mathbb{R}^d)}$. In other words, T is nonexpansive.*

(4) *For all $f \in L^p(\mathbb{R}^d)$ and all $C \in \mathbb{R}^{\geq 0}$, $T(Cf) = CTf$.*

(5) *There exists a strictly positive function $h \in L^p(\mathbb{R}^d)$ such that:*

- $\|h\|_{L^p(\mathbb{R}^d)} = 1$.
- *h is strictly decreasing (radially).*
- $Th = h$.
- $\{f \in L^p(\mathbb{R}^d) : Rf = f, RTf = Tf\} = \{Ch : C \in \mathbb{R}^{\geq 0}\}$ *(i.e., up to constant multiplication, h is the unique radially symmetric function such that $RTf = Tf$).*

Then, for any nonnegative function $f \in L^p(\mathbb{R}^d)$, the sequence $\{(RT)^n f\}_{n=1}^{\infty}$ converges to $h_f = \|f\|_{L^p(\mathbb{R}^d)} h$ in $L^p(\mathbb{R}^d)$.

Proof Sketch. The first thing to note about this theorem's proof is that it is a density argument, which is to say that rather than show the theorem's truth directly for all $f \in L^p(\mathbb{R}^d)$, we instead show its truth for a dense subset of $L^p(\mathbb{R}^d)$ (in this case, the set of bounded functions that also have bounded support), and then afterwards extend the statement to the rest of $L^p(\mathbb{R}^d)$.

There are two broad stages to the proof that Theorem 4.6 holds for this dense subset of $L^p(\mathbb{R}^d)$. The first stage is showing that there does, in fact, exist a radially symmetric function g (sharing f's norm) such that some subsequence of $\{(RT)^n f\}_{n=1}^{\infty}$ converges to it in $L^p(\mathbb{R}^d)$. The outline of this stage of the proof will look something like this:

- Use Helly's selection principle to show that $\{(RT)^n f\}_{n=1}^{\infty}$ has a subsequence $\{(RT)^{n_i} f\}_{i=1}^{\infty}$ that converges pointwise to some function g.
- Show that this g is radially symmetric and shares f's norm.
- Use dominated convergence to show that $\{(RT)^{n_i} f\}_{i=1}^{\infty}$ converges to g not just pointwise, but also in $L^p(\mathbb{R}^d)$.

The second stage, then, is showing that this g is equal to our h_f (and that the whole sequence $\{(RT)^n f\}_{n=1}^{\infty}$ converges to it, of course, not just a subsequence $\{(RT)^{n_i} f\}_{i=1}^{\infty}$). The outline of this stage of the proof will look something like this:

- Using property (3) (nonexpansivity) and the fact that $Rh = Th = h$, show that the sequence $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$ is monotone decreasing, and then use the monotone convergence theorem to show that it is convergent.
- Use the fact that $\{(RT)^{n_i} f\}_{i=1}^{\infty}$ converges to g in $L^p(\mathbb{R}^d)$ to show that the value to which $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$ converges is, in fact, $\|h_f - g\|_{L^p(\mathbb{R}^d)}$.
- Use property (3) to show that $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$'s limit can also be written as $\|h_f - RTg\|_{L^p(\mathbb{R}^d)}$.
- Use property (3) and the fact that $Rh = Th = h$ to show that because $\|h_f - g\|_{L^p(\mathbb{R}^d)} = \|h_f - RTg\|_{L^p(\mathbb{R}^d)}$, it must be the case that $\|Rh_f - g\|_{L^p(\mathbb{R}^d)} = \|h_f - Tg\|_{L^p(\mathbb{R}^d)}$.

- Use Theorem 4.5 to conclude that $RTg = Tg$, and use h's uniqueness to show that $g = h$.

From this point, it suffices to complete the density argument in order to complete the proof.

Note the importance of the function h – both in its existence and its uniqueness – in this proof. Intuitively, the existence of such a unique function h satisfying property (5) is what allows the two symmetries R and T to "compete" in a meaningful way.

To understand why this is the case, consider two simple functional operators, the translation T_b given by $T_b f(x) = f(x+b)$ (where $b \in \mathbb{R}^d \setminus \{0\}$ is a nonzero vector) and the rotation T_U given by $T_U f(x) = f(Ux)$ (where $U \in \mathsf{SO}(d)$ is a rotation matrix). It can be readily seen that T_b and T_U both satisfy properties (1) through (4) of this theorem; however, neither satisfy property (5). In T_b's case, there are no radially decreasing functions h such that $T_b h = h$ (as "centering" the function at b rather than 0 will make the function no longer radially decreasing), and in T_U's case, the set $\{f \in L^p(\mathbb{R}^d) : Rf = f, RT_U f = T_U f\}$ consists of every single radially symmetric function f (as rotating any radially symmetric), meaning that any possible h fails to be unique.

And, indeed, we obtain nothing interesting when we consider the sequences $\{(RT_b)^n f\}_{n=1}^{\infty}$ and $\{(RT_U)^n f\}_{n=1}^{\infty}$. Since the Lebesgue measure is invariant under translation, R is as well: $Rf = RT_b f$. This means that repeatedly applying R and T_b to a function will just result in the function Rf over and over again: we translate f, take the symmetric decreasing rearrangement to get $RT_b f = Rf$, translate that, take the symmetric decreasing rearrangement to just shift the function back to the origin, translate again to shift it to b, take R again to move it back, and so on. And since the Lebesgue measure is invariant under rotation, R is as well: $Rf = RT_U f$. Repeatedly applying R and T_u to a function will have the same result as with T_b: we rotate f, take the symmetric decreasing rearrangement to get $RT_U f = Rf$, rotate it and get back the same function due to its symmetry, take R again and get back the same function because it's already symmetric decreasing, rotate again and get back the same function again, take R again and get back the same function again, and so on.

If we want $\{(RT)^n f\}_{n=1}^{\infty}$ to converge to something interesting – to the same function for all f – T has to actually compete with R. It has to scramble functions to the extent that the symmetric decreasing rearrangement of Tf will be completely different from that of f (and thus so that $(RT)^{k+1} f$

is completely different from the preceding sequence element $(RT)^k f$ for all k), instead of "cooperating" with R and leaving the sequence $\{(RT)^n f\}_{n=1}^{\infty}$ stagnant after the first term. And the way we ensure that this competition happens is with property (5): the existence and uniqueness of h.

Proof. Let f be a nonnegative function in $L^p(\mathbb{R}^d)$ that is bounded and whose support on $\mathbb{R}^d$ is bounded, and let $h_f = \|f\|_p h$ be the constant multiple of h that shares f's norm. Note that there must exist a constant $C > 0$ such that

$$f \leq Ch_f.$$

After all, if $f = 0$, then $h_f = \|0\|_p h = 0$ as well, and the statement is true. Otherwise, when $f \neq 0$, so $\|f\|_p \neq 0$, we can say that f is bounded, so there must be some $M \in \mathbb{R}^{>0}$ such that $f \leq M$. Furthermore, f's support is a bounded set, meaning that there exists some radius $K \in \mathbb{R}^{>0}$ such that f is 0 outside of the ball of that radius centered at the origin. Because $Rh_f = h_f$ (by virtue of h_f being a constant multiple of h), h_f is radially symmetric and nonincreasing, meaning that if $h_f(x) = m$ for some (and thus all) x where $|x| = K$, then $h_f(x) \geq m$ for all x where $|x| \leq K$, which is to say for all x in the ball of radius K centered at the origin. And as h (and thus any positive constant multiple of h, like h_f or Ch_f) is strictly positive, $m > 0$. Letting $C = \frac{M}{m}$, we may write, for all x in the ball of radius K centered at the origin, that

$$Ch_f(x) = \frac{M}{m} h_f(x) \geq \frac{M}{m} m = M \geq f(x),$$

and for all x outside that ball, that

$$Ch_f(x) > 0 = f(x),$$

as desired.

Now, consider the sequence $\{(RT)^n f\}_{n=1}^{\infty}$, and specifically consider any element $(RT)^n f$ of it. Because any $(RT)^n f = R(T(RT)^{n-1} f)$ is the symmetric decreasing rearrangement of some function, it must be (symmetric) nonincreasing. Furthermore, because R and T are both order-preserving (and both fix Ch_f), any element of this sequence must, like f, be bounded above by Ch_f. After all, $(RT)^0 f = f \leq Ch_f$, and if $(RT)^k f \leq Ch_f$ for some $k \leq 0$, then $T(RT)^k f \leq TCh_f = Ch_f$, which in turn means that $(RT)^{k+1} f = R(T(RT)^k f) \leq RCh_f = Ch_f$; so, by induction, $(RT)^n f \leq Ch_f$ for all $n \geq 1$.

From here, because $Ch_f \leq Ch_f(0)$ (due to the fact that Ch_f is nonincreasing, so for any $x \in \mathbb{R}^d$, $|x| \geq 0 = |0|$, implying that $Ch_f(x) \leq Ch_f(0)$), $(RT)^n f \leq Ch_f \leq Ch_f(0)$ holds for all $n \in \mathbb{N}$, making $\{(RT)^n f\}_{n=1}^{\infty}$ a uniformly bounded sequence. Since $\{(RT)^n f\}_{n=1}^{\infty}$ is uniformly bounded and every member of it is nonincreasing, Helly's selection principle (Lemma 4.1) allows us to conclude that there exists a subsequence $\{(RT)^{n_i} f\}_{i=1}^{\infty}$ ($\{n_i\}_{i=1}^{\infty}$ being a strictly increasing subsequence of the natural numbers) that converges pointwise to some function $g : \mathbb{R}^d \to \mathbb{R}$.

Note that g is a radially symmetric decreasing function. After all, $|x| \leq |y|$ implies that $(RT)^{n_i} f(x) \geq (RT)^{n_i} f(y)$ for all $i \in \mathbb{N}$ (because each $(RT)^{n_i} f$ is radially symmetric decreasing), which in turn implies that

$$g(x) = \lim_{i\to\infty} (RT)^{n_i} f(x) \geq \lim_{i\to\infty} (RT)^{n_i} f(y) = g(y).$$

Furthermore, note that $\|g\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$. After all, since R and T are both norm-preserving operations, $\|(RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$ for all i (by induction); so, as $\lim_{i\to\infty} (RT)^{n_i} f = g$, we can conclude (as the nonnegative

$$\|(RT)^{n_i} f - g\|_{L^p(\mathbb{R}^d)} \geq \left| \|(RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} - \|g\|_{L^p(\mathbb{R}^d)} \right|$$

goes to 0) that $\|g\|_{L^p(\mathbb{R}^d)} = \lim_{i\to\infty} \|(RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} = \lim_{i\to\infty} \|f\|_{L^p(\mathbb{R}^d)}$. Finally, note that because every $(RT)^{n_i} f$ is less than or equal to $Ch_f \in L^p(\mathbb{R}^d)$, dominated convergence furthermore allows us to conclude that $g \in L^p(\mathbb{R}^d)$ as well, and $\{(RT)^{n_i} f\}_{i=1}^{\infty}$ converges to g under the $L^p(\mathbb{R}^d)$ norm.

Remember that $Th = h$, and that h is symmetric decreasing (strictly so, in fact), meaning that $Rh = h$. Furthermore, by property (5) of Theorem 4.5,

$$Rh_f = R(\|f\|_{L^p(\mathbb{R}^d)} h) = \|f\|_{L^p(\mathbb{R}^d)} Rh = \|f\|_{L^p(\mathbb{R}^d)} h = h_f,$$

and by property (4) of T,

$$Th_f = T(\|f\|_{L^p(\mathbb{R}^d)} h) = \|f\|_{L^p(\mathbb{R}^d)} Th = \|f\|_{L^p(\mathbb{R}^d)} h = h_f.$$

Thus, h_f is fixed under T and R.

