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Transition Matrices for Young's Representations of the Symmetric Group

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TRANSITION MATRICES FOR YOUNG'S REPRESENTATIONS OF THE SYMMETRIC GROUP

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ABSTRACT

The irreducible modules of the symmetric group S_n are indexed by the integer partitions $\{\lambda \mid \lambda \vdash n\}$. In the 1920's, Alfred Young defined representations on these modules according to the action of permutations $\sigma \in S_n$ on the standard Young tableaux of shape λ, denoted $\mathcal{SYT}(\lambda)$. In this paper, we solve an open problem by determining the change-of-basis matrix $\mathcal{A}_\lambda$ between two of these representations — the seminormal representation and the natural representation — by relating the entries of $\mathcal{A}_\lambda$ to paths on the crystal graph Γ_λ, which is the Hasse diagram for weak Bruhat order on $\mathcal{SYT}(\lambda)$. We then describe a recursive rule for computing these entries that puts the computational complexity for determining $\mathcal{A}_\lambda$ on the order of $|\mathcal{SYT}(\lambda)|^2$, and we abstract our formula to the Iwahori-Hecke algebra $H_n(q)$ and the generalized symmetric group $G_{r,n}$.

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1. INTRODUCTION

This paper concerns the representation theory of S_n, the symmetric group on n letters. We work on the irreducible representations of S_n which are indexed by integer partitions $\lambda \vdash n$, denoting the corresponding irreducible representation S_n^λ. We give a combinatorial description of the entries in the change-of-basis matrix between two well-known bases of S_n^λ.

It can be shown that $\dim\left(S_n^\lambda\right) = |\mathcal{SYT}(\lambda)|$, where $\mathcal{SYT}(\lambda)$ denotes the set of standard Young tableaux of shape λ. A standard Young tableau $\mathsf{T} \in \mathcal{SYT}(\lambda)$ is created by assigning $1, 2, \ldots, n$ to the boxes of λ such that each row is increasing from left to right, and each column is increasing from top to bottom. A basis for S_n^λ can therefore be indexed by standard Young tableaux. There are two well-known representations of S_n^λ created by Alfred Young in the 1920's — the natural representation and the seminormal representation — which are defined using the combinatorics of tableaux. We provide a brief discussion of the representation theory of the symmetric group in Chapter 2, and we define these particular representations rigorously in Chapter 3. Our goal is to solve for the matrix $\mathcal{A}_\lambda$ which changes basis between the seminormal and the natural representations, which has been an open problem since Young's initial definition of the two representations. This matrix has some nice, seemingly generalizable properties. Figure 1.1 displays the change-of-basis matrix between Young's natural basis $\mathcal{N}_{(3,2)} = \{\mathbf{n}_\mathsf{T} \mid \mathsf{T} \in \mathsf{SYT}(3,2)\}$ and Young's seminormal basis $\mathcal{V}_{(3,2)} = \{\mathbf{v}_\mathsf{T} \mid \mathsf{T} \in \mathsf{SYT}(3,2)\}$, in the case where $\lambda = (3,2)$ for the symmetric group S_5.

Notice that the rows and columns of $\mathcal{A}_{(3,2)}$ are indexed by the standard Young tableaux of shape $(3,2)$, and the matrix is upper-triangular when the tableaux are given this particular ordering. It turns out that for any n and any integer partition λ, such an ordering which preserves the upper-triangularity of $\mathcal{A}_\lambda$ exists.

To compute the entries of $\mathcal{A}_\lambda$ we use a graph Γ_λ, called a crystal graph, which we introduce in Chapter 4. The vertices of Γ_λ are $\mathcal{SYT}(\lambda)$, and two tableaux are adjacent if and only if they differ by an adjacent swap $(i, i+1)$. In Figure 1.2, edge color denotes a specific adjacent swap. We also show in Chapter 4 that this graph is isomorphic to an interval in the Hasse diagram for weak (left) Bruhat order, a partial ordering on S_n, so it follows that Γ_λ is connected and has a well-defined

$$\mathcal{A}_{(3,2)} = \begin{array}{c} \\ \mathbf{v}_{\young(135,24)} \\ \mathbf{v}_{\young(125,34)} \\ \mathbf{v}_{\young(134,25)} \\ \mathbf{v}_{\young(124,35)} \\ \mathbf{v}_{\young(123,45)} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{n}_{\young(135,24)} & \mathbf{n}_{\young(125,34)} & \mathbf{n}_{\young(134,25)} & \mathbf{n}_{\young(124,35)} & \mathbf{n}_{\young(123,45)} \end{array} \\ \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \cdot & \frac{3}{2} & \cdot & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \frac{9}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix} \end{array}.$$

Figure 1.1: The change-of-basis matrix $\mathcal{A}_{(3,2)}$.

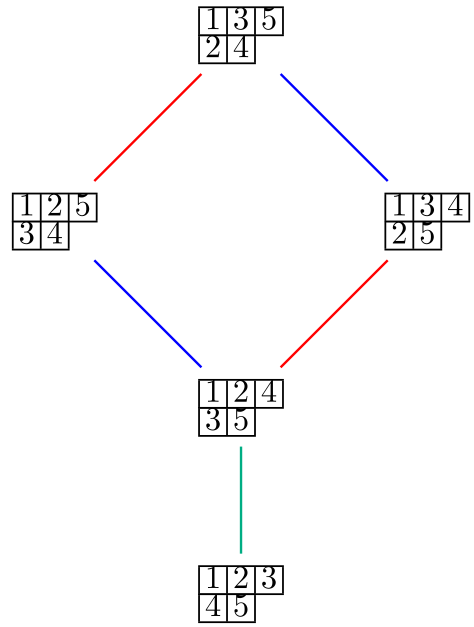


Figure 1.2: The crystal graph $\Gamma_{(3,2)}$.

notion of depth for any integer partition λ.

The entries in $\mathcal{A}_\lambda$ are given by summing weighted paths on Γ_λ. To find the (S, T)-entry of $\mathcal{A}_\lambda$, we take a path to the tableau T and consider all subpaths which terminate at S. We assign a weight to each of these subpaths, determined by a product of coefficients given by the seminormal representation. In Chaper 5, we prove that the (S, T)-entry of $\mathcal{A}_\lambda$ is the sum of the weights on these subpaths. It follows from this construction that $\mathcal{A}_\lambda$ is upper-triangular when indexed by depth in Γ_λ.

In Chapter 6, we show that our method for computing $\mathcal{A}_\lambda$ can be easily generalized. First, we provide the change-of-basis formula between a third representation defined by Young — the orthogonal representation — and the natural representation. Then, we provide the analogous change-of-basis formula for the Iwahori-Hecke algebra $H_n(q)$, a q-analog of S_n, and for the generalized symmetric group $G_{r,n} = C_r \wr S_n$, where the natural and seminormal representations are indexed by r-tuples of tableaux.