Now, consider the sequence $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$. Note that $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$ is decreasing, as for any n, the nonexpansivity of T and R, as well as h_f's fixedness under T and R, tells us that

$$\begin{aligned} \|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)} &\geq \|Th_f - T(RT)^n f\|_{L^p(\mathbb{R}^d)} = \|h_f - T(RT)^n f\|_{L^p(\mathbb{R}^d)} \\ &\geq \|Rh_f - RT(RT)^n f\|_{L^p(\mathbb{R}^d)} = \|h_f - (RT)^{n+1} f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Since $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$ is decreasing and bounded below by 0 (norms are always nonnegative, after all), the monotone convergence theorem tells us that $\lim_{n\to\infty}\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$ exists. Furthermore, because $\{(RT)^{n_i} f\}_{i=1}^{\infty}$ converges to g under the $L^p(\mathbb{R}^d)$ norm, $\{h_f - (RT)^{n_i} f\}_{i=1}^{\infty}$ converges to $h_f - g$ in $L^p(\mathbb{R}^d)$, which in turn (as the nonnegative

$$\|(h_f - (RT)^{n_i} f) - (h_f - g)\|_{L^p(\mathbb{R}^d)} \geq \left| \|h_f - (RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} - \|h_f - g\|_{L^p(\mathbb{R}^d)} \right|$$

goes to 0) means that the subsequence $\{\|h_f - (RT)^{n_i} f\|_{L^p(\mathbb{R}^d)}\}_{i=1}^{\infty}$ must converge to $\|h_f - g\|_{L^p(\mathbb{R}^d)}$. Since the subsequence $\{\|h_f - (RT)^{n_i} f\|_{L^p(\mathbb{R}^d)}\}_{i=1}^{\infty}$ converges to $\|h_f - g\|_{L^p(\mathbb{R}^d)}$, and $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$ converges to something, it must be the case that

$$\lim_{n\to\infty} \|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)} = \|h_f - g\|_{L^p(\mathbb{R}^d)}.$$

Now, note that, by the nonexpansivity of R and T, $RTg = \lim_{i\to\infty}(RT)^{n_i+1} f$ in $L^p(\mathbb{R}^d)$. After all, if $\|g - (RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} < \epsilon$ for an arbitrarily large i, then

$$\begin{aligned}\epsilon > \|g - (RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} &\geq \|Tg - T(RT)^{n_i} f\|_{L^p(\mathbb{R}^d)} \\ &\geq \|RTg - (RT)^{n_i+1} f\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

as well. By similar logic to above, the fact that $RTg = \lim_{i\to\infty}(RT)^{n_i+1} f$ in $L^p(\mathbb{R}^d)$ means that $\lim_{i\to\infty} \|h_f - (RT)^{n_i+1} f\|_{L^p(\mathbb{R}^d)} = \|h_f - RTg\|_{L^p(\mathbb{R}^d)}$, which (as $\{\|h_f - (RT)^{n_i+1} f\|_{L^p(\mathbb{R}^d)}\}_{i=1}^{\infty}$ is a subsequence of the convergent $\{\|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)}\}_{n=1}^{\infty}$) means that

$$\lim_{n\to\infty} \|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)} = \|h_f - RTg\|_{L^p(\mathbb{R}^d)},$$

and thus that

$$\|h_f - g\|_{L^p(\mathbb{R}^d)} = \|h_f - RTg\|_{L^p(\mathbb{R}^d)}.$$

Via repeated application of the fact that $Rh_f = Th_f = h_f$, as well as the nonexpansivity of R and T, we take this statement and write

$$\begin{aligned}&\|h_f - g\|_{L^p(\mathbb{R}^d)} = \|h_f - RTg\|_{L^p(\mathbb{R}^d)} = \|Rh_f - RTg\|_{L^p(\mathbb{R}^d)} \\ &\leq \|h_f - Tg\|_{L^p(\mathbb{R}^d)} = \|Th_f - Tg\|_{L^p(\mathbb{R}^d)} \leq \|h_f - g\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

and conclude from this sort of circular inequality chain that

$$\|Rh_f - RTg\|_{L^p(\mathbb{R}^d)} = \|h_f - Tg\|_{L^p(\mathbb{R}^d)}.$$

Since h (and thus h_f) is strictly decreasing, statement (6) of Theorem 4.5 allows us to conclude that $RTg = Tg$; and as we showed earlier, g is symmetric decreasing, meaning that (by part (3) of Theorem 4.5) $Rg = g$. But per our assumptions, constant multiples of h are the only functions for which those two statements can both be true; thus, $g = kh$ for some $k \in \mathbb{R}^{\geq 0}$. And as $\|kh\|_{L^p(\mathbb{R}^d)} = k\|h\|_{L^p(\mathbb{R}^d)} = k(1) = k$, while $\|g\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$, we can conclude that $g = \|f\|_{L^p(\mathbb{R}^d)}h = h_f$. With this fact, we may write that

$$\lim_{n\to\infty} \|h_f - (RT)^n f\|_{L^p(\mathbb{R}^d)} = \|h_f - g\|_{L^p(\mathbb{R}^d)} \\ = \|h_f - h_f\|_{L^p(\mathbb{R}^d)} = \|0\|_{L^p(\mathbb{R}^d)} = 0,$$

which in turn means that

$$\lim_{n\to\infty} (RT)^n f = h_f,$$

as desired.

All that remains to show is that this convergence holds for all nonnegative $f \in L^p(\mathbb{R}^d)$, and not just the fs that are bounded and have bounded support. Let $\epsilon > 0$. Note that the set of functions in $L^p(\mathbb{R}^d)$ that are bounded and have bounded support is dense; so, for any $f \in L^p(\mathbb{R}^d)$, we may select a function $\bar{f} \in L^p(\mathbb{R}^d)$ that is bounded and has bounded support such that $\|f - \bar{f}\|_{L^p(\mathbb{R}^d)} < \epsilon/3$. (Since R and T are nonexpansive, $\|(RT)^n f - (RT)^n \bar{f}\|_{L^p(\mathbb{R}^d)} < \epsilon/3$ for all n as well.) Furthermore, because $\lim_{n\to\infty}(RT)^n \bar{f} = h_{\bar{f}}$, we know there must exist some $N \in \mathbb{N}$ such that for all $k \geq N$, $\|h_{\bar{f}} - (RT)^k \bar{f}\|_{L^p(\mathbb{R}^d)} < \epsilon/3$. Finally, note that

$$\|h_f - h_{\bar{f}}\|_{L^p(\mathbb{R}^d)} = \left\| \|f\|_{L^p(\mathbb{R}^d)} h - \|\bar{f}\|_{L^p(\mathbb{R}^d)} h \right\|_{L^p(\mathbb{R}^d)} \\ = \left| \|f\|_{L^p(\mathbb{R}^d)} - \|\bar{f}\|_{L^p(\mathbb{R}^d)} \right| \|h\|_{L^p(\mathbb{R}^d)} = \left| \|f\|_{L^p(\mathbb{R}^d)} - \|\bar{f}\|_{L^p(\mathbb{R}^d)} \right| \\ \leq \|f - \bar{f}\|_{L^p(\mathbb{R}^d)} < \epsilon/3.$$

Thus, for all $k \geq N$,

$$\|h_f - (RT)^k f\|_{L^p(\mathbb{R}^d)} \leq \|h_f - h_{\bar{f}}\|_{L^p(\mathbb{R}^d)} + \|(RT)^k f - (RT)^k \bar{f}\|_{L^p(\mathbb{R}^d)} \\ + \|h_{\bar{f}} - (RT)^k \bar{f}\|_{L^p(\mathbb{R}^d)} < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus, $\lim_{n\to\infty}(RT)^n f = h_f$, for all $f \in L^p(\mathbb{R}^d)$, as desired. □

With the first part – the central part – of the competing symmetries algorithm demonstrated, we may now show how this convergence of $\{(RT)^n f\}_{n=1}^{\infty}$ can be used to find extremizers of Φ.

Theorem 4.7. *Let $d > 1$ be a given integer, let $q \in (1, d+1]$ be given, and let $p = \frac{dq}{d+q-1}$. If there exists a functional operator (a symmetry) $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ such that:*

(1) For all $f \in L^p(\mathbb{R}^d)$, $\|Tf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$. In other words, T is norm-preserving.

(2) For all $f, g \in L^p(\mathbb{R}^d)$, if $f \leq g$, then $Tf \leq Tg$. In other words, T is order-preserving.

(3) For all $f, g \in L^p(\mathbb{R}^d)$, $\|Tf - Tg\|_{L^p(\mathbb{R}^d)} \leq \|f - g\|_{L^p(\mathbb{R}^d)}$. In other words, T is nonexpansive.

(4) For all $f \in L^p(\mathbb{R}^d)$ and all $C \in \mathbb{R}^{\geq 0}$, $T(Cf) = CTf$.

(5) There exists a strictly positive function $h \in L^p(\mathbb{R}^d)$ such that:

- $\|h\|_{L^p(\mathbb{R}^d)} = 1$.
- *h is strictly decreasing (radially).*
- $Th = h$.
- $\{f \in L^p(\mathbb{R}^d) : Rf = f, RTf = Tf\} = \{Ch : C \in \mathbb{R}^{\geq 0}\}$.

(6) For all $f \in L^p(\mathbb{R}^d)$, $\Phi(Tf) \geq \Phi(f)$.

Then h and all positive constant multiples of h are extremizers of Φ.

Proof. Let f be a nonzero, nonnegative function in $L^p(\mathbb{R}^d)$. For any natural m, define the function $f_m = \min\{f(x), mh(x)\}$; note that f_m is still nonnegative (as f and mh are), and that as it is bounded by functions in $L^p(\mathbb{R}^d)$, it is also in $L^p(\mathbb{R}^d)$. Thus, by Theorem 4.6, the sequence $\{(RT)^n f_m\}_{n=1}^{\infty}$ must converge to $h_{f_m} = \|f_m\|_{L^p(\mathbb{R}^d)} h$ (the constant multiple of h whose norm is equal to $\|f_m\|_{L^p(\mathbb{R}^d)}$) in $L^p(\mathbb{R}^d)$. Since $\{(RT)^n f_m\}_{n=1}^{\infty}$ converges to h_{f_m} in $L^p(\mathbb{R}^d)$, by Lemma 4.2 there must exist a subsequence $\{(RT)^{n_i} f_m\}_{i=1}^{\infty}$ that converges to h_{f_m} pointwise.

Note that $(RT)^0 f_m = f_m \leq mh$. Furthermore note that if $(RT)^k f_m \leq mh$ for some $k \geq 0$, then $T(RT)^k f_m \leq T(mh) = mh$, which in turn means that $(RT)^{k+1} f_m = R(T(RT)^k f_m) \leq Rmh = mh$ (as as R and T are order-preserving and fix constant multiples of h), so, by induction, $(RT)^n f_m \leq mh$

for all $n \geq 1$, and thus the sequence $\{(RT)^{n_i} f_m\}_{i=1}^{\infty}$ is bounded above by the function $mh \in L^p(\mathbb{R}^d)$.

Since $\mathcal{X}$ is a bounded linear operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathcal{M})$, we know that $\mathcal{X}mh \in L^q(\mathcal{M})$, and that $\mathcal{X}$ is continuous. Therefore, with $mh \in L^p(\mathbb{R}^d)$ as the dominating function for the sequence $\{(RT)^{n_i} f_m\}_{i=1}^{\infty}$ that converges to h_{f_m} pointwise, we may use dominated convergence to conclude that $\{\|\mathcal{X}(RT)^{n_i} f_m\|_{L^q(\mathcal{M})}\}_{i=1}^{\infty}$ converges to $\|\mathcal{X}h_{f_m}\|_{L^q(\mathcal{M})}$.

Dominated convergence also lets us show that $\{\|(RT)^{n_i} f_m\|_{L^p(\mathbb{R}^d)}\}_{i=1}^{\infty}$ converges to $\|h_{f_m}\|_{L^p(\mathbb{R}^d)} = \|f_m\|_{L^p(\mathbb{R}^d)}$. Thus, the quotient of these two sequences $\{\Phi((RT)^{n_i} f_m)\}_{i=1}^{\infty}$ must converge to $\Phi(h_{f_m})$ – and, because Φ is invariant under multiplication by a positive constant (as discussed in the paragraph following the proof of Lemma4.4), we can say that it converges to $\Phi(Ch)$ for any $C > 0$.

Now, consider the sequence $\{\Phi((RT)^n f_m)\}_{n=1}^{\infty}$. Note that because $\Phi(Rg) \geq \Phi(g)$ and $\Phi(Tg) \geq \Phi(g)$ for all f, this sequence is nondecreasing – $\Phi((RT)^n f_m) \leq \Phi(T(RT)^n f_m) \leq \Phi((RT)^{n+1} f_m)$ for all $n \geq 1$ (and, indeed, for $n = 0$ as well, meaning that every element of $\{\Phi((RT)^n f_m)\}_{n=1}^{\infty}$ is greater than or equal to $\Phi(f_m)$). Since this sequence of real numbers has a subsequence $\{\Phi((RT)^{n_i} f_m)\}_{i=1}^{\infty}$ that converges to $\Phi(Ch)$, and is nondecreasing, we can conclude that $\{\Phi((RT)^n f_m)\}_{n=1}^{\infty}$ itself converges to $\Phi(Ch)$ from below. And as $\Phi(f_m)$ is less than or equal to every element in $\{\Phi((RT)^n f_m)\}_{n=1}^{\infty}$, it must be less than or equal to that sequence's limit. Thus, for any $m \in \mathbb{N}$,

$$\Phi(f_m) \leq \Phi(Ch).$$

Now, consider the sequence $\{f_m\}_{m=1}^{\infty} = \{\min\{f(x), mh(x)\}\}_{m=1}^{\infty}$. It converges pointwise to f (as for any $x \in \mathbb{R}^d$, $f_m(x) = \min\{f(x), mh(x)\} = f(x)$ for all $m \geq \frac{f(x)}{h(x)}$), and, due to h's strict positivity, is furthermore monotone nondecreasing pointwise. By the monotone convergence theorem, then (taking functions to the pth power, taking functions to the qth power, and taking the X-ray transform of functions preserves their monotonicity, after all), it must be the case that $\{\Phi(f_m)\}_{m=1}^{\infty}$ converges to $\Phi(f)$; and as $\Phi(Ch)$ is greater than or equal to every element in $\{\Phi(f_m)\}_{m=1}^{\infty}$, it must be greater than or equal to that sequence's limit. Thus,

$$\Phi(f) \leq \Phi(Ch).$$

All that remains to extend this proof from all nonnegative functions $f \in L^p(\mathbb{R}^d)$ where $\|f\|_{L^p(\mathbb{R}^d} \neq 0$ to all functions $f \in L^p(\mathbb{R}^d)$ where $\|f\|_{L^p(\mathbb{R}^d} \neq 0$. By Lemma 3.3, for any $f \in L^p(\mathbb{R}^d)$ (with nonzero norm), $\Phi(f) \leq \Phi(|f|)$.

And as $|f|$ is nonnegative, we may use what we concluded above to state that $\Phi(|f|) \leq \Phi(Ch)$. Thus, for all $f \in L^p(\mathbb{R}^d)$ with nonzero $L^p(\mathbb{R}^d)$ norm,

$$\Phi(f) \leq \Phi(Ch).$$

Thus, as h and all positive constant multiples of h are elements of $L^p(\mathbb{R}^d)$ with nonzero norm whose value under Φ is greater than or equal to that of every single member of $L^p(\mathbb{R}^d)$ (with nonzero norm), we can conclude that h and its positive constant multiples are extremizers of Φ, as desired. □

(The above proofs were mostly derived from [2] and from Chapter 4 of [6].)

4.4 Applying the Competing Symmetries Argument to Prove the Baernstein-Loss Conjecture

Now that we have defined and demonstrated the competing symmetries proof in general, we will demonstrate how it has been concretely applied to prove the Baernstein-Loss Conjecture in two separate cases.

Because the competing symmetries argument requires a particular transformation $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ (for a given q and d) in order to function, the two successful competing symmetry-based proofs of the Baernstein-Loss conjecture – the one for $q = 2$ and the one for $q = d + 1$ – are each associated with a different such transformation (a sphere-based one we call D in the $q = 2$ case and a hemisphere-based one we call J in the $q = d + 1$ case).

The transformations needed for these proofs are, as discussed in Theorem 4.7, required to satisfy several properties (norm-preservation, order-preservation, and so on); however, because many of these properties are very broad and hard to study on their own terms, rather than try and search for useful transformations T from the hard-to-describe space of all maps $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ satisfying the required properties in their weakest possible form, analysts usually restrict their attention to certain well-understood subclasses of transformations $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ that can easily be shown to satisfy the required properties, or even stronger versions of them (even if it is technically an open question whether we could possibly find transformations satisfying the required properties outside of these classes). For instance, one of the requirements for a transformation T to apply to the competing symmetries argument is that $\Phi(Tf) \geq \Phi(f)$ for all $f \in L^p(\mathbb{R}^d)$

with nonzero norm. But because finding and classifying such transformations where $\Phi(Tf) > \Phi(f)$ for some f is difficult, we will (as per past research) restrict our attention to symmetries of Φ.

Definition 9. *A transformation $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is called a symmetry of $\mathbb{R}^d$ if, for all $f \in L^p(\mathbb{R}^d)$ where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$,*

$$\Phi(Tf) = \Phi(f).$$

In a similar vein, there are potentially many types of transformations that are norm-preserving, order-preserving, and nonexpansive. However, again, we will restrict our attention to transformations of the following form (since this subclass of transformations is easier to analyze, still quite broad, and guaranteed to be norm-preserving, order-preserving, homogenous of degree 1, and not just nonexpansive, but fully isometric).

Theorem 4.8. *Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_d(x))$ be a bijective map from $\mathbb{R}^d$ to itself that is continuous and differentiable almost everywhere. The functional transformation $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ given by*

$$Tf(x) = \gamma^* f(x) = |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x))$$

(where $\mathcal{J}$ represents the Jacobian of a function, or rather, the determinant of the Jacobian) satisfies the following properties:

(1) For all $f \in L^p(\mathbb{R}^d)$, $\|Tf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$. In other words, T is norm-preserving.

(2) For all $f, g \in L^p(\mathbb{R}^d)$, if $f \leq g$, then $Tf \leq Tg$. In other words, T is order-preserving.

(3) For all $f, g \in L^p(\mathbb{R}^d)$, $\|Tf - Tg\|_{L^p(\mathbb{R}^d)} = \|f - g\|_{L^p(\mathbb{R}^d)}$. In other words, T is an isometry on $L^p(\mathbb{R}^d)$ (and is also nonexpansive by implication).

(4) For all $f \in L^p(\mathbb{R}^d)$ and $C \in \mathbb{R}^{\geq 0}$, $T(Cf) = CTf$.

Proof. First, let us show that statement (1) holds, which is to say that T preserves functions' L^p norms. For any $f \in L^p(\mathbb{R}^d)$, note that:

$$\|Tf\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |Tf(x)|^p dx\right)^{1/p} = \left(\int_{\mathbb{R}^d} ||\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x))|^p dx\right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^d} |f(\gamma^{-1}(x))|^p \, ||\mathcal{J}_{\gamma^{-1}}(x)|^{1/p}|^p dx\right)^{1/p} = \left(\int_{\mathbb{R}^d} |f(\gamma^{-1}(x))|^p \, |\mathcal{J}_{\gamma^{-1}}(x)| dx\right)^{1/p}$$

We now change variables so that $u = \gamma^{-1}(x)$, meaning that $du = |\mathcal{J}_{\gamma^{-1}}(x)|dx$, allowing us to write that:

$$\left(\int_{\mathbb{R}^d} |f(\gamma^{-1}(x))|^p \, |\mathcal{J}_{\gamma^{-1}}(x)|dx\right)^{1/p} = \left(\int_{\mathbb{R}^d} |f(u)|^p du\right)^{1/p} = \|f\|_{L^p(\mathbb{R}^d)}.$$

Thus, we see that $\|Tf\|_{L^p(X)} = \|f\|_{L^p(X)}$, as desired.

Next, let us show that statement (2) holds, which is to say that T preserves the order of functions in $L^p(\mathbb{R}^d)$. Let $f, g \in L^p(\mathbb{R}^d)$, such that $f \leq g$ (i.e., $f(x) \leq g(x)$ for all $x \in \mathbb{R}^d$). Then, for all $x \in \mathbb{R}^d$, we may write that:

$$\begin{aligned}
&f \leq g \\
&f(\gamma^{-1}(x)) \leq g(\gamma^{-1}(x)) && (\text{as } \gamma^{-1}(x) \in \mathbb{R}^d) \\
&|\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x)) \leq |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} g(\gamma^{-1}(x)) && (\text{as } |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} \geq 0) \\
&Tf(x) \leq Tg(x)
\end{aligned}$$

Thus, as $Tf(x) \leq Tg(x)$ for all $x \in \mathbb{R}^d$, $Tf \leq Tg$, as desired.

Next, let us show that statement (3) holds, which is to say that T preserves the distance between functions in $L^p(\mathbb{R}^d)$. Let $f, g \in L^p(\mathbb{R}^d)$. Note that for all $x \in \mathbb{R}^d$,

$$\begin{aligned}
Tf(x) - Tg(x) &= |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x)) - |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} g(\gamma^{-1}(x)) \\
&= |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} (f(\gamma^{-1}(x)) - g(\gamma^{-1}(x))) = |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} (f - g)(x) \\
&= T(f - g)(x).
\end{aligned}$$

So, $Tf - Tg = T(f - g)$. Since, as per statement (1), T preserves functions' $L^p(\mathbb{R}^d)$ norms, we then have that

$$\|Tf - Tg\|_{L^p(\mathbb{R}^d)} = \|T(f - g)\|_{L^p(\mathbb{R}^d)} = \|f - g\|_{L^p(\mathbb{R}^d)}$$

for all $f, g \in L^p(\mathbb{R}^d)$, as desired.

Finally, let us show that statement (4) holds. Let $f \in L^p(\mathbb{R}^d)$ and $C \in \mathbb{R}^{\geq 0}$. Then, for all $x \in \mathbb{R}^d$:

$$Tf(x) = |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} \left(Cf(\gamma^{-1}(x))\right) = C \left(|\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x)) = CTf(x).\right)$$

□

Thus, as $T(Cf(x)) = CTf(x)$ for all $x \in \mathbb{R}^d$, $T(Cf) = CTf$, as desired.

As we shall discuss below, the two successful transformations that have been discovered for this argument are constructed using this "γ" method.

4.4.1 Using the Transformation D to Prove the Baernstein-Loss Conjecture when $q = 2$

As has been mentioned several times previously, proving the Baernstein-Loss conjecture when $q = 2$ involves using a transformation $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ called D. To be more specific, D is a transformation that is defined such that for any $f \in L^p(\mathbb{R}^d)$,

$$Df(x) = \left(\frac{2}{|x+a|^2}\right)^{\frac{d}{p}} f\left(\frac{2x_1}{|x+a|^2}, ..., \frac{2x_{d-1}}{|x+a|^2}, \frac{|x|^2-1}{|x+a|^2}\right)$$

(where $a = (0, \ldots, 0, 1) \in \mathbb{R}^d$). As per Theorem 4.8, D is based on the mapping

$$\gamma(x) = \left(\frac{2x_1}{|x-a|^2}, ..., \frac{2x_{d-1}}{|x-a|^2}, \frac{1-|x|^2}{|x-a|^2}\right),$$

which is a continuous, differentiable, invertible map from $\mathbb{R}^d \setminus \{a\}$ to itself (as the singleton set $\{a\}$ has measure 0 in $\mathbb{R}^d$, we can ignore it).

The map γ is, in turn, derived from the following sphere-based process:

- Given a point $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ (with $x_1 \neq 0$), map it onto the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ via a stereographic projection:

$$(x_1, \ldots, x_d) \mapsto \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_d}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right).$$

- Rotate this hemisphere 90 degrees "down" (so that the "north pole" at $(0, \ldots, 0, 1)$ rotates to the position of the standard basis vector $(0, \ldots, 0, 1, 0) \in \mathbb{S}^d$, while the point $(0, \ldots, 0, 1, 0)$ is itself mapped to the "south pole" $(0, \ldots, 0, -1) \in \mathbb{S}^d$, with all other standard basis vectors being fixed). This rotation corresponds to the map:

$$(s_1, s_2, \ldots, s_d, s_{d+1}) \mapsto (s_1, s_2, \ldots, s_{d+1}, -s_d),$$

  and applying it to our point x after the aforementioned modified stereographic projection gives us:

$$(x_1, \ldots, x_d) \mapsto \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_{d-1}}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}, \frac{-2x_d}{1+|x|^2}\right).$$

- Finally, undo the stereographic projection. The inverse of our stereographic projection is

$$(s_1, s_2, \ldots, s_d, s_{d+1}) \mapsto \left(\frac{s_1}{1+s_{d+1}}, \frac{s_2}{1+s_{d+1}}, \ldots, \frac{s_d}{1+s_{d+1}}\right).$$

With this in mind, our final step in transforming points $x \in \mathbb{R}^d$ – points which, after these first two steps, have so far been mapped to points in the sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ – will be to use this inverse stereographic projection to move them from that sphere back to $\mathbb{R}^d$. Doing so, we obtain

$$(x_1, \ldots, x_d) \mapsto \left(\frac{2x_1}{1+|x|^2 - 2x_d}, \ldots, \frac{2x_{d-1}}{1+|x|^2 - 2x_d}, \frac{1-|x|^2}{1+|x|^2-2x_d} \right)$$
$$= \left(\frac{2x_1}{|x-a|^2}, \ldots, \frac{2x_{d-1}}{|x-a|^2}, \frac{1-|x|^2}{|x-a|^2} \right),$$

which is indeed our γ.

As we would hope, D satisfies all of the requirements posed by Theorem 4.7. By Theorem 4.8, it satisfies requirements (1), (2), (3), and (4); as we shall see in section 5.2, it satisfies requirement (6) by being a symmetry of Φ when $q = 2$; and there is indeed a function $h \in L^p(\mathbb{R}^d)$ satisfying the conditions of requirement (5), namely,

$$h(x) = 2^{\frac{d-1}{p}} \left(\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \right)^{-\frac{1}{p}} (1+|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}.$$

While a proof of the last part of requirement (5) (the uniqueness of h and its constant multiples among radially symmetric decreasing functions that remain radially symmetric decreasing under D) is nontrivial (and can be seen in [6]), the fact that h is a radially symmetric, strictly decreasing function of unit norm that is fixed under D can be immediately verified.

In any case, note that as this h is a constant multiple of $(1+|x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}$, it is one of the conjectured extremizers f_0. By Theorem 4.7, h is an extremizer of Φ when $q = 2$, which, by Lemma 4.4, means that every f_0 is an extremizer of Φ when $q = 2$, and the Baernstein-Loss conjecture is true when $q = 2$, as desired.

4.4.2 Using the Transformation J to Prove the Baernstein-Loss Conjecture when $q = d + 1$

In a similar vein, proving the Baernstein-Loss conjecture when $q = d + 1$ involves using a transformation $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ called J. Specifically, J is a transformation that is defined such that for any $f \in L^p(\mathbb{R}^d)$,

$$Jf(x) = x_1^{-2} f(x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_d).$$

It is based on the mapping

$$\gamma(x) = (x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_d),$$

which is a continuous, differentiable, invertible map from $\mathbb{R}^d \setminus \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = 0\}$ to itself (as the $d-1$-dimensional hyperplane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = 0\}$ has measure 0 in $\mathbb{R}^d$, we can ignore it).