2. REPRESENTATION THEORY OF THE SYMMETRIC GROUP

The symmetric group S_n is the group of permutations on n letters under the operation of permutation composition. The simplest of these permutations are the adjacent transpositions $s_1, \dots, s_{n-1}$ for which s_i swaps i and $i+1$. The symmetric group is generated by the set of adjacent transpositions, meaning that any permutation in the group can be written as the product of adjacent transpositions.

Example 2.0.1. *In S_7,*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}$$

is a permutation (written in two-line notation) which sends $1 \to 3, 2 \to 1, \dots, 7 \to 6$. The permutation σ can also be written in cycle notation, and can be further decomposed into the product of adjacent transpositions:

$$\sigma = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}}_{\textit{two-line}} = \underbrace{(1352)(476)}_{\textit{cycle}} = \underbrace{s_2 s_1 s_4 s_3 s_6 s_5 s_4}_{\textit{product of transpositions}}.$$

Any product of adjacent transpositions which produces the permutation σ is called a **word** for σ. The generators $\{s_i \mid 1 \le i \le n-1\}$ are subject to the relations

$$\begin{aligned} &\text{(S1)} \quad s_i^2 = 1, && 1 \le i \le n-1, \\ &\text{(S2)} \quad s_i s_j = s_j s_i, && 1 \le i, j \le n-1, |i-j| > 1, \\ &\text{(S3)} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, && 1 \le i \le n-2. \end{aligned} \tag{2.0.2}$$

Relations (S2) and (S3) are called the **braid relations**.

A **matrix representation** of S_n is a group homomorphism $\rho : S_n \to \mathsf{GL}_r(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0. For each $\sigma \in S_n$, ρ maps σ to an invertible $r \times r$ matrix M_σ over $\mathbb{F}$ in a way which preserves the group operation of permutation composition. This means that for all $\sigma, \tau, \sigma\tau \in S_n$ where $\rho(\sigma) = M_\sigma$, $\rho(\tau) = M_\tau$, and $\rho(\sigma\tau) = M_{\sigma\tau}$,

$$M_\sigma \cdot M_\tau = M_{\sigma\tau}.$$

One matrix representation of S_n is the permutation representation, where for $\sigma \in S_n$ the representation puts an entry of 1 in the $\sigma(i)$-th row of the i-th column of $\rho(\sigma)$ and a 0 everywhere else.

Example 2.0.3. *In the permutation representation of S_5,*

$$\begin{pmatrix}1&2&3&4&5\\3&1&5&4&2\end{pmatrix} \mapsto \begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}1&2&3&4&5\\2&3&4&5&1\end{pmatrix} \mapsto \begin{pmatrix}0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix},$$

and we can see that this map accurately preserves permutation composition:

$$\begin{array}{ccccc}\begin{pmatrix}1&2&3&4&5\\3&1&5&4&2\end{pmatrix} & \circ & \begin{pmatrix}1&2&3&4&5\\2&3&4&5&1\end{pmatrix} & = & \begin{pmatrix}1&2&3&4&5\\1&5&4&2&3\end{pmatrix}\\ \updownarrow & & \updownarrow & & \updownarrow\\ \begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix} & \cdot & \begin{pmatrix}0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix} & = & \begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}.\end{array}$$

By Maschke's theorem (see, for instance, [13]), any representation can be decomposed into the direct sum of **irreducible** representations. As the name suggests, the irreducible representations are those that cannot be decomposed further into the direct sum of any smaller representations.

Each matrix representation of S_n has a corresponding **S_n-module**, which is the underlying vector space V of column vectors on which the matrix representation acts. For instance, if ρ is the permutation representation of S_n, then $V = \mathbb{F}^n = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is the standard basis.

Example 2.0.4. *For the permutation representation ρ,*

$$\rho\left(\begin{pmatrix}1&2&3&4&5\\3&1&5&4&2\end{pmatrix}\right) = \begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix} = \begin{pmatrix}\mathbf{v}_3 & \mathbf{v}_1 & \mathbf{v}_5 & \mathbf{v}_4 & \mathbf{v}_2\end{pmatrix}.$$

The action of this permutation on the associated module V is given as:

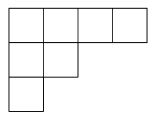
$$\sigma \cdot \mathbf{v}_1 = \mathbf{v}_3, \quad \sigma \cdot \mathbf{v}_2 = \mathbf{v}_1, \quad \sigma \cdot \mathbf{v}_3 = \mathbf{v}_5, \quad \sigma \cdot \mathbf{v}_4 = \mathbf{v}_4, \quad \sigma \cdot \mathbf{v}_5 = \mathbf{v}_2.$$

We will speak of irreducible representations and irreducible S_n-modules interchangeably.

3. YOUNG'S REPRESENTATIONS OF THE SYMMETRIC GROUP

Alfred Young, a pioneer of combinatorial representation theory, realized that the irreducible S_n-modules can be described by **integer partitions** of n. An integer partition $\lambda \vdash n$ is a non-increasing k-tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in (\mathbb{Z}^+)^k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The partition $\lambda \vdash n$ is associated with its corresponding **Young diagram**, a left-justified array of boxes where row i has λ_i boxes.

Example 3.0.1. *The partition $(4, 2, 1) \vdash 7$ has the Young diagram given below.*



The irreducible representations of S_n are indexed by the integer partitions $\{\lambda \mid \lambda \vdash n\}$ (see [8]), so the complete set of irreducible S_n-modules is given as $\{S_n^\lambda \mid \lambda \vdash n\}$.

Now, a **tableau** of shape $\lambda \vdash n$ is a filling of the boxes of λ with $1, 2, \ldots, n$. A related yet stronger notion is that of a **standard Young tableau** (also simply referred to as a standard tableau), which is a tableau in which the entries are increasing across rows and down columns.

Example 3.0.2. *For $\lambda = (4, 2, 1)$, a tableau t and a standard Young tableau T of shape λ.*

$$\mathsf{t} = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 5 & 7 \\ \hline 4 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \mathsf{T} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

A basis for S_n^λ can be indexed by the set of standard Young tableaux of shape λ, denoted $\mathcal{SYT}(\lambda)$ (see [12] and [5], Theorem 1.3).

Example 3.0.3. *The standard tableaux of shape $\lambda = (3, 2) \vdash 5$ are*

$$\mathcal{SYT}(\lambda) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \right\},$$

so a basis for $S_5^{(3,2)}$ can be indexed by the above set.

The number of standard Young tableaux of shape λ is denoted f_λ, and can be computed by the hook length formula defined by Young in [14].

For a given partition $\lambda \vdash n$, we emphasize two special standard tableaux — the column-reading tableau, denoted C, and the row-reading tableau, denoted R. They are created by filling in the boxes of λ with $1, 2, \ldots, n$ down the columns and the rows, respectively.