The map γ is, in turn, derived from the following hemisphere-based process:

- Given a point $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ (with $x_1 \neq 0$), map it onto the top half of the sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ (i.e., the part of the sphere that lies in the region $\{(s_1, \ldots, s_{d+1}) \in \mathbb{R}^{d+1} : s_{d+1} > 0\}$) via a modified stereographic projection:

$$(x_1, \ldots, x_d) \mapsto \left(\frac{x_1}{\sqrt{1+|x|^2}}, \ldots, \frac{x_d}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}} \right).$$

- Rotate this hemisphere 90 degrees "down" (so that the "north pole" at $(0, \ldots, 0, 1)$ rotates to the position of the standard basis vector $(1, 0, \ldots, 0) \in \mathbb{S}^d$, while the point $(1, 0, \ldots, 0)$ is itself mapped to the "south pole" $(0, \ldots, 0, -1) \in \mathbb{S}^d$, with all other standard basis vectors being fixed), and then reflect it around the hyperplane $\{(s_1, \ldots, s_{d+1}) : s_{d+1} = 0\}$ (so that the "north pole" $(0, \ldots, 0, 1)$ and "south pole" $(0, \ldots, 0, -1)$ swap positions, but no other axes are affected). This rotation and reflection corresponds to the map:

$$(s_1, s_2, \ldots, s_d, s_{d+1}) \mapsto (s_{d+1}, s_2, \ldots, s_d, s_1),$$

  and applying it to our point x after the aforementioned modified stereographic projection gives us:

$$(x_1, \ldots, x_d) \mapsto \left(\frac{1}{\sqrt{1+|x|^2}}, \ldots, \frac{x_d}{\sqrt{1+|x|^2}}, \frac{x_1}{\sqrt{1+|x|^2}} \right).$$

- Finally, undo the modified stereographic projection. The inverse of that modified stereographic projection is

$$(s_1, s_2, \ldots, s_d, s_{d+1}) \mapsto \left(\frac{s_1}{s_{d+1}}, \frac{s_2}{s_{d+1}}, \ldots, \frac{s_d}{s_{d+1}} \right).$$

Note that even though the modified stereographic projection we defined specifically mapped $\mathbb{R}^d$ to the hemisphere of $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ that is contained in the region $\{(s_1, \ldots, s_{d+1}) \in \mathbb{R}^{d+1} : s_{d+1} > 0\}$, this inverse projection can be used to bijectively map *any* hemisphere of $\mathbb{S}^d$ to $\mathbb{R}^d$.

With this in mind, our final step in transforming points $x \in \mathbb{R}^d$ – points which, after these first two steps, have so far been mapped to points in the hemisphere of $\mathbb{S}^d$ that is contained in the region $\{(s_1, \ldots, s_{d+1}) \in \mathbb{R}^{d+1} : s_1 > 0\}$ – will be to use this inverse modified stereographic projection to move them from that hemisphere back to $\mathbb{R}^d$. Doing so, we obtain

$$(x_1, \ldots, x_d) \mapsto (x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_d),$$

which is indeed our γ.

Once more, as we would hope, J satisfies all of the requirements posed by Theorem 4.7. By Theorem 4.8, it satisfies requirements (1), (2), (3), and (4); as we shall see in section 5.3, it satisfies requirement (6) by being a symmetry of Φ when $q = d + 1$; and there is indeed a function $h \in L^p(\mathbb{R}^d)$ satisfying the conditions of requirement (5). And that function is once again

$$h(x) = 2^{\frac{d-1}{p}} \left(\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \right)^{-\frac{1}{p}} (1 + |x|^2)^{-\frac{1}{2}\frac{d-1}{p-1}}.$$

A proof of the uniqueness of h and its constant multiples among radially symmetric decreasing functions that remain radially symmetric decreasing under J remains nontrivial, but it remains clear that h is a radially symmetric, strictly decreasing function of unit norm that is fixed under J.

Chapter 5

A Sufficient Condition for a Transformation to be a Symmetry

As discussed previously, when coming up with transformations T for the competing symmetries argument, it is easiest to focus on symmetries of Φ derived via maps $\gamma : \mathbb{R}^d \to \mathbb{R}^d$. Although such transformations may not be the only ones that would work for this argument, they are the easiest to create and analyze. This fact is born out by the following theorem, original to this paper, which seeks to provide a condition that must be satisfied in order for a transformation derived from a $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ to be a symmetry of Φ, and, in doing so, further narrow the scope of analysts' search even further (perhaps even, as Chapter 7 hints at, narrowing that scope down to nothing).

Theorem 5.1. *Let d be an integer such that $d > 1$, let q be any real number such that $q \geq 2$, and let $p = \frac{dq}{d+q-1}$; furthermore, let $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ be a functional operator. If T satisfies the following conditions:*

(1) There exists a bijective map $\gamma(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_d(x))$ from $\mathbb{R}^d$ to itself (that is continuous and bijective almost everywhere) such that

$$Tf(x) = \gamma^* f(x) = |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x)). \tag{5.1}$$

(2) The equation

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}$$
$$= |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2} \tag{5.2}$$

holds for almost every $u, v \in \mathbb{R}^d$, almost every $s \in \mathbb{R}$, and all $i \in \{1, \ldots, d\}$.

Then T is a symmetry of $\Phi(f) = \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}}$, which is to say that $\|Tf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$, $\|\mathcal{X}Tf\|_{L^q(\mathcal{M})} = \|\mathcal{X}f\|_{L^q(\mathcal{M})}$, and $\Phi(Tf) = \Phi(f)$.

Proof. From Theorem 4.8, we already know that $\|Tf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$. So, it suffices to show that $\|\mathcal{X}Tf\|_{L^q(\mathcal{M})} = \|\mathcal{X}f\|_{L^q(\mathcal{M})}$ in order to show that $\Phi(f) = \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}}$ and $\Phi(Tf) = \frac{\|\mathcal{X}Tf\|_{L^q(\mathcal{M})}}{\|Tf\|_{L^p(\mathbb{R}^d)}}$ are equal.

First, consider the case where $q = 2$. In this case, our condition (5.2) on $T = \gamma^*$ can be written as:

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{1-d}$$
$$= |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+1}{2d}} \left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{0}$$
$$= |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+1}{2d}}. \tag{5.3}$$

Now, using Lemma 4.3 (Blaschke's identity), we may write that:

$$\|\mathcal{X}Tf\|_{L^2(\mathcal{M})} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Tf(x)Tf(y)|x-y|^{1-d}\left(\int_{\mathbb{R}} Tf((1-t)x+ty)dt\right)^0 dxdy$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Tf(x)Tf(y)|x-y|^{1-d}dxdy$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x))|\mathcal{J}_{\gamma^{-1}}(y)|^{1/p} f(\gamma^{-1}(y))|x-y|^{1-d}dxdy.$$

Now, let us change variables so that $u = \gamma^{-1}(x)$ (meaning that $du = |\mathcal{J}_{\gamma^{-1}}(x)|dx \Rightarrow dx = |\mathcal{J}_{\gamma^{-1}}(x)|^{-1}du$) and $v = \gamma^{-1}(y)$ (meaning that $dv = |\mathcal{J}_{\gamma^{-1}}(y)|dy \Rightarrow dy = |\mathcal{J}_{\gamma^{-1}}(y)|^{-1}dv$), allowing us to write that:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x))|\mathcal{J}_{\gamma^{-1}}(y)|^{1/p} f(\gamma^{-1}(y))|x-y|^{1-d}dxdy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}|\mathcal{J}_{\gamma^{-1}}(x)|^{1/p}|\mathcal{J}_{\gamma^{-1}}(y)|^{1/p}|\mathcal{J}_{\gamma^{-1}}(x)|^{-1}|\mathcal{J}_{\gamma^{-1}}(y)|^{-1}dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}|\mathcal{J}_{\gamma^{-1}}(x)|^{-1+\frac{1}{p}}|\mathcal{J}_{\gamma^{-1}}(y)|^{-1+\frac{1}{p}}dudv.
\end{aligned}$$

Using the fact that $p = \frac{dq}{d+q-1} = \frac{2d}{d+1}$, the fact that for any continuous, invertible function γ; $|\mathcal{J}_{\gamma^{-1}}(z)| = |\mathcal{J}_{\gamma}(\gamma^{-1}(z))|^{-1}$; and equation (5.3) above, we find that:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}|\mathcal{J}_{\gamma^{-1}}(x)|^{-1+\frac{1}{p}}|\mathcal{J}_{\gamma^{-1}}(y)|^{-1+\frac{1}{p}}dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}|\mathcal{J}_{\gamma^{-1}}(x)|^{-1+\frac{d+1}{2d}}|\mathcal{J}_{\gamma^{-1}}(y)|^{-1+\frac{d+1}{2d}}dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}|\mathcal{J}_{\gamma}(\gamma^{-1}(x))|^{1-\frac{d+1}{2d}}|\mathcal{J}_{\gamma}(\gamma^{-1}(y))|^{1-\frac{d+1}{2d}}dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}|\mathcal{J}_{\gamma}(u)\mathcal{J}_{\gamma}(v)|^{1-\frac{d+1}{2d}}dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{1-d}\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{1-d}dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|u-v|^{1-d}dudv,
\end{aligned}$$

which, by Blaschke's identity once more, is simply equal to

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|u-v|^{1-d}dudv = \|\mathcal{X}f\|_{L^2(\mathcal{M})}.$$

Thus, we see that $\|\mathcal{X}Tf\|_{L^2(\mathcal{M})} = \|\mathcal{X}f\|_{L^2(\mathcal{M})}$, as desired.

Now, consider the case where $q > 2$. Note that as $q-2 \neq 0$, the quantity

$$\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_{\gamma}((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}$$

cannot depend on the value of s. Otherwise,

$$|\mathcal{J}_{\gamma}(u)\mathcal{J}_{\gamma}(v)|^{1-\frac{d+q-1}{dq}}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_{\gamma}((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2},$$

the whole right-hand side of the condition (5.2), would have to depend on s – meaning that it could not equal $\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}$, an expression which does not depend on s at all.

Now, once more using our corollary of Blaschke's identity, we may write:

$$\begin{aligned}\|\mathcal{X}Tf\|_{L^q(\mathcal{M})} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Tf(x)Tf(y)|x-y|^{q-1-d}\left(\int_{\mathbb{R}} Tf((1-t)x+ty)dt\right)^{q-2} dxdy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x))|\mathcal{J}_{\gamma^{-1}}(y)|^{1/p} f(\gamma^{-1}(y))|x-y|^{q-1-d} \\ &\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)x+ty)|^{1/p} f(\gamma^{-1}((1-t)x+ty))dt\right)^{q-2} dxdy.\end{aligned}$$

Now, let us once more change variables so that $u=\gamma^{-1}(x)$ (meaning that $du=|\mathcal{J}_{\gamma^{-1}}(x)|dx \Rightarrow dx=|\mathcal{J}_{\gamma^{-1}}(x)|^{-1}du$) and $v=\gamma^{-1}(y)$ (meaning that $dv=|\mathcal{J}_{\gamma^{-1}}(y)|dy \Rightarrow dy=|\mathcal{J}_{\gamma^{-1}}(y)|^{-1}dv$). With this change of variables (and once again remembering that $|\mathcal{J}_{\gamma^{-1}}(x)|=|\mathcal{J}_{\gamma}(\gamma^{-1}(x))|^{-1}=|\mathcal{J}_{\gamma}(u)|^{-1}$ and $|\mathcal{J}_{\gamma^{-1}}(y)|=|\mathcal{J}_{\gamma}(\gamma^{-1}(y))|^{-1}=|\mathcal{J}_{\gamma}(v)|^{-1}$), we may write that:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\mathcal{J}_{\gamma^{-1}}(x)|^{1/p} f(\gamma^{-1}(x))|\mathcal{J}_{\gamma^{-1}}(y)|^{1/p} f(\gamma^{-1}(y))|x-y|^{q-1-d} \\ &\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)x+ty)|^{1/p} f(\gamma^{-1}((1-t)x+ty))dt\right)^{q-2} dxdy \\ &= \int_{\gamma^{-1}(\mathbb{R}^d)}\int_{\gamma^{-1}(\mathbb{R}^d)} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_{\gamma^{-1}}(x)|^{1/p}|\mathcal{J}_{\gamma^{-1}}(y)|^{1/p} \\ &\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)\gamma(u)+t\gamma(v))|^{1/p} f(\gamma^{-1}((1-t)\gamma(u)+t\gamma(v)))dt\right)^{q-2} \\ &\qquad\qquad\times|\mathcal{J}_{\gamma^{-1}}(x)|^{-1}|\mathcal{J}_{\gamma^{-1}}(y)|^{-1}dudv \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_{\gamma^{-1}}(x)|^{-1+1/p}|\mathcal{J}_{\gamma^{-1}}(y)|^{-1+1/p} \\ &\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)\gamma(u)+t\gamma(v))|^{1/p} f(\gamma^{-1}((1-t)\gamma(u)+t\gamma(v)))dt\right)^{q-2} dudv \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_{\gamma}(u)\mathcal{J}_{\gamma}(v)|^{1-1/p} \\ &\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)\gamma(u)+t\gamma(v))|^{1/p} f(\gamma^{-1}((1-t)\gamma(u)+t\gamma(v)))dt\right)^{q-2} dudv \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_{\gamma}(u)\mathcal{J}_{\gamma}(v)|^{1-\frac{d+q-1}{dq}} \\ &\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)\gamma(u)+t\gamma(v))|^{\frac{d+q-1}{dq}} f(\gamma^{-1}((1-t)\gamma(u)+t\gamma(v)))dt\right)^{q-2} dudv\end{aligned}$$

Now, let us define a variable $s \in \mathbb{R}$ dependent on t such that $\gamma((1-s)u + sv) = (1-t)\gamma(u) + t\gamma(v)$ for all $t \in \mathbb{R}$. In this way, $t = \frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$, and $dt = \left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right| ds$ (where $\cdot$ here is the vector dot product, and i is any number in the range $\{1,\ldots,d\}$). Note that while t is not necessarily well-defined for arbitrary functions γ, condition (5.2) necessarily implies that dt will be the same regardless of the i we pick, which (via the integral $\int_0^{s_0} \nabla\gamma_i((1-s)u+sv)\cdot(u-v)\,ds + \gamma_i(u)$) in turn implies that t will be the same regardless of the i we pick, so they are well-defined in our case.

In any case, changing variables from t to s, and once again using the inverse function theorem, we get:

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}} \\
&\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}((1-t)\gamma(u)+t\gamma(v))|^{\frac{d+q-1}{dq}} f(\gamma^{-1}((1-t)\gamma(u)+t\gamma(v)))dt\right)^{q-2} dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}} \\
&\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma^{-1}}(\gamma^{-1}((1-s)u+sv))|^{\frac{d+q-1}{dq}} f((1-s)u+sv)\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right| ds\right)^{q-2} dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}} \\
&\quad\times\left(\int_{\mathbb{R}} |\mathcal{J}_{\gamma}((1-s)u+sv)|^{-\frac{d+q-1}{dq}} f((1-s)u+sv)\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right| ds\right)^{q-2} dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}} \\
&\quad\times\left(\int_{\mathbb{R}} f((1-s)u+sv)\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_{\gamma}((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right| ds\right)^{q-2} dudv.
\end{aligned}
$$

And since $\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_{\gamma}((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|$ is constant with respect to s, we may remove it from our integral over s, and then use condition (5.2) as follows:

$$
\begin{aligned}
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)f(v)|\gamma(u)-\gamma(v)|^{q-1-d}|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}} \\
&\quad\times\left(\int_{\mathbb{R}} f((1-s)u+sv)\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_{\gamma}((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right| ds\right)^{q-2} dudv
\end{aligned}
$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)f(v)|\gamma(u) - \gamma(v)|^{q-1-d} |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}}$$
$$\times \left(\left| \frac{\nabla \gamma_i((1-s)u + sv) \cdot (v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}} (\gamma_i(v) - \gamma_i(u))} \right| \int_{\mathbb{R}} f((1-s)u+sv)ds \right)^{q-2} dudv$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)f(v)|\gamma(u) - \gamma(v)|^{q-1-d}$$
$$\times |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{-1+\frac{d+q-1}{dq}} \left| \frac{\nabla \gamma_i((1-s)u + sv) \cdot (v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}} (\gamma_i(v) - \gamma_i(u))} \right|^{q-2}$$
$$\times \left(\int_{\mathbb{R}} f((1-s)u+sv)ds \right)^{q-2} dudv$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)f(v)|\gamma(u) - \gamma(v)|^{q-1-d} \left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|} \right)^{q-1-d}$$
$$\times \left(\int_{\mathbb{R}} f((1-s)u+sv)ds \right)^{q-2} dudv$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)f(v)|u-v|^{q-1-d} \left(\int_{\mathbb{R}} f((1-s)u+sv)ds \right)^{q-2} dudv$$

But of course, by Blaschke's identity once more, we may write that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)f(v)|u-v|^{q-1-d} \left(\int_{\mathbb{R}} f((1-s)u+sv)ds \right)^{q-2} dudv = \|\mathcal{X}f\|_{L^q(\mathcal{M})}. \tag{5.4}$$

Thus, we have that $\|\mathcal{X}Tf\|_{L^q(\mathcal{M})} = \|\mathcal{X}f\|_{L^q(\mathcal{M})}$, as desired.

Thus, we see that for any $T = \gamma^*$ satisfying our given conditions, it must be the case that $\|Tf\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$ and $\|\mathcal{X}Tf\|_{L^q(\mathcal{M}_X)} = \|\mathcal{X}f\|_{L^q(\mathcal{M})}$, and thus that $\Phi(Tf) = \Phi(f)$, as desired. □

To understand this condition a bit better, we will now turn our attention to three basic symmetries of Φ – the aforementioned transformations D and J (which we will now prove to be symmetries of Φ), as well as many fundamental affine transformations (which cannot be used in the competing symmetries argument due to a lack of a unique h satisfying requirement (5) of Theorem 4.7). We will show how each one is derived from a continuous, differentiable (almost everywhere) bijection $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ that (as we would hope from a symmetry of Φ) satisfies condition (5.2) from Theorem 5.1.

5.1 The Sufficient Condition and Rotations, Reflections, Translations, and Dilations

Consider the following maps $\gamma : \mathbb{R}^d \to \mathbb{R}^d$:

- Let $\gamma(x) = x - b$ for some constant $b \in \mathbb{R}^d$. Then $\gamma^{-1}(x) = x + b$, and that inverse's Jacobian is $\mathcal{J}_{\gamma^{-1}}(x) = 1$. Thus, our functional transformation is

  $$\gamma^* f(x) = |1|^{1/p} f(x+b) = f(x+b),$$

  a translation of the function f.

- Let $\gamma(x) = \frac{1}{r}x$ for some constant $r \in \mathbb{R}$ (or, equivalently, let $\gamma(x) = \frac{1}{r}Ix$, where $r \in \mathbb{R}$ is a constant and I is the identity matrix of dimension d). Then $\gamma^{-1}(x) = rx$, and that inverse's Jacobian is $\mathcal{J}_{\gamma^{-1}}(x) = r^d$. Thus, our functional transformation is

  $$\gamma^* f(x) = |r^d|^{1/p} f(rx) = |r|^{d/p} f(rx),$$

  a dilation of the function f.

- Let $\gamma(x) = R^{-1}x$ for some matrix $R \in \mathsf{O}(d)$ – which is to say, a matrix representing a rotation or a reflection. Then $\gamma^{-1}(x) = Rx$, and that inverse's Jacobian is $\mathcal{J}_{\gamma^{-1}}(x) = \det(R) = 1$. Thus, our functional transformation is

  $$\gamma^* f(x) = |1|^{1/p} f(Rx) = f(Rx),$$

  a rotation of the function f.

Any composition of these transformations in $\mathbb{R}^d$ can be written as an affine transformation of the form

$$\gamma(x) = kUx + b$$

where $k \in \mathbb{R} \setminus \{0\}$ is a nonzero real number, $U \in \mathsf{O}(d)$ is an orthogonal matrix, and $b \in \mathbb{R}^d$ is a vector. (Note that as all orthogonal matrices are invertible, a transformation of this form always has an inverse over the totality of $\mathbb{R}^d$ in the form of $\gamma^{-1}(x) = \frac{1}{k}U^{-1}(x-b)$; note furthermore that this transformation is continuous over all of $\mathbb{R}^d$.) And indeed, any such transformation also induces a symmetry γ^* of Φ for all possible values of d and q.

Let us now show as such using the condition (5.2) whose soundness we just proved. Let $\gamma(x) = kUx + b$ (with $k \in \mathbb{R} \setminus \{0\}$, $U \in \mathsf{O}(d)$, and $b \in \mathbb{R}^d$) be a continuous, invertible map from $\mathbb{R}^d$ to $\mathbb{R}^d$ (as above). First, consider the right half of equation (5.2).

$$|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left| \frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))} \right|^{q-2}.$$

The Jacobian of our map γ is $\mathcal{J}_\gamma(x) = \det(kU)$ (after all, the Jacobian of any function is the determinant of the matrix that best approximates the function's tangent space – which, here, is just the matrix kU itself). And since U is an orthogonal matrix, its determinant must be either 1 or -1, making the absolute value of its determinant 1; thus, we may further write that $\mathcal{J}_\gamma(x) = \det(kU) = k^d \det(U) = k^d$.

Furthermore, if we let $U_{\bullet i}$ represent the ith row of the matrix U and b_i represent the ith element of the vector b, then the function $\gamma_i : \mathbb{R}^d \to \mathbb{R}$ (representing the ith element of γ's output) is given by $\gamma_i(x) = kU_{\bullet i}x + b_i$, or $\gamma_i(x_1, x_2, \ldots, x_d) = kU_{1i}x_1 + kU_{2i}x_2 + \cdots + kU_{di}x_d + b_i$, making its gradient $\nabla\gamma_i(x) = (kU_{1i}, kU_{2i}, \ldots, kU_{di}) = kU_{\bullet i}$.

With this in mind, we may write for all $i \in \{1, \ldots, d\}$, all $s \in \mathbb{R}$, and all $u, v \in \mathbb{R}^d$:

$$\begin{aligned}
&|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left| \frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))} \right|^{q-2} \\
&= |(|k^d|)(|k^d|)|^{1-\frac{d+q-1}{dq}} \left| \frac{kU_{\bullet i}\cdot(v-u)}{|k^d|^{\frac{d+q-1}{dq}}((kU_{\bullet i}v+b_i)-(kU_{\bullet i}u+b_i))} \right|^{q-2} \\
&= |k^{2d}|^{1-\frac{d+q-1}{dq}} \left| \frac{kU_{\bullet i}(v-u)}{|k^d|^{\frac{d+q-1}{dq}} kU_{\bullet i}(v-u)} \right|^{q-2} = |k|^{2d-2\frac{d+q-1}{q}} \left| \frac{1}{|k|^{\frac{d+q-1}{q}}} \right|^{q-2} \\
&= |k|^{2d-2\frac{d+q-1}{q}} |k|^{-\frac{d+q-1}{q}(q-2)} = |k|^{2d-2\frac{d+q-1}{q}-(d+q-1)+2\frac{d+q-1}{q}} \\
&= |k|^{d-q+1}
\end{aligned}$$

Meanwhile, turning to the left side of equation (5.2), we may write for any $u, v \in \mathbb{R}^d$:

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d} = \left(\frac{|\gamma(u)-\gamma(v)|}{|u-v|}\right)^{d-q+1} = \left(\frac{|(kUu+b)-(kUv+b)|}{|u-v|}\right)^{d-q+1}$$

$$= \left(\frac{|(kU(u-v)|}{|u-v|}\right)^{d-q+1} = \left(\frac{|kU(u-v)|}{|u-v|}\right)^{d-q+1} = \left(|k|\frac{|U(u-v)|}{|u-v|}\right)^{d-q+1}.$$

At this point, remembering that U is an orthogonal matrix, and thus an isometry, we note that $|U(u-v)| = |u-v|$, meaning that $\frac{|U(u-v)|}{|u-v|} = 1$ and the left side of the equation simplifies to just $|k|^{d-q+1}$.

Thus, as

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}$$
$$= |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2}$$

holds for transformations γ of the form $kUx+b$ (with both sides of the equation equalling $|k|^{d-q+1}$), we can conclude that the derived functional γ^* is a symmetry of Φ, as desired.

5.2 The Sufficient Condition and the Spherical Inversion D

To show that D is indeed a symmetry of Φ when $q=2$, we will show that

$$\gamma(x) = \left(\frac{2x_1}{|x-a|^2}, ..., \frac{2x_{d-1}}{|x-a|^2}, \frac{1-|x|^2}{|x-a|^2}\right),$$

the transformation from which D is derived, satisfies the condition described above. First, consider the left-hand side of the equation. Since we are only considering the case where $q=2$, we have (for any $u, v \in \mathbb{R}^d$):

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{2-1-d} = |u-v|^{1-d}|\gamma(u)-\gamma(v)|^{d-1}$$
$$=|u-v|^{1-d}\left|\sqrt{\left(\frac{2u_1}{|u-a|^2}-\frac{2v_1}{|v-a|^2}\right)^2+\cdots+\left(\frac{2u_{d-1}}{|u-a|^2}-\frac{2v_{d-1}}{|v-a|^2}\right)^2+\left(\frac{1-|u|^2}{|u-a|^2}-\frac{1-|v|^2}{|v-a|^2}\right)^2}\right|^{d-1}$$
$$=|u-v|^{1-d}\left|\left(\frac{2u_1|v-a|^2-2v_1|u-a|^2}{|u-a|^2|v-a|^2}\right)^2+\cdots+\left(\frac{2u_{d-1}|v-a|^2-2v_{d-1}|u-a|^2}{|u-a|^2|v-a|^2}\right)^2\right.$$