Example 3.0.4. *The column- and row-reading tableaux for $\lambda = (4, 3, 2)$.*

$$\mathsf{C} = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 3 & 6 \\ \cline{1-2}\end{array} \qquad \mathsf{R} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}$$

We can now describe two different bases for S_n^λ, both indexed by $\mathcal{SYT}(\lambda)$, which were defined by Young in the 1920's.

3.1 The Seminormal Representation

The basis for Young's seminormal representation of S_n^λ is given as $\mathcal{V}_\lambda = \{\mathbf{v}_\mathsf{T} \mid \mathsf{T} \in \mathcal{SYT}(\lambda)\}$, so that

$$S_n^\lambda = \mathbb{Q}\text{-span}\{\mathbf{v}_\mathsf{T}\},$$

where the action defined on s_i is

$$s_i \cdot \mathbf{v}_\mathsf{T} = \frac{1}{\delta_i(\mathsf{T})}\mathbf{v}_\mathsf{T} + \left(1 + \frac{1}{\delta_i(\mathsf{T})}\right)\mathbf{v}_{s_i(\mathsf{T})}. \tag{3.1.1}$$

We have $\mathbf{v}_{s_i(\mathsf{T})} = \mathbf{0}$ if $s_i(\mathsf{T})$ is non-standard, and $\delta_i(\mathsf{T})$ denotes the **axial distance** from i to $i+1$ in T. The axial distance is calculated as the minimum number of boxes one must traverse in T, moving only horizontally or vertically, to get from i to $i+1$ (see [12]). Distance is positive when moving up and to the right; it is negative when moving down and to the left.

Example 3.1.2.

$$\mathsf{S} = \young(1245,36,7) \qquad \mathsf{T} = \young(1246,35,7)$$

The axial distance from 3 to 4 in S *is* $\delta_3(\mathsf{S}) = 3$*, and the axial distance from 6 to 7 in* T *is* $\delta_6(\mathsf{T}) = -5$.

For ease of notation going forward, we will define

$$w_i(\mathsf{T}) := \frac{1}{\delta_i(\mathsf{T})} \qquad \text{and} \qquad m_i(\mathsf{T}) := 1 + \frac{1}{\delta_i(\mathsf{T})}. \tag{3.1.3}$$

Now, the seminormal action is only defined on the generating set $s_1, \ldots, s_{n-1}$, so to obtain the representation of any other permutation $\sigma \in S_n$, σ must first be expressed as the product of adjacent transpositions and then we must compute the representations of each of these transpositions.

Example 3.1.4. *To compute the representation of* $s_3 = (34)$ *in the seminormal basis for the partition shape* $\lambda = (3,2)$*, we consider the action of* s_3 *on each seminormal basis element (each standard tableau of shape* λ*):*

$$\begin{aligned}
s_3 \cdot \mathbf{v}_{\young(135,24)} &= -\mathbf{v}_{\young(135,24)} + 0\mathbf{v}_{\young(145,23)} \\
&= -\mathbf{v}_{\young(135,24)} \\
s_3 \cdot \mathbf{v}_{\young(125,34)} &= \mathbf{v}_{\young(125,34)} + 2\mathbf{v}_{\young(125,43)} \\
&= \mathbf{v}_{\young(125,34)} \\
s_3 \cdot \mathbf{v}_{\young(134,25)} &= \mathbf{v}_{\young(134,25)} + 2\mathbf{v}_{\young(143,25)} \\
&= \mathbf{v}_{\young(134,25)} \\
s_3 \cdot \mathbf{v}_{\young(124,35)} &= \frac{1}{3}\mathbf{v}_{\young(124,35)} + \frac{4}{3}\mathbf{v}_{\young(123,45)} \\
s_3 \cdot \mathbf{v}_{\young(123,45)} &= -\frac{1}{3}\mathbf{v}_{\young(123,45)} + \frac{2}{3}\mathbf{v}_{\young(124,35)}
\end{aligned}$$

So,

$$
[s_3]_{\mathcal{V}_\lambda} = \begin{array}{c} \\ \mathbf{v}_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} \end{array}
\begin{array}{c} \begin{array}{ccccc} \mathbf{v}_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} & \mathbf{v}_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} & \mathbf{v}_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} & \mathbf{v}_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} & \mathbf{v}_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} \end{array} \\
\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \frac{4}{3} & -\frac{1}{3} \end{pmatrix} \end{array}.
$$

Some notable properties of the seminormal representation are:

1. The coefficients in (3.1.1) are rational, so this representation is defined over $\mathbb{Q}$.
2. The action of s_i can be computed quickly.
3. The matrices of the generators are sparse, and close to diagonal. There is at most one off-diagonal entry in each row and column.
4. To compute the matrix of σ, we need the matrices for each term in the word $\sigma = s_{i_1} \dots s_{i_k}$. So, we must be able to decompose σ into the product of adjacent transpositins.

As a testament to its speed, the seminormal representation is often used in the discrete Fourier transform on S_n (see, for instance, [4]).

3.2 The Natural Representation

Young's natural representation has traditionally been defined using tabloids (in [13]) or using Young symmetrizers (in [5]). We use a different construction, following Ram [11]. Ram proves in Section 5 that we can define Young's natural basis in terms of the seminormal basis as follows: first, fix a partition $\lambda \vdash n$ and consider $\mathcal{SYT}(\lambda)$ along with the seminormal basis $\mathcal{V}_\lambda$. Then,

(1) For the column-reading tableau C, $\mathbf{n}_\mathsf{C} = \mathbf{v}_\mathsf{C}$ (up to a scalar, which we choose to be 1).

(2) For all other $\mathsf{T} \in \mathcal{SYT}(\lambda)$ and the permutation σ for which $\sigma(\mathsf{C}) = \mathsf{T}$,

$$\mathbf{n}_\mathsf{T} = \mathbf{n}_{\sigma(\mathsf{C})} =: \sigma \cdot \mathbf{n}_\mathsf{C}. \tag{3.2.1}$$

So, we compute the natural representation of π using the action

$$\pi \cdot \mathbf{n}_\mathsf{T} = \mathbf{n}_{\pi(\mathsf{T})} \tag{3.2.2}$$

for any $\pi \in S_n$. When $\pi(\mathsf{T})$ is standard, then $\mathbf{n}_{\pi(\mathsf{T})}$ stays in the basis. When $\pi(\mathsf{T})$ is non-standard, though, $\mathbf{n}_{\pi(\mathsf{T})}$ must be re-expressed recursively in the basis of standard tableaux. There are multiple ways to do this, from Garnir relations (see [13]) to tableau intersection (see [5]), which we will not discuss in detail here.

Ram shows that the Garnir relations are satisfied under this definition, so that $\mathcal{N}_\lambda = \{\mathbf{n}_\mathsf{T} \mid \mathsf{T} \in \mathcal{SYT}(\lambda)\}$ is a natural basis for S_n^λ.