$$
\begin{aligned}
&\qquad + \left(\frac{(1-|u|^2)|v-a|^2 - (1-|v|^2)|u-a|^2}{|u-a|^2|v-a|^2} \right)^2 \Bigg|^{\frac{d-1}{2}} \\
=&|u-v|^{1-d} \Big| |u-a|^{-4}|v-a|^{-4} \big(4u_1^2|v-a|^4 - 8u_1v_1|u-a|^2|v-a|^2 + 4v_1|u-a|^4 \\
&\qquad + \cdots + 4u_{d-1}^2|v-a|^4 - 8u_{d-1}v_{d-1}|u-a|^2|v-a|^2 \\
&\qquad + 4v_{d-1}|u-a|^4 + (|u|^4 - 2|u|^2 + 1)|v-a|^4 \\
&\qquad - 2(|u|^2|v|^2 - |u|^2 - |v|^2 + 1)|u-a|^2|v-a|^2 \\
&\qquad + (|v|^4 - 2|v|^2 + 1)|u-a|^4\big) \Big|^{\frac{d-1}{2}} \\
=&|u-v|^{1-d} \Big| |u-a|^{-4}(4u_1^2 + \cdots + 4u_{d-1}^2 + |u|^4 - 2|u|^2 + 1) \\
&\qquad + |v-a|^{-4}(4v_1^2 + \cdots + 4v_{d-1}^2 + |v|^4 - 2|v|^2 + 1) \\
&\qquad - |u-a|^{-2}|v-a|^{-2}(8u_1v_1 + \cdots + 8u_{d-1}v_{d-1} + 2|u|^2|v|^2 - 2|u|^2 - 2|v|^2 + 2) \Big|^{\frac{d-1}{2}} \\
=&|u-v|^{1-d} \Big| |u-a|^{-4}((4|u|^2 - 4u_d^2) + |u|^4 - 2|u|^2 + 1) \\
&\qquad + |v-a|^{-4}((4|v|^2 - 4v_d^2) + |v|^4 - 2|v|^2 + 1) \\
&\qquad - |u-a|^{-2}|v-a|^{-2}((-4|u-v|^2 + 4|u|^2 + 4|v|^2 - 8u_dv_d) \\
&\qquad + 2|u|^2|v|^2 - 2|u|^2 - 2|v|^2 + 2) \Big|^{\frac{d-1}{2}} \\
=&|u-v|^{1-d} \Big| |u-a|^{-4}(|u|^4 + 2|u|^2 - 4u_d^2 + 1) + |v-a|^{-4}(|v|^4 + 2|v|^2 - 4v_d^2 + 1) \\
&\qquad - |u-a|^{-2}|v-a|^{-2}\big(-4|u-v|^2 \\
&\qquad\quad + (|u|^2|v|^2 + 2u_d|v|^2 - 2v_d|u|^2 + |u|^2 + |v|^2 \\
&\qquad\quad - 4u_dv_d + 2u_d - 2v_d + 1) \\
&\qquad\quad + (|u|^2|v|^2 - 2u_d|v|^2 + 2v_d|u|^2 + |u|^2 + |v|^2 \\
&\qquad\quad - 4u_dv_d - 2u_d + 2v_d + 1)\big) \Big|^{\frac{d-1}{2}} \\
=&|u-v|^{1-d} \Big| |u-a|^{-4}(|u|^2 - 2u_d + 1)(|u|^2 + 2u_d + 1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |v-a|^{-4}(|v|^2 - 2v_d + 1)(|v|^2 + 2v_d + 1) \\
&\quad - |u-a|^{-2}|v-a|^{-2}\big(-4|u-v|^2 + (|u|^2 + 2u_d + 1)(|v|^2 - 2v_d + 1) \\
&\qquad\qquad + (|u|^2 - 2u_d + 1)(|v|^2 + 2v_d + 1)\big)\Big|^{\frac{d-1}{2}} \\
&= |u-v|^{1-d}\Big||u-a|^{-4}|u-a|^2|u+a|^2 + |v-a|^{-4}|v-a|^2|v+a|^2 \\
&\quad - |u-a|^{-2}|v-a|^{-2}\big(-4|u-v|^2 + |u+a|^2|v-a|^2 + |u-a|^2|v+a|^2\big)\Big|^{\frac{d-1}{2}} \\
&= |u-v|^{1-d}\Big||u-a|^{-2}|u+a|^2 + |v-a|^{-2}|v+a|^2 + 4|u-a|^{-2}|v-a|^{-2}|u-v|^2 \\
&\quad - |u-a|^{-2}|u+a|^2 - |v-a|^{-2}|v+a|^2\Big|^{\frac{d-1}{2}} \\
&= |u-v|^{1-d}\left|\frac{4|u-v|^2}{|u-a|^2|v-a|^2}\right|^{\frac{d-1}{2}} = \left(\frac{1}{|u-v|}\right)^{d-1}\left(\frac{2|u-v|}{|u-a||v-a|}\right)^{d-1} \\
&= \left(\frac{2}{|u-a||v-a|}\right)^{d-1}.
\end{aligned}
$$

Next, consider the right-hand side of the equation. Note that the Jacobian for the transformation γ is $\mathcal{J}_\gamma(x) = \left(\frac{2}{|u-a|^2}\right)^d$. Since we are only considering the case where $q = 2$, we have for any $u, v \in \mathbb{R}^d$:

$$
\begin{aligned}
&|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+2-1}{2d}}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{2-2} \\
&= \left|\left(\frac{2}{|u-a|^2}\right)^d\left(\frac{2}{|v-a|^2}\right)^d\right|^{1-\frac{d+1}{2d}}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{0} \\
&= \left|\left(\frac{2}{|u-a|^2}\right)\left(\frac{2}{|v-a|^2}\right)\right|^{d-\frac{d+1}{2}} = \left|\frac{4}{|u-a|^2|v-a|^2}\right|^{\frac{d}{2}-\frac{1}{2}} = \left(\frac{2}{|u-a||v-a|}\right)^{d-1}.
\end{aligned}
$$

Thus, as $\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d} = |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2}$ holds for the transformation γ when $q = 2$ (with both sides of the equation equalling $\left(\frac{2}{|u-a||v-a|}\right)^{d-1}$), we can conclude that the derived functional D is a symmetry of Φ, as desired.

5.3 The Sufficient Condition and the Spherical Inversion J

To show that J is indeed a symmetry of Φ when $q = d+1$, we will show that $\gamma(x) = (x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_d)$, the transformation from which J is derived, satisfies the condition described above. First, consider the left-hand side of the equation. Since we are only considering the case where $q = d+1$, we have (for any $u, v \in \mathbb{R}^d$:

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d} = \left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{(d+1)-1-d} = \left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{0} = 1.$$

Next, we will consider the right-hand side of the equation. Note that the Jacobian determinant of the map $\gamma(x) = (x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_d)$ is $-x_1^{-(d+1)}$. Meanwhile, $\gamma_i(x)$ is equal to x_1^{-1} when $i = 1$ and $x_1^{-1}x_i$ when $i \in \{2, \ldots, d\}$; this means that $\nabla\gamma_1(x) = (-x_1^{-2}, 0, 0, \ldots, 0)$ and $\nabla\gamma_i(x) = (-x_1^{-2}x_i, 0, \ldots, 0, x_1^{-1}, 0, \ldots, 0)$ for $i \in \{2, \ldots, d\}$ (with the x_1^{-1} occurring in the ith entry of the vector).

With this in mind, and still assuming that $q = d+1$, we may (for any $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, any $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, and any $s \in \mathbb{R}$) write the right half of the equation as follows when $i = 1$:

$$\begin{aligned}
&|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left| \frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))} \right|^{q-2} \\
&= \begin{array}{l} |(-u_1^{-(d+1)})(-v_1^{-(d+1)})|^{1-\frac{d+(d+1)-1}{d(d+1)}} \\ \times \left| \dfrac{(-((1-s)u_1+sv_1)^{-2}, 0, 0, \ldots, 0)\cdot(v_1-u_1, \ldots, v_d-u_d)}{|-((1-s)u_1+sv_1)^{-(d+1)}|^{\frac{d+(d+1)-1}{d(d+1)}}(v_1^{-1}-u_1^{-1})} \right|^{(d+1)-2} \end{array} \\
&= |u_1v_1|^{-(d+1)(1-\frac{2d}{d(d+1)})} \left| \frac{-((1-s)u_1+sv_1)^{-2}(v_1-u_1)}{|(1-s)u_1+sv_1|^{-(d+1)\frac{2d}{d(d+1)}}(v_1^{-1}-u_1^{-1})} \right|^{d-1} \\
&= |u_1v_1|^{-((d+1)-2)} \left| \frac{-((1-s)u_1+sv_1)^{-2}(v_1-u_1)}{((1-s)u_1+sv_1)^{-2}(v_1^{-1}-u_1^{-1})} \right|^{d-1} \\
&= |u_1v_1|^{-(d-1)} \left| \frac{u_1-v_1}{(v_1^{-1}-u_1^{-1})} \right|^{d-1} = \left| \frac{u_1-v_1}{u_1v_1(v_1^{-1}-u_1^{-1})} \right|^{d-1} \\
&= \left| \frac{u_1-v_1}{u_1-v_1} \right|^{d-1} = |1|^{d-1} = 1,
\end{aligned}$$

and as follows when $i \in \{2, \ldots, d\}$:

$$|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left| \frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))} \right|^{q-2}$$

$$= \begin{matrix} |(-u_1^{-(d+1)})(-v_1^{-(d+1)})|^{1-\frac{d+(d+1)-1}{d(d+1)}} \\ \times \left| \frac{(-\frac{(1-s)u_i+sv_i}{((1-s)u_1+sv_1)^2}, 0, \ldots, 0, ((1-s)u_1+sv_1)^{-1}, 0, \ldots, 0)\cdot(v_1-u_1, \ldots, v_i-u_i, \ldots)}{|-((1-s)u_1+sv_1)^{-(d+1)}|^{\frac{d+(d+1)-1}{d(d+1)}}(v_1^{-1}v_i-u_1^{-1}u_i)} \right|^{(d+1)-2} \end{matrix}$$

$$= |u_1v_1|^{-(d+1)(1-\frac{2d}{d(d+1)})} \left| \frac{-\frac{(1-s)u_i+sv_i}{((1-s)u_1+sv_1)^2}(v_1-u_1)+((1-s)u_1+sv_1)^{-1}(v_i-u_i)}{|(1-s)u_1+sv_1|^{-(d+1)\frac{2d}{d(d+1)}}(v_1^{-1}v_i-u_1^{-1}u_i)} \right|^{d-1}$$

$$= |u_1v_1|^{-((d+1)-2)} \left| \frac{-\frac{(1-s)u_i+sv_i}{((1-s)u_1+sv_1)^2}(v_1-u_1)+((1-s)u_1+sv_1)^{-1}(v_i-u_i)}{((1-s)u_1+sv_1)^{-2}(v_1^{-1}v_i-u_1^{-1}u_i)} \right|^{d-1}$$

$$= |u_1v_1|^{-(d-1)} \left| \frac{-((1-s)u_i+sv_i)(v_1-u_1)+((1-s)u_1+sv_1)(v_i-u_i)}{v_1^{-1}v_i-u_1^{-1}u_i} \right|^{d-1}$$

$$= \left| \frac{-(1-s)v_1u_i - sv_1v_i + (1-s)u_1u_i + su_1v_i + (1-s)u_1v_i + sv_1v_i - (1-s)u_1u_i - sv_1u_i}{u_1v_1(v_1^{-1}v_i-u_1^{-1}u_i)} \right|^{d-1}$$

$$= \left| \frac{-(1-s)v_1u_i + su_1v_i + (1-s)u_1v_i - sv_1u_i}{u_1v_i - v_1u_i} \right|^{d-1} = \left| \frac{(s+(1-s))u_1v_i - (s+(1-s))v_1u_i}{u_1v_i - v_1u_i} \right|^{d-1}$$

$$= \left| \frac{u_1v_i - v_1u_i}{u_1v_i - v_1u_i} \right|^{d-1} = |1|^{d-1} = 1.$$

Thus, as $\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d} = |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left| \frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))} \right|^{q-2}$ holds for the transformation γ when $q = d+1$ (with both sides of the equation equalling 1), we can conclude that the derived functional J is a symmetry of Φ, as desired.

Chapter 6

Intuition for the Sufficient Condition

At first glance, the sufficient condition we have found for a transformation to be a symmetry appears inelegant and complicated.

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}$$
$$= |\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2}$$

And indeed, with all of the complicated elements of this equation (especially on the right-hand side) – gradients and Jacobians taken to weird powers and combined in awkward fashions – there does not seem to be any easy intuition that describes exactly which transformations γ do and do not satisfy this condition.

However, the process of heuristically interpreting this condition becomes easier with a key realization: Since the two sides of the equation must equal each other, if evaluating one side of the equation for a specific transformation γ_0 results in an expression that cannot result from the other side of the equation being evaluated for any transformation γ (because that expression derived from γ_0 depends on variables that do not appear on the other side of the equation, for instance), then γ_0 cannot be a transformation that satisfies the condition.

Using this logic, we can come up with two surprisingly elegant heuristic requirements – one for each side of the equation – that it is necessary for a transformation to fulfill in order to satisfy condition (5.2).

6.1 The Left-Hand Side: Mapping Distances Nicely

First, let us consider the left-hand side of the equation:

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}.$$

Conveniently, this expression has a relatively simple intuitive interpretation: for any points $u, v \in \mathbb{R}^d$, it represents the ratio between the distance from u to v and the distance from $\gamma(u)$ to $\gamma(v)$, taken to the power of $q-d-1$; in other words, it represents how the transformation γ maps distances between points.

As discussed above, if a transformation γ is to satisfy condition (5.2), then each side of the equation must evaluate to the sort of expression that can be "captured" by the other side of the equation. So, looking at the left-hand side of the equation, we can get an intuitive sense of under what circumstances $\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}$ evaluates to such a sufficiently simple expression. Those circumstances are described the following statement, which describes a (somewhat vague, admittedly) condition that is equivalent to $\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d}$ being simple enough, and thus is necessary (though not sufficient) for γ is to satisfy condition (5.2).

Statement 6.1. *If γ is a transformation that satisfies the conditions in Theorem 5.1, and thus induces a symmetry γ^* of $\Phi(f) = \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}}$, then one of the two following things must be true:*

- $q = d + 1$
- *The transformation γ maps distances between points in $\mathbb{R}^d$ in a sufficiently "nice" fashion.*

This heuristic interpretation is, admittedly, not stated in the most rigorous or unambiguous fashion, so let us look at a few examples of this intuition in action.

6.1.1 Affine Transformations

As we discussed earlier in section 5.1, if γ is an affine transformation of the form $\gamma(x) = kUx + b$, where b is a vector in $\mathbb{R}^d$ and kU is a scalar multiple of an orthogonal matrix, then it satisfies the conditions in Theorem 5.1 for all possible values of q and d. And, indeed, because both orthogonal matrices and translations are distance-preserving transformations (isometries), while scalar multiplication simply multiplies all distances by a scalar multiple, transformations of this form do map distances in an extremely "nice" way: for any $u, v \in \mathbb{R}^d$, the distance between $\gamma(u)$ and $\gamma(v)$ will simply be k times the distance between u and v.

But what about affine transformations $\gamma(x) = Ax + b$ (with $A \in \mathsf{GL}(\mathbb{R}^d)$ not a scalar multiple of a matrix $U \in \mathsf{O}(d)$, and $b \in \mathbb{R}^d$) that are not isometries or scalar multiples thereof? For arbitrary values of q and d, these transformations do not satisfy condition (5.2), and we can see why by appealing to the intuition from Statement 6.1. There is no simple way to describe the way in which an arbitrary affine transformation maps the distance between two points, and therefore no way to simplify the left-hand side of condition (5.2) for this γ beyond:

$$\left(\frac{|u-v|}{|\gamma(u)-\gamma(v)|}\right)^{q-1-d} = \left(\frac{|u-v|}{|(Au+b)-(Av-b)|}\right)^{q-1-d} = \left(\frac{|u-v|}{|A(u-v)|}\right)^{q-1-d}$$

when $q \neq d+1$. And, indeed, when the right-hand side of condition (5.2) ends up being constant for these γ:

$$\begin{aligned}
&|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2} \\
&= |\det(A)^2|^{1-\frac{d+q-1}{dq}} \left|\frac{A_{\bullet i}\cdot(v-u)}{|\det(A)|^{\frac{d+q-1}{dq}}((A_{\bullet i}v+b_i)-(A_{\bullet i}u+b_i))}\right|^{q-2} \\
&= |\det(A)|^{2-2\frac{d+q-1}{dq}} \left|\frac{A_{\bullet i}(v-u)}{|\det(A)|^{\frac{d+q-1}{dq}}A_{\bullet i}(v-u)}\right|^{q-2} = |\det(A)|^{2-2\frac{d+q-1}{dq}} \left|\frac{1}{|\det(A)|^{\frac{d+q-1}{dq}}}\right|^{q-2} \\
&= |\det(A)|^{2-2\frac{d+q-1}{dq}}|\det(A)|^{-\frac{d+q-1}{dq}(q-2)} = |\det(A)|^{2-2\frac{d+q-1}{dq}-\frac{d+q-1}{d}+2\frac{d+q-1}{dq}} \\
&= |\det(A)|^{2-\frac{d+q-1}{d}} = |\det(A)|^{\frac{d-q+1}{d}}
\end{aligned}$$

it becomes clear that the only way these affine transformations can satisfy these conditions is if $q = d+1$, so that

$$\left(\frac{|u-v|}{|A(u-v)|}\right)^{q-1-d} = \left(\frac{|u-v|}{|A(u-v)|}\right)^{d+1-1-d} = \left(\frac{|u-v|}{|A(u-v)|}\right)^{0} = 1$$

and

$$|\det(A)|^{\frac{d-q+1}{d}} = |\det(A)|^{\frac{d-d-1+1}{d}} = |\det(A)|^0 = 1.$$

In both of these cases, a transformation γ only satisfies condition (5.2) when either it maps distances "nicely" or when $q = d + 1$, as described in Statement 6.1.

6.1.2 D and J

Let us now contrast the two rotation-based functional operators we discussed earlier in the context of this heuristic requirement.

First, consider D, the sphere-based inversion based on the transformation $\gamma(x) = \left(\frac{2x_1}{|x-a|^2}, ..., \frac{2x_{d-1}}{|x-a|^2}, \frac{1-|x|^2}{|x-a|^2}\right)$ which is a symmetry of Φ if and only if $q = 2$. Following a similar derivation to one described in section 5.2, we may determine a relatively simple way to describe how this γ maps distances: namely, that the distance between the points $\gamma(u)$ and $\gamma(v)$ will be equal to $\frac{|u-a||v-a|}{2}$ (where $a = (0, \ldots, 0, 1)$) times the distance from u to v. And as discussed in the rest of section 5.2, the "niceness" of this ratio between $|u - v|$ and $|\gamma(u) - \gamma(v)|$ indeed ends up being sufficient for it to be captured by the right-hand side of condition (5.2).

Next, consider J, the hemisphere-based inversion based on the transformation $\gamma(x) = (x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1}x_d)$, which is a symmetry of Φ if and only if $q = d + 1$. Unlike with D's corresponding transformation, there is no simple way to describe how this γ maps distances; the expression $\frac{|\gamma(u)-\gamma(v)|}{|u-v|}$ cannot be further simplified. And indeed, it turns out that for this γ, there is no way to for the right-hand side of condition (5.2) to be able to capture the ratio between $|\gamma(u) - \gamma(v)|$ and $|\gamma(u) - \gamma(v)|$, unless that ratio is taken to the power of 0.

These two examples serve as a perfect illustration of the dual nature of Statement 6.1. To be specific, Statement 6.1 claims that for a mapping $T = \gamma^*$ to be a symmetry of Φ for given values of p, q, and d, it must either be the case $q = d + 1$, or that γ maps distances nicely. To wit, D is not a symmetry of Φ when $q = d + 1$, but since there are values of p, q, and d for which it is a symmetry of Φ (namely, when $q = 2$), its associated γ must map distances nicely – and indeed, it does. Contrawise, J's associated γ does not map distances in a nice, easy-to-describe manner, so J can only be a symmetry of Φ when $q = d + 1$ – and indeed, the cases where $q = d + 1$ are the only ones where J is a symmetry.

6.2 The Right-Hand Side: Mapping Lines to Lines

Now, let us consider the right-hand side of the equation:

$$|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}} \left| \frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))} \right|^{q-2}.$$

Compared to the condition's left-hand side, it is much more difficult to succinctly and heuristically describe what this entire expression equals for any given mapping γ. However, for our purposes, it will end up being sufficient to focus mainly on just one part of this expression.

As before, for a given $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ to satisfy condition (5.2), this side of the equation simplify to something that can be "captured" by the other side of condition (5.2). In this case, one obvious situation in which the right side of the expression can be incapable of being "captured" by the left side (for some γ) is when it depends on the value of $i \in \{1, \ldots, d\}$. And, indeed, if we isolate the parts of this expression that can potentially depend on i (such as $\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}$), and determine what transformations γ make that expression not depend on i (and thus are eligible to satisfy (5.2)), we end up with the following, surprisingly elegant requirement:

Statement 6.2. *If γ is a transformation that satisfies the conditions in Theorem 5.1, and thus induces a symmetry γ^* of $\Phi(f) = \frac{\|\mathcal{X}f\|_{L^q(\mathcal{M})}}{\|f\|_{L^p(\mathbb{R}^d)}}$, then one of the two following things must be true:*

- $q = 2$
- *The transformation γ maps every set ℓ in $\mathbb{R}^d$ that is equal almost everywhere to a line to a set $\gamma(\ell)$ in $\mathbb{R}^d$ that is equal almost everywhere to a line. Specifically, for almost every $u, v \in \mathbb{R}^d$, if $\ell = \{(1-s)u + sv : s \in \mathbb{R}\} \subseteq \mathbb{R}^d$ is the line through u and v, then it is the case that*

  $$\gamma(\ell) = \{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$$

  almost everywhere.

Proof. As noted before, for γ to satisfy condition (5.2), the right-hand side of condition (5.2) must not depend on the value of i almost everywhere (as the left-hand side does not depend on the value of i, and the two must be equal). So,

$$|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2}$$

does not depend on i almost everywhere. And as

$$|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}}$$

and

$$\left(\frac{1}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}}\right)^{q-2}$$

intrinsically do not depend on i,

$$\begin{aligned}&|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}(\gamma_i(v)-\gamma_i(u))}\right|^{q-2}\\ &=|\mathcal{J}_\gamma(u)\mathcal{J}_\gamma(v)|^{1-\frac{d+q-1}{dq}}\left(\frac{1}{|\mathcal{J}_\gamma((1-s)u+sv)|^{\frac{d+q-1}{dq}}}\right)^{q-2}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{q-2}\end{aligned}$$

depends on i if and only if $\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{q-2}$ does. So, $\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{q-2}$ does not depend on i almost everywhere. There are two cases from here:

- First, consider the case where $q=2$. Then,

$$\begin{aligned}\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{q-2}&=\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{2-2}\\ &=\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{0}=1,\end{aligned}$$

  which, as a constant, never depends on i, regardless of what the transformation γ is.

- Next, consider the case where $q\neq 2$. In this case, $\left|\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}\right|^{q-2}$ will depend on i if and only if $\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}$ depends on i; so, we can conclude that $\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}$ does not depend on i almost everywhere. The expression $\frac{\nabla\gamma_i((1-s)u+sv)\cdot(v-u)}{\gamma_i(v)-\gamma_i(u)}$, meanwhile, is the derivative of $\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$ with respect to s, so $\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$ does not depend on i either (almost everywhere).

Since $\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$ does not depend on i, we know that

$$\frac{\gamma_1((1-s)u+sv)-\gamma_1(u)}{\gamma_1(v)-\gamma_1(u)} = \frac{\gamma_2((1-s)u+sv)-\gamma_2(u)}{\gamma_2(v)-\gamma_2(u)} = \cdots = \frac{\gamma_d((1-s)u+sv)-\gamma_d(u)}{\gamma_d(v)-\gamma_d(u)}$$

(almost everywhere). Let t represent the value that all of these expressions are equal to; i.e., let $t = \frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$ for all $i \in \{1,\ldots,d\}$ (almost everywhere). This means that we may write, for almost every $s \in \mathbb{R}$ and $u, v \in \mathbb{R}^d$, that for all $i \in \{1,\ldots,d\}$,

$$\begin{aligned} t &= \frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)} \\ (\gamma_i(v)-\gamma_i(u))t &= \gamma_i((1-s)u+sv)-\gamma_i(u) \\ (\gamma_i(v)-\gamma_i(u))t+\gamma_i(u) &= \gamma_i((1-s)u+sv) \\ (1-t)\gamma_i(u)+t\gamma_i(v) &= \gamma_i((1-s)u+sv) \end{aligned}$$

and thus that $(1-t)\gamma(u)+t\gamma(v) = \gamma((1-s)u+sv)$.

Now, for almost any $u, v \in \mathbb{R}^d$, consider the line $\ell = \{(1-s)u+sv : s \in \mathbb{R}\}$, and furthermore consider its image $\gamma(\ell)$ under our transformation. Let $(1-s_0)u+s_0v$ be a point on ℓ (where $s_0 \in \mathbb{R}$). As we have just shown for almost any given $u, v \in \mathbb{R}$, and for almost any $s_0 \in \mathbb{R}$, there must exist a value $t_0 \in \mathbb{R}$ such that $(1-t_0)\gamma(u)+t_0\gamma(v) = \gamma((1-s_0)u+s_0v)$. Therefore, it is true almost everywhere that the point $\gamma((1-s_0)u+s_0v) \in \gamma(\ell)$ is on the line $\{(1-t)\gamma(u)+t\gamma(v) : t \in \mathbb{R}\}$; and thus, $\gamma(\{(1-s)u+sv : s \in \mathbb{R}\}) \subseteq \{(1-t)\gamma(u)+t\gamma(v) : t \in \mathbb{R}\}$ almost everywhere. And as γ satisfying condition (5.2) implies that γ^{-1} satisfies it as well, we may apply this same logic to see that

$$\begin{aligned} &\gamma^{-1}(\{(1-t)\gamma(u)+t\gamma(v) : t \in \mathbb{R}\}) \\ &\subseteq \{(1-s)\gamma^{-1}(\gamma(u))+s\gamma^{-1}(\gamma(v)) : s \in \mathbb{R}\} \\ &= \gamma(\{(1-s)u+sv : s \in \mathbb{R}\}) \end{aligned}$$

almost everywhere, which, as γ is bijective, means that $\{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\} \subseteq \ell$ almost everywhere. By double inclusion, this means that

$$\gamma(\ell) = \{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$$

almost everywhere.

And so we see that if γ satisfies condition (5.2), then either $q = 2$, or

$$\gamma(\{(1-s)u + sv : s \in \mathbb{R}\}) = \{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$$

almost everywhere for almost every $u, v \in \mathbb{R}^d$, as desired. □

This statement is less easy to derive from the right-hand side of the equation than Statement 6.1 is from the left-hand side, but what it lacks in intuitive clarity it makes up for in rigor: Any transformation γ can definitively be said – proven, in fact, as we just saw – to either satisfy the conditions of Statement 6.2 or fail to do so (in contrast to Statement 6.1, which is a lot more subjective).

Like Statement 6.1, Statement 6.2 also possesses a dual nature, one that is best understood by looking at some examples of this "mapping lines to lines" condition in action.

6.2.1 Affine Transformations

Because all affine transformations of the form $\gamma(x) = kUx + b$, where b is a vector in $\mathbb{R}^d$ and kU is a scalar multiple of an orthogonal matrix, satisfy condition (5.2) (for all q), we would hope that such γs satisfy Statement 6.2. And, indeed, not only do these affine γs satisfy Statement 6.2, but in a sense, they do so the best out of any possible transformation γ.