Example 3.2.3. *To compute the representation of $s_3 = (34)$ in the natural basis, we calculate the action of s_3 on each natural basis element:*

$$
\begin{aligned}
s_3 \cdot \mathbf{n}_{\ytableaushort{135,24}} &= \mathbf{n}_{\ytableaushort{145,23}} \\
&= -\mathbf{n}_{\ytableaushort{135,24}} \\
s_3 \cdot \mathbf{n}_{\ytableaushort{125,34}} &= \mathbf{n}_{\ytableaushort{125,43}} \\
&= \mathbf{n}_{\ytableaushort{125,34}} - \mathbf{n}_{\ytableaushort{135,24}} \\
s_3 \cdot \mathbf{n}_{\ytableaushort{134,25}} &= \mathbf{n}_{\ytableaushort{143,25}} \\
&= \mathbf{n}_{\ytableaushort{134,25}} - \mathbf{n}_{\ytableaushort{135,24}} \\
s_3 \cdot \mathbf{n}_{\ytableaushort{124,35}} &= \mathbf{n}_{\ytableaushort{123,45}} \\
s_3 \cdot \mathbf{n}_{\ytableaushort{123,45}} &= \mathbf{n}_{\ytableaushort{124,35}}
\end{aligned}
$$

For the second and fourth basis elements above, we must re-express the right-hand side in terms of standard tableaux when we act with s_3, since we obtain a non-standard tableau. We omit this computation as it is unimportant to the task at hand. With this, we get,

$$
[s_3]_{\mathcal{N}_\lambda} =
\begin{array}{c}
\\
\mathbf{n}_{\ytableaushort{135,24}} \\
\mathbf{n}_{\ytableaushort{125,34}} \\
\mathbf{n}_{\ytableaushort{134,25}} \\
\mathbf{n}_{\ytableaushort{124,35}} \\
\mathbf{n}_{\ytableaushort{123,45}}
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
\mathbf{n}_{\ytableaushort{135,24}} & \mathbf{n}_{\ytableaushort{125,34}} & \mathbf{n}_{\ytableaushort{134,25}} & \mathbf{n}_{\ytableaushort{124,35}} & \mathbf{n}_{\ytableaushort{123,45}}
\end{array} \\
\begin{pmatrix}
-1 & -1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}.
$$

Some notable properties of the natural representation are:

1. The coefficients in (6.3.12) are integral even when $\pi(\mathsf{T})$ is non-standard, so this representation is defined over $\mathbb{Z}$.

2. The action is simple to state and $\pi \cdot \mathbf{n}_{\mathsf{T}}$ is immediately given when $\pi(\mathsf{T})$ is another standard tableau.

3. We require a recursive straightening algorithm to express $\mathbf{n}_{\pi(\mathsf{T})}$ in terms of standard tableaux when $\pi(\mathsf{T})$ is non-standard.

The complementary upsides and downsides of the seminormal and natural bases — the seminormal action is straightforward to compute, but only on the adjacent transpositions, while the natural action can be tedious to compute, but is defined on the entire group — further motivate the desire for a change-of-basis matrix between the two representations. In the following chapter, we formalize the combinatorial structure of $\mathcal{SYT}(\lambda)$ with a graph that will allow us to compute entries in this matrix.

4. ORDERING $\mathcal{SYT}(\lambda)$

4.1 Bruhat Order on S_n

Following Björner and Wachs [3], we give a partial ordering on $\mathcal{SYT}(\lambda)$ that is inherited from a well-known partial ordering on S_n, called Bruhat order. In this section, we define Bruhat order and a closely related partial ordering, weak (left) Bruhat order. We describe these partial orderings by their **covering relations**. For a poset $(S, \leq)$ and $s_1, s_2 \in S$ we say that s_1 covers s_2, denoted $s_1 \longrightarrow s_2$, if $s_1 \leq s_2$ and there does not exist $s_3 \in S$ such that $s_1 \leq s_3 \leq s_2$. It follows that $s \leq t$ for $s, t \in S$ if and only if there exist $s_i \in S$ such that

$$s = s_0 \longrightarrow s_1 \longrightarrow \cdots \longrightarrow s_k = t,$$

so we say that the covering relation generates the partial order. Equivalently, the partial ordering $\leq$ is the transitive and reflexive closure of the covering relation $\longrightarrow$.

Both of the partial orderings with which we are concerned depend on the notion of reduced words. A **reduced word** for $\sigma \in S_n$ is a product of adjacent transpositions $s_{i_1}s_{i_2}\cdots s_{i_k}$ such that

1. $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ (that is, $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a word for σ) and
2. σ cannot be expressed as a product of fewer than k adjacent transpositions.

Reduced words are not unique, but the length of a reduced word is. If $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced word, then we say that σ has **length** k, denoted $\ell(\sigma) = k$.

Bruhat order is a partial order on the symmetric group S_n generated by the covering relation

$$\sigma \xrightarrow{i,j} \tau \quad \text{if} \quad \tau = (i,j)\sigma \quad \text{for} \quad i < j$$

for all $\sigma, \tau \in S_n$. So, we say that σ precedes τ in Bruhat order, denoted $\sigma \leq \tau$, if there exist $\sigma_d \in S_n$ and transpositions (i_d, j_d) such that

$$\sigma = \sigma_0 \xrightarrow{i_1,j_1} \sigma_1 \xrightarrow{i_2,j_2} \cdots \xrightarrow{i_k,j_k} \sigma_k = \tau.$$

Permutations can also be compared in Bruhat order based on their reduced words. For $\sigma, \tau \in S_n$ where $\tau = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced word, $\sigma \leq \tau$ if there exists a reduced word $\sigma = s_{i_a}\cdots s_{i_j}$ such that $1 \leq a < \cdots < j \leq k$. This is called the **subword property** (see [2] Theorem 2.2.2). In other words, $\sigma \leq \tau$ if a subword of some reduced word for τ is a word for σ.

Weak (left) Bruhat order, which we will simply call weak Bruhat order, is a closely related partial ordering generated by the covering relation

$$\sigma \xrightarrow{i} \tau \quad \text{if} \quad \tau = s_i\sigma \quad \text{and} \quad \ell(\tau) = \ell(\sigma) + 1$$

for $\sigma, \tau \in S_n$. We again say that σ precedes τ in weak Bruhat order (denoted $\sigma \leq_W \tau$) if there exist $\sigma_d \in S_n$ and i_d such that

$$\sigma = \sigma_0 \xrightarrow{i_1} \sigma_1 \xrightarrow{i_2} \cdots \xrightarrow{i_k} \sigma_k = \tau,$$

or equivalently if $\tau = s_{i_k} \cdots s_{i_1}\sigma$ for some sequence of adjacent transpositions such that $\ell(\tau) = \ell(\sigma) + k$. Weak Bruhat order is a special case of Bruhat order, so

$$\sigma \leq_W \tau \Rightarrow \sigma \leq \tau.$$

A poset $(S, \leq)$ is given graphically by a **Hasse diagram**, a graph whose vertices are the elements of S and whose edges are given by the covering relations. In this paper, we are interested in the Hasse diagram for weak Bruhat order on S_n. This is the graph H whose vertex set is S_n where there is an (undirected) edge between $\sigma, \tau \in V(H)$ if and only if $\sigma = s_i\tau$ (equivalently $\tau = s_i\sigma$) for $1 \leq i \leq n-1$. Note that for any $\sigma \in S_n$, all possible reduced words for σ are encoded in the Hasse diagram as the possible paths of shortest length from the identity permutation e (which precedes every other permutation in weak Bruhat order) to σ.