To wit, in order to satisfy Statement 6.2 when $q \neq 2$, the transformations $\gamma(x) = kUx + b$ must preserve (almost all) lines. And we can confirm that they do so by noting that for any γ of this form,

$$\begin{aligned}\frac{\gamma_i((1-s)u + sv) - \gamma_i(u)}{\gamma_i(v) - \gamma_i(u)} &= \frac{(kU_{\bullet i}((1-s)u + sv) + b) - (kU_{\bullet i}u + b)}{(kU_{\bullet i}v + b) - (kU_{\bullet i}u + b)} \\ &= \frac{kU_{\bullet i}((1-s)u + sv) - kU_{\bullet i}u}{kU_{\bullet i}v - kU_{\bullet i}u} = \frac{kU_{\bullet i}(u - su + sv - u)}{kU_{\bullet i}(v - u)} \\ &= \frac{skU_{\bullet i}(v-u)}{kU_{\bullet i}(v-u)} = s\end{aligned}$$

for all i, s, u, and v, which means that for any point $(1-s_0)u+s_0v$ on a line $\{(1-s)u+sv : s \in \mathbb{R}\}$, that point's image $\gamma((1-s_0)u+s_0v)$ is on the line $\{(1-t)\gamma(u)+t\gamma(v) : t \in \mathbb{R}\}$ (being specifically equal to the point $(1-t_0)\gamma(u)+t_0\gamma(v)$, where $t_0 = s_0$), with a similar statement holding for any point $(1-t_0)\gamma(u)+t_0\gamma(v)$ on $\{(1-t)\gamma(u)+t\gamma(v) : t \in \mathbb{R}\}$. Thus, using these facts alongside the bijectiveness of γ, we can conclude that γ maps any (and thus almost any) line $\{(1-s)u+sv : s \in \mathbb{R}\}$ to a set that is exactly (and thus almost exactly) equal to the line $\{(1-t)\gamma(u)+t\gamma(v) : t \in \mathbb{R}\}$, and thus that it satisfies Statement 6.2.

However, we can also confirm that these γs satisfy Statement 6.2 just by noting that all transformations $\gamma(x) = kUx + b$ are affine transformations. After all, it is a fundamental property of affine transformations that not only do they preserve lines, but affine transformations $\mathbb{R}^d \to \mathbb{R}^d$ are also the only ones that map not just *almost all* lines in $\mathbb{R}^d$ to lines in $\mathbb{R}^d$, but *all* lines in $\mathbb{R}^d$ to lines in $\mathbb{R}^d$. And indeed, we shall soon see a non-affine transformation that satisfies Statement 6.2 by mapping almost all lines to lines – but which will not map lines to lines on the set (of measure 0) where it is discontinuous.

6.2.2 D and J

The two previously-discussed inversions, D and J, should also satisfy Statement 6.2 (after all, their respective γs both satisfy condition (5.2), at least for certain values of q). And indeed, it can be shown that they do.

The spherical rotation-based symmetry D is based on a γ that does not map almost all lines to lines. After all, it is immediately visible that $\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$, which is equal to

$$\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)} = \frac{\dfrac{2((1-s)u_i+sv_i)}{|(1-s)u+sv-a|^2} - \dfrac{2u_i}{|u-a|^2}}{\dfrac{2v_i}{|v-a|^2} - \dfrac{2u_i}{|u-a|^2}}$$

when $i \in \{1, \ldots, d-1\}$ and

$$\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)} = \frac{\dfrac{1-|(1-s)u+sv|^2}{|(1-s)u+sv-a|^2} - \dfrac{1-|u|^2}{|u-a|^2}}{\dfrac{1-|v|^2}{|v-a|^2} - \dfrac{1-|u|^2}{|u-a|^2}}$$

when $i = d$, will vary depending on i for almost all u, v, and s, which in turn means that for almost every point $(1-s_0)u + s_0 v$ on almost any given line $\{(1-s)u + sv : s \in \mathbb{R}\}$ (with $s_0 \in \mathbb{R}$), we cannot write $\gamma((1-s_0)u + s_0 v)$ as a point on the line $\{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$. However, because D is specifically only a symmetry of Φ when $q = 2$, it still satisfies Statement 6.2, as that statement allows for non-line-preserving transformations γ to induce symmetries of Φ when $q = 2$.

Now, consider the hemispherical rotation-based transformation J. It is only a symmetry of Φ when $q = d+1$ case. However, in that $q = d+1$ case, it still satisfies Statement 6.2 by virtue of its corresponding γ preserving almost all lines. To wit, for any $u, v \in \mathbb{R}^d \setminus \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 = 0\}$, and any $s \in \mathbb{R}$ such that $(1-s)u + sv \in \mathbb{R}^d \setminus \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 = 0\}$ (which will be almost every s, as the line through u and v will cross $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 = 0\}$ at most once), we may write that

$$\frac{\gamma_i((1-s)u+sv) - \gamma_i(u)}{\gamma_i(v) - \gamma_i(u)} = \frac{\frac{1}{(1-s)u_1+sv_1} - \frac{1}{u_1}}{\frac{1}{v_1} - \frac{1}{u_1}} = \frac{\frac{u_1 - ((1-s)u_1+sv_1)}{u_1((1-s)u_1+sv_1)}}{\frac{u_1 - v_1}{u_1 v_1}}$$
$$= \frac{\frac{su_1 - sv_1}{(1-s)u_1+sv_1}}{\frac{u_1-v_1}{v_1}} = \frac{sv_1}{(1-s)u_1+sv_1}$$

when $i = 1$ and

$$\frac{\gamma_i((1-s)u+sv) - \gamma_i(u)}{\gamma_i(v) - \gamma_i(u)} = \frac{\frac{(1-s)u_i+sv_i}{(1-s)u_1+sv_1} - \frac{u_i}{u_1}}{\frac{v_i}{v_1} - \frac{u_i}{u_1}} = \frac{\frac{u_1((1-s)u_i+sv_i) - u_i((1-s)u_1+sv_1)}{u_1((1-s)u_1+sv_1)}}{\frac{u_1 v_i - u_i v_1}{u_1 v_1}}$$
$$= \frac{\frac{(1-s)u_1u_i + su_1v_i - (1-s)u_1u_i - su_iv_1}{(1-s)u_1+sv_1}}{\frac{u_1v_i - u_iv_1}{v_1}} = \frac{\frac{su_1v_i - su_iv_1}{(1-s)u_1+sv_1}}{\frac{u_1v_i-u_iv_1}{v_1}} = \frac{sv_1}{(1-s)u_1+sv_1}$$

when $i \in \{2, \dots, d\}$ – which is to say that $\frac{\gamma_i((1-s)u+sv)-\gamma_i(u)}{\gamma_i(v)-\gamma_i(u)}$ is the same for all i. Thus, for any point $(1-s_0)u + s_0v$ on a line $\{(1-s)u + sv : s \in \mathbb{R}\}$, that point's image $\gamma((1-s_0)u + s_0v)$ is on the line $\{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$ (being specifically equal to the point $(1-t_0)\gamma(u) + t_0\gamma(v)$, where $t_0 = \frac{s_0v_1}{(1-s_0)u_1+s_0v_1}$), with a similar statement holding for any point $(1-t_0)\gamma(u) + t_0\gamma(v)$ on $\{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$. Thus, using these facts alongside the bijectiveness of γ, we can conclude that γ maps almost every line $\{(1-s)u + sv : s \in \mathbb{R}\}$ (ignoring the ones on the hyperplane $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 = 0\}$) to a set that is almost exactly equal to the line $\{(1-t)\gamma(u) + t\gamma(v) : t \in \mathbb{R}\}$ (again, ignoring points on the hyperplane $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 = 0\}$), and thus that it satisfies Statement 6.2.

Again, the duality of this statement – and thus of condition (5.2) itself – is visible in these two example symmetries. Statement 6.1 claims that for a mapping $T = \gamma^*$ to be a symmetry of Φ for given values of p, q, and d, it must either be the case $q = 2$, or that γ lines to lines (almost everywhere, at least). To wit, J is not a symmetry of Φ when $q = 2$, but since there are values of p, q, and d for which it is a symmetry of Φ (namely, when $q = 2$), its associated γ must map lines to lines – and, indeed, it does. Contrawise, D's associated γ does not map lines to lines, so D can only be a symmetry of Φ when $q = 2$ – and indeed, the cases where $q = 2$ are the only ones where D is a symmetry.

Chapter 7

A Conjectured Further Restriction

The fact that any symmetry of Φ (to which Theorem 5.1 applies) that we may attempt to create must map distances nicely and map lines to lines almost everywhere, on the face of it, significantly restricts what sort of (easy-to-create) γ-derived transformations T can be symmetries (for values of q where we don't have an "out" for one of the two requirements, as we do in the $q = 2$ and $q = d + 1$ cases). Very few maps $\mathbb{R}^d \to \mathbb{R}^d$ map distances nicely, and very few maps $\mathbb{R}^d \to \mathbb{R}^d$ map lines to lines almost everywhere; it would seem very difficult, if not impossible, to have a non-affine transformation that does both. And indeed, we now present a conjecture that, if true, would establish that it is in fact impossible to, within the constraints of the techniques we have been using, prove the competing symmetries argument for $2 < q < d + 1$.

Conjecture 2. *If $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ is bijective, is continuous almost everywhere, is differentiable almost everywhere, and maps lines to lines almost everywhere, then γ is either an affine transformation, the γ associated with J, or a composition thereof.*

Affine transformations cannot be used in the competing symmetries argument (as previously discussed, they cannot satisfy condition (5) of Theorem 4.7), and J (as well as affine transformations of J) is only able to be used in the competing symmetries argument when $q = d + 1$ (as previously discussed, it does not map distances nicely). Thus, if this conjecture is true, no symmetry of Φ (to which Theorem 5.1 applies) that is useful for the competing symmetry argument can exist when $2 < q < d + 1$.

While, of course, we do not have any formal proof of this conjecture (hence why we are still labelling it as a conjecture), we suspect such a proof may take the following form:

Proof Sketch.

- Consider (almost) any set of parallel lines in $\mathcal{M}$, and apply γ to each element of this set to get a new set of lines. Show that this new set must be (equal almost everywhere to) either another set of parallel lines, or a set of lines that intersect at exactly one point. If the new set of lines has more than two points of intersection (perhaps excepting a set of lines with measure zero in $\mathcal{M}$), then γ will either be non-bijective and discontinuous on a set of nonzero measure in $\mathbb{R}^d$, or will fail to map lines to lines on a set of nonzero measure in $\mathbb{R}^d$.

- Show that if every set of parallel lines in $\mathcal{M}$ maps to another set of parallel lines (or, at least, a set that differs from a set of parallel lines by a set of measure zero), then γ is an affine transformation. (This is a minor extension of a well-known theorem concerning affine transformations.)

- Show that in the other case, if there exists some set of parallel lines in $\mathcal{M}$ that maps to a set of lines that intersect at exactly one point (or something equal almost everywhere to one), then every set of parallel lines in $\mathcal{M}$ maps to a set of lines that intersect at exactly one point (or something equal almost everywhere to one). (Otherwise, γ will either be non-bijective and discontinuous on a set of nonzero measure in $\mathbb{R}^d$, or will fail to map lines to lines on a set of nonzero measure in $\mathbb{R}^d$.)

- Show that if every set of parallel lines in $\mathcal{M}$ maps to a set of lines that intersect at exactly one point(or something equal almost everywhere to one), then those points all have to lie along the same $d-1$-dimensional hyperplane in $\mathbb{R}^d$.

- In that case, compose γ with an affine transformation to so that the aforementioned $d-1$-dimensional hyperplane of points becomes

  $$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = 0\}.$$

  Then, show that based on everything we know about how γ acts on points and lines, γ composed with this affine transformation must be equal (almost everywhere, at least) to J.

The following conjecture pertaining to this "almost everywhere collineation" problem may also be of use in proving, disproving, or otherwise exploring Conjecture 2.

Conjecture 3. *Let $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ be bijective, be continuous almost everywhere, be differentiable almost everywhere, and map lines to lines almost everywhere. If we represent lines in $\mathcal{M}$ via the slope-intercept parametrization (i.e., as vectors in $\mathbb{R}^{2(d-1)}$), then, when treated as a function acting on these vectors $(m, b) \in \mathbb{R}^{2(d-1)}$ (with $\gamma(m, b)$ representing the line to which the line (m, b) maps under γ), γ will either act as an affine transformation, J, or a composition thereof.*

Chapter 8

Conclusion and Potential Future Work

In this thesis, we have introduced all concepts necessary for understanding the Baernstein-Loss conjecture (not to mention the conjecture itself). We have recounted a proof the most general form of competing symmetries argument (as it applies to the Baernstein-Loss conjecture). We have proven a sufficient condition for certain types of transformations to be able to be used in the competing symmetries argument. We have conjectured that that condition restricts our ability to choose useful transformations for the argument to such an extent that the argument is impossible to extend to other values of q.

And while we would like to confidently say that we have conclusively shown that the competing symmetries argument cannot be generalized to other values of q, we cannot actually do so. We have not really proven any hard facts about under what circumstances the competing symmetries argument can or cannot be used to prove the Baernstein-Loss conjecture. It is still entirely possible that our conjecture is wrong, and that it is possible to come up with a useful transformation that maps lines to lines and maps distances nicely. It is still entirely possible that there exists a useful symmetry of Φ based on a transformation γ that doesn't satisfy our condition (it was a sufficient condition, not a necessary one, remember). It is still entirely possible that there exists a useful symmetry of Φ that isn't based on a γ at all. It is still entirely possible that there exists a transformation we can use in the competing symmetries argument that isn't a symmetry of Φ – where $\Phi(Tf) \geq \Phi(f)$ for all $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$), but it is not the case that $\Phi(Tf) = \Phi(f)$ for all $f \in L^p(\mathbb{R}^d)$ (where $\|f\|_{L^p(\mathbb{R}^d)} \neq 0$).

However, what we can say with confidence is that previous intuitive hopes that the competing symmetries argument may be extended to other values of q via transformations falling "between" D and J have now been dashed. It is not mere coincidence that the $q = 2$ and $q = d + 1$ cases were the first to fall to the competing symmetries argument; as the implications of Theorem 5.1 show, there is a sense in which it is inherently easier to find transformations that work for them as compared to other values of q. And the techniques that were used to find such suitable transformations for $q = 2$ and $q = d + 1$ will not work without significant modification (if at all).

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