We also rely on the notion of an **interval** in the Hasse diagram H. For $\sigma, \tau \in S_n$, we denote an interval in the Hasse diagram as $[\sigma, \tau]$, which is the subgraph of H induced by the vertices

$$[\sigma, \tau] = \{\gamma \in S_n \mid \sigma \leq_W \gamma \leq_W \tau\}. \tag{4.1.1}$$

4.2 Bruhat Order on $\mathcal{SYT}(\lambda)$

We can extend the notions of Bruhat order and weak Bruhat order to define analogous partial orderings on $\mathcal{SYT}(\lambda)$.

Bruhat order on $\mathcal{SYT}(\lambda)$ is a partial ordering generated by the covering relation

$$\mathsf{S} \xrightarrow{i,j} \mathsf{T} \quad \text{if} \quad \mathsf{T} = (i,j)(\mathsf{S}) \quad \text{for} \quad i < j \quad \text{and } i \text{ is in a lower row of } \mathsf{S} \text{ than } j$$

for all $\mathsf{S}, \mathsf{T} \in \mathcal{SYT}(\lambda)$. So, we say that S precedes T in Bruhat order on $\mathcal{SYT}(\lambda)$, denoted $\mathsf{S} \leq \mathsf{T}$, if there exist $\mathsf{S}_d \in \mathcal{SYT}(\lambda)$ and transpositions (i_d, j_d) such that

$$\mathsf{S} = \mathsf{S}_0 \xrightarrow{i_1,j_1} \mathsf{S}_1 \xrightarrow{i_2,j_2} \cdots \xrightarrow{i_k,j_k} \mathsf{S}_k = \mathsf{T}.$$

Similarly, **weak Bruhat order on** $\mathcal{SYT}(\lambda)$ is generated by the covering relation

$$\mathsf{S} \xrightarrow{i} \mathsf{T} \quad \text{if} \quad \mathsf{T} = s_i(\mathsf{S}) \quad \text{and } i \text{ is in a lower row of } \mathsf{S} \text{ than } i+1$$

for all $\mathsf{S}, \mathsf{T} \in \mathcal{SYT}(\lambda)$. We again say that S precedes T in weak Bruhat order on $\mathcal{SYT}(\lambda)$, denoted $\mathsf{S} \leq_W \mathsf{T}$, if there exist $\mathsf{S}_d \in \mathcal{SYT}(\lambda)$ and i_d such that

$$\mathsf{S} = \mathsf{S}_0 \xrightarrow{i_1} \mathsf{S}_1 \xrightarrow{i_2} \cdots \xrightarrow{i_k} \mathsf{S}_k = \mathsf{T},$$

or equivalently if $\mathsf{T} = s_{i_k} \cdots s_{i_1}(\mathsf{S})$ for some minimal-length sequence of adjacent transpositions.

Now, Björner and Wachs [3] establish a correspondence between S_n and $\mathcal{SYT}(\lambda)$ by defining the **word** of a tableau $\mathsf{T} \in \mathcal{SYT}(\lambda)$, denoted w_{T}, to be the permutation (in two-line notation) given by

$$w_{\mathsf{T}} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \mathsf{T}(\text{box}_1) & \mathsf{T}(\text{box}_2) & \cdots & \mathsf{T}(\text{box}_n) \end{pmatrix},$$

where $\mathsf{T}(\text{box}_i)$ denotes the entry in the i-th box of T and the boxes are read in column order.

Example 4.2.1. *In S_7,*

$$\mathsf{T} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

has the word

$$w_{\mathsf{T}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 5 & 2 & 7 & 3 & 6 \end{pmatrix} = (24)(3576) = s_3 s_2 s_4 s_3 s_6 s_5.$$

Importantly, $w_{\mathsf{C}} = e$, the identity permutation, and w_{R} is the permutation of longest length in $\{w_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{SYT}(\lambda)\}$. Furthermore, notice that w_{T} is the permutation for which $w_{\mathsf{T}}(\mathsf{C}) = \mathsf{T}$. By associating standard tableaux with permutations in this way, we formalize the similarities between our partial orderings on S_n and on $\mathcal{SYT}(\lambda)$.

Proposition 4.2.2. *For $\mathsf{S}, \mathsf{T} \in \mathcal{SYT}(\lambda)$ and $\sigma, \tau \in S_n$, $\mathsf{S} \leq \mathsf{T}$ (S precedes T in Bruhat order) if and only if $w_{\mathsf{S}} \leq w_{\mathsf{T}}$. Similarly, $\mathsf{S} \leq_W \mathsf{T}$ (S precedes T in weak Bruhat order) if and only if $w_{\mathsf{S}} \leq_W w_{\mathsf{T}}$.*

This follows from the following theorem of Björner and Wachs.

Theorem 4.2.3. *([3] Theorem 7.2) The map $\mathsf{T} \mapsto w_{\mathsf{T}}$ is a bijection from weak Bruhat order on $\mathcal{SYT}(\lambda)$ to the interval $[w_{\mathsf{C}}, w_{\mathsf{R}}] = [e, w_{\mathsf{R}}]$ in weak Bruhat order on S_n.*

This theorem applies to Bruhat order more generally, but we will be most interested in the interval $[e, w_{\mathsf{R}}]$ in weak Bruhat order.

4.3 The Crystal Graph Γ_λ

For $\lambda \vdash n$ we define a graph Γ_λ on $\mathcal{SYT}(\lambda)$, which we refer to as a **crystal graph** for its connection to the crystal graphs of Kashiwara and Lusztig for quantum groups and semisimple Lie algebras (see [9] and [10]).

Definition 4.3.1. *For a partition $\lambda \vdash n$, we let Γ_λ be the graph with vertex set $\mathcal{SYT}(\lambda)$ and an undirected edge connecting $\mathsf{S}, \mathsf{T} \in \mathcal{SYT}(\lambda)$ if there exists $1 \leq i \leq n-1$ such that $s_i(\mathsf{S}) = \mathsf{T}$; this edge is given the label s_i. Note that $s_i(\mathsf{S}) = \mathsf{T}$ if and only if $s_i(\mathsf{T}) = \mathsf{S}$.*

Observe that this graph is the Hasse diagram for weak Bruhat order on $\mathcal{SYT}(\lambda)$.

Now, for $\mathsf{T} \in \mathcal{SYT}(\lambda)$ and $1 \leq i \leq n-1$ there is at most one edge labeled s_i incident to T in Γ_λ. Thus, a **path** from C to T in Γ_λ can be identified uniquely with a sequence of edge labels $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k}) \in \{s_1, \ldots, s_{n-1}\}^k$.

To further drive home the structural similarities between S_n and $\mathcal{SYT}(\lambda)$, we provide the following propositions. We substitute i_j for s_{i_j} as edge labels to save ink. (Note that these statements follow from the poset structure of Γ_λ. So, they are ultimately unnecessary as they simply solidify existing notions, but they are left for the interested reader.)

Proposition 4.3.2. *Paths obey the braid relations (S2) and (S3), for which we provide analogous definitions in terms of Γ_λ.*

(B2) *If $|i-j| > 1$ and $p = (a_1, \ldots, a_t, i, j, a_{t+3}, \ldots, a_k)$ is a path from C to T in Γ_λ, then $p' = (a_1, \ldots, a_t, j, i, a_{t+3}, \ldots, a_k)$ is a path from C to T in Γ_λ.*

(B3) *If* $p = (a_1, \ldots, a_t, i, i+1, i, a_{t+4}, \ldots, a_k)$ *is a path from* C *to* T *in* Γ_λ*, then* $p' = (a_1, \ldots, a_t, i+1, i, i+1, a_{t+4}, \ldots, a_k)$ *is a path from* C *to* T *in* Γ_λ*.*

Recall that for $\sigma \in S_n$, a **reduced word** for σ is a minimal-length product $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ of adjacent transpositions. Reduced words are not necessarily unique but if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} = s_{j_1} s_{j_2} \cdots s_{j_t}$ then $k = t = \ell(\sigma)$. Any word for σ can be converted to a reduced word by applying a sequence of the relations in (2.0.2), and any two reduced words can be transformed into one another using a sequence of braid relations (S2) and (S3).

Proposition 4.3.3. *If* $\mathsf{T} \in \mathcal{SYT}(\lambda)$*, then* $p = (i_1, \ldots, i_k)$ *is a minimal-length path from* C *to* T *in* Γ_λ *if and only if* $s_{i_k} \cdots s_{i_2} s_{i_1}$ *is a reduced word for the permutation* σ *determined by* $\sigma(\mathsf{C}) = \mathsf{T}$*. Thus, the number of minimal-length paths from* C *to* T *equals the number of reduced words for* σ*.*

The next proposition gives an algebraic underpinning of the notion of the **depth** of $\mathsf{T} \in \mathcal{SYT}(\lambda)$ in Γ_λ, which is traditionally defined as the length of the shortest path from C to T.

Proposition 4.3.4. *For all* $\mathsf{T} \in \mathcal{SYT}(\lambda)$*, the depth of* T *in* Γ_λ *is given by* $\ell(w_{\mathsf{T}})$*.*

We can demonstrate these ideas in the crystal graph $\Gamma_{(3,2,1)}$. For instance, the tableau T (labeled below) has

$$w_{\mathsf{T}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 6 & 4 \end{pmatrix}.$$

There are five reduced words for w_{T}, namely

$$w_{\mathsf{T}} = s_4 s_5 s_2 s_3 = s_4 s_2 s_5 s_3 = s_4 s_2 s_3 s_5 = s_2 s_4 s_3 s_5 = s_2 s_4 s_5 s_3.$$

There are five paths of shortest length from C to T, each of which corresponds directly to one of the reduced words listed above. Any of these paths can be transformed into another path by applying a combination of (B2) and (B3). Furthermore, each reduced word has length 4 ($\ell(w_{\mathsf{T}}) = 4$), and T has depth 4 in $\Gamma_{(3,2,1)}$.

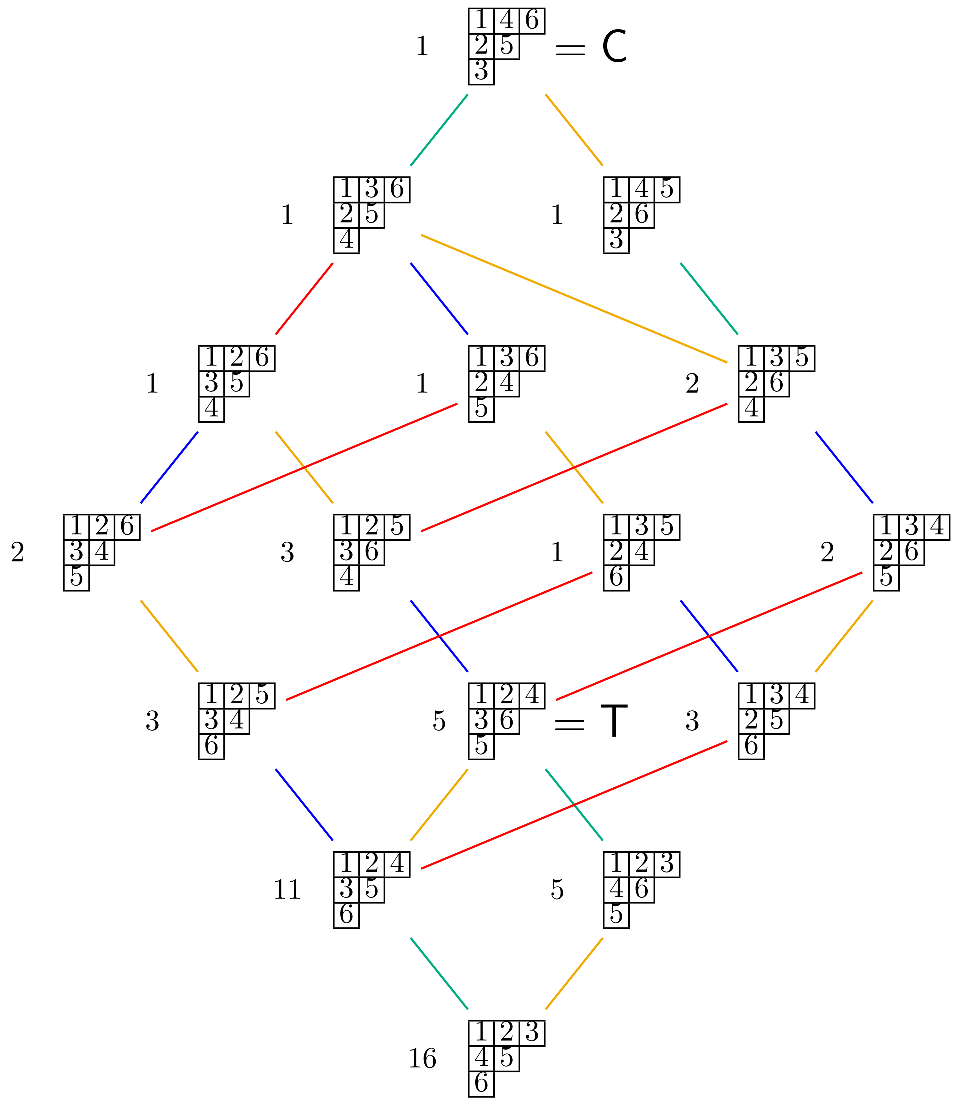


Figure 4.1: The crystal graph $\Gamma_{(3,2,1)}$ with each tableau labeled by the number of reduced expressions for its word w.

5. CHANGING BASIS

5.1 From Seminormal to Natural

Now, we describe how to use the graph Γ_λ to compute natural basis elements in the seminormal basis. This will determine the entries in the matrix $\mathcal{A}_\lambda$ which satisfies

$$\mathcal{A}_\lambda^{-1}[\sigma]_{\mathcal{V}_\lambda}\mathcal{A}_\lambda = [\sigma]_{\mathcal{N}_\lambda}$$

for all $\sigma \in S_n$.

Even without a general formula, we can solve for this matrix numerically. For instance, the change-of-basis matrix for the partition $\lambda = (3,2) \vdash 5$ is given by:

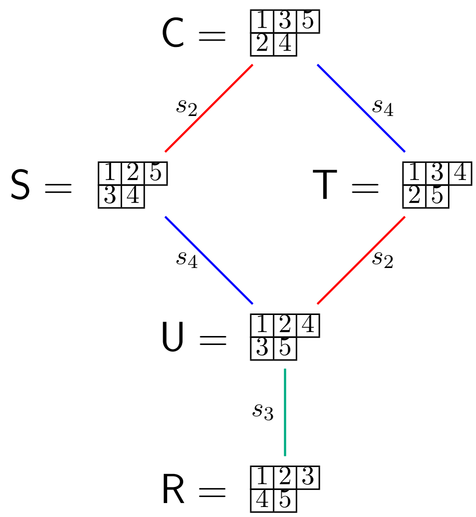
$$\mathcal{A}_{(3,2)} = \begin{array}{c} \\ \mathbf{v}_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} \\ \mathbf{v}_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} \end{array} \begin{array}{c} \begin{matrix} \mathbf{n}_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} & \mathbf{n}_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} & \mathbf{n}_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} & \mathbf{n}_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} & \mathbf{n}_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} \end{matrix} \\ \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \cdot & \frac{3}{2} & \cdot & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \frac{9}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix} \end{array}. \tag{5.1.1}$$

Note that this matrix has rational entries, and it is upper triangular under this ordering of rows and columns. We give a method for computing entries of $\mathcal{A}_\lambda$ using Γ_λ, first providing an example which illustrates our method.

Example 5.1.2. *Let $\lambda = (3,2)$ and label the tableaux as below. To compute $\mathbf{n}_{\mathsf{R}}$ in the seminormal basis, we first consider the permutation $\sigma \in S_n$ for which $\sigma(\mathsf{C}) = \mathsf{R}$ and a word for σ. The word $\sigma = s_3s_4s_2$ is one possibility. Note that this is also a reduced word for w_{R}, and this corresponds directly to a (minimal-length) path in Γ_λ from C to R given by the edge sequence (s_2, s_4, s_3). Now,*

$$\begin{aligned}
\mathsf{R} &= s_3s_4s_2(\mathsf{C}) \\
\mathbf{n}_{\mathsf{R}} &= \mathbf{n}_{s_3s_4s_2(\mathsf{C})} = s_3s_4s_2 \cdot \mathbf{n}_{\mathsf{C}} = s_3s_4s_2 \cdot \mathbf{v}_{\mathsf{C}} = s_3s_4\left(s_2 \cdot \mathbf{v}_{\mathsf{C}}\right) \\
&= s_3s_4\left(w_2(\mathsf{C})\mathbf{v}_{\mathsf{C}} + m_2(\mathsf{C})\mathbf{v}_{\mathsf{S}}\right) \\
&= s_3\left(w_2(\mathsf{C})s_4 \cdot \mathbf{v}_{\mathsf{C}} + m_2(\mathsf{C})s_4 \cdot \mathbf{v}_{\mathsf{S}}\right) \\
&= s_3(w_2(\mathsf{C})\left(w_4(\mathsf{C})\mathbf{v}_{\mathsf{C}} + m_4(\mathsf{C})\mathbf{v}_{\mathsf{T}}\right) + m_2(\mathsf{C})\left(w_4(\mathsf{S})\mathbf{v}_{\mathsf{S}} + m_4(\mathsf{S})\mathbf{v}_{\mathsf{U}}\right)) \\
&= w_2(\mathsf{C})\, w_4(\mathsf{C})\, s_3 \cdot \mathbf{v}_{\mathsf{C}} + m_2(\mathsf{C})\, w_4(\mathsf{S})\, s_3 \cdot \mathbf{v}_{\mathsf{S}} + w_2(\mathsf{C})\, m_4(\mathsf{C})\, s_3 \cdot \mathbf{v}_{\mathsf{T}} + m_2(\mathsf{C})\, m_4(\mathsf{S})\, s_3 \cdot \mathbf{v}_{\mathsf{U}} \\
&= w_2(\mathsf{C})\, w_4(\mathsf{C})\, (w_3(\mathsf{C})\mathbf{v}_{\mathsf{C}} + \mathbf{0}) + m_2(\mathsf{C})\, w_4(\mathsf{S})\, (w_3(\mathsf{S})\mathbf{v}_{\mathsf{S}} + \mathbf{0}) + w_2(\mathsf{C})\, m_4(\mathsf{C})\, (w_3(\mathsf{T})\mathbf{v}_{\mathsf{T}} + \mathbf{0}) \\
&\quad + m_2(\mathsf{C})\, m_4(\mathsf{S})\, (w_3(\mathsf{U})\mathbf{v}_{\mathsf{U}} + m_3(\mathsf{U})\mathbf{v}_{\mathsf{R}}) \\
&= w_2(\mathsf{C})\, w_4(\mathsf{C})\, w_3(\mathsf{C})\mathbf{v}_{\mathsf{C}} + m_2(\mathsf{C})\, w_4(\mathsf{S})\, w_3(\mathsf{S})\mathbf{v}_{\mathsf{S}} + w_2(\mathsf{C})\, m_4(\mathsf{C})\, w_3(\mathsf{T})\mathbf{v}_{\mathsf{T}} \\
&\quad + m_2(\mathsf{C})\, m_4(\mathsf{S})\, w_3(\mathsf{U})\mathbf{v}_{\mathsf{U}} + m_2(\mathsf{C})\, m_4(\mathsf{S})\, m_3(\mathsf{U})\mathbf{v}_{\mathsf{R}}
\end{aligned}$$

Notice that



1. *The coefficients in this expansion are derived from the seminormal action.*
2. *The coefficients are influenced by the chosen path* (s_2, s_4, s_3) *(that is, each coefficient has a term with a subscript of 2, 4, and 3).*
3. *The subset of* m_i *terms attached to each* $\mathbf{v}_\mathsf{T}$ *describes a path from* C *to* T*.*

So,

$$\begin{aligned}
\mathbf{n}_\mathsf{R} &= w_2(\mathsf{C})\, w_4(\mathsf{C})\, w_3(\mathsf{C})\mathbf{v}_\mathsf{C} + m_2(\mathsf{C})\, w_4(\mathsf{S})\, w_3(\mathsf{S})\mathbf{v}_\mathsf{S} + w_2(\mathsf{C})\, m_4(\mathsf{C})\, w_3(\mathsf{T})\mathbf{v}_\mathsf{T} \\
&\quad + m_2(\mathsf{C})\, m_4(\mathsf{S})\, w_3(\mathsf{U})\mathbf{v}_\mathsf{U} + m_2(\mathsf{C})\, m_4(\mathsf{S})\, m_3(\mathsf{U})\mathbf{v}_\mathsf{R} \\
&= \frac{1}{\delta_2(\mathsf{C})} \cdot \frac{1}{\delta_4(\mathsf{C})} \cdot \frac{1}{\delta_3(\mathsf{C})}\mathbf{v}_\mathsf{C} + \left(1 + \frac{1}{\delta_2(\mathsf{C})}\right) \cdot \frac{1}{\delta_4(\mathsf{S})} \cdot \frac{1}{\delta_3(\mathsf{S})}\mathbf{v}_\mathsf{S} + \frac{1}{\delta_2(\mathsf{C})} \cdot \left(1 + \frac{1}{\delta_4(\mathsf{C})}\right) \cdot \frac{1}{\delta_3(\mathsf{T})}\mathbf{v}_\mathsf{T} \\
&\quad + \left(1 + \frac{1}{\delta_2(\mathsf{C})}\right) \cdot \left(1 + \frac{1}{\delta_4(\mathsf{S})}\right) \cdot \frac{1}{\delta_3(\mathsf{U})}\mathbf{v}_\mathsf{U} + \left(1 + \frac{1}{\delta_2(\mathsf{C})}\right) \cdot \left(1 + \frac{1}{\delta_4(\mathsf{S})}\right) \cdot \left(1 + \frac{1}{\delta_3(\mathsf{U})}\right)\mathbf{v}_\mathsf{R} \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot (-1)\mathbf{v}_\mathsf{C} + \frac{3}{2} \cdot \frac{1}{2} \cdot 1\mathbf{v}_\mathsf{S} + \frac{1}{2} \cdot \frac{3}{2} \cdot 1\mathbf{v}_\mathsf{T} + \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{3}\mathbf{v}_\mathsf{U} + \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{4}{3}\mathbf{v}_\mathsf{R} \\
&= -\frac{1}{4}\mathbf{v}_\mathsf{C} + \frac{3}{4}\mathbf{v}_\mathsf{S} + \frac{3}{4}\mathbf{v}_\mathsf{T} + \frac{3}{4}\mathbf{v}_\mathsf{U} + 3\mathbf{v}_\mathsf{R}
\end{aligned}$$

computes the R*-th column of* (5.1.1)*.*

By abstracting this method, we can find a general formula for $\mathcal{A}_\lambda$. We first fix a path to T — in the previous example, this path was given by $\pi = (s_2, s_4, s_3)$.

Definition 5.1.3. *If* $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k}) \in \{s_1, \ldots, s_{n-1}\}^k$ *is a path to* $\mathsf{T} \in \mathcal{SYT}(\lambda)$*, then a* ***subpath of*** π *is a* k*-tuple* $\omega = (z_1, z_2, \ldots z_k)$ *where either* $z_j = e$ *or* $z_j = s_{i_j}$ *such that* ω *defines a path in* Γ_λ*, where an entry of* e *corresponds to a null move.*

In the previous example, the subpaths of π are $(s_2, e, e), (e, s_4, e), (s_2, s_4, e)$, and (s_2, s_4, s_3). The sequence (e, e, s_3) is **not** a subpath, though, because C is not adjacent to an s_3 edge.

Any subpath ω of π gives rise to a unique sequence of standard Young tableaux ($\mathsf{S}_1 = \mathsf{C}, \mathsf{S}_2, \ldots,$ $\mathsf{S}_{k+1} =: \mathsf{S}$) such that for each $1 \leq j \leq k$, either $\mathsf{S}_{j+1} = \mathsf{S}_j$ or $\mathsf{S}_{j+1} = s_{i_j}(\mathsf{S}_j)$. We say that ω

terminates at S. Considering the above example, the subpath (e, s_4, e) has the tableau sequence $(\mathsf{C}, \mathsf{C}, \mathsf{T}, \mathsf{T})$, and this subpath terminates at T.

Definition 5.1.4. *Given a path $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k})$, every subpath $\omega = (z_1, z_2, \ldots, z_k)$ of π with tableau sequence $(\mathsf{S}_1 = \mathsf{C}, \mathsf{S}_2, \ldots, \mathsf{S}_{k+1})$ has an associated weight. We assign a cost to each component in the subpath, given piecewise as*

$$\mathsf{c}_j = \begin{cases} w_{i_j}(\mathsf{S}_j) & \text{if } z_j = e, \\ m_{i_j}(\mathsf{S}_j) & \text{if } z_j = s_{i_j}. \end{cases} \tag{5.1.5}$$

The weight on ω, denoted $\mathsf{wt}_\pi(\omega)$, is found by multiplying together the constituent costs along the subpath:

$$\mathsf{wt}_\pi(\omega) = \prod_{j=1}^{k} \mathsf{c}_j \tag{5.1.6}$$

By definition we let $\mathsf{wt}_\emptyset(\emptyset) = 1$.

Notice that the coefficient on any particular term in the subpath is derived directly from the seminormal action. With this new terminology, we now conjecture a general formula for $\mathcal{A}_\lambda$.

Theorem 5.1.7. *For each $\lambda \vdash n$ and $\mathsf{T} \in \mathcal{SYT}(\lambda)$, the expression of the natural basis element $\mathbf{n}_\mathsf{T}$ in terms of the seminormal basis of S_n^λ is given by*

$$\mathbf{n}_\mathsf{T} = \sum_{\mathsf{S} \leq \mathsf{T}} A_{\mathsf{S},\mathsf{T}} \mathbf{v}_\mathsf{S} \tag{5.1.8}$$

where the coefficients $A_{\mathsf{S},\mathsf{T}} \in \mathbb{Q}$ are given by

$$A_{\mathsf{S},\mathsf{T}} = \sum \mathsf{wt}_\pi(\omega), \tag{5.1.9}$$

where we fix a path π to T and sum over all subpaths ω of π which terminate at S.

Before we prove this theorem, notice that $A_{\mathsf{S},\mathsf{T}}$ does not depend on the particular path to T we use.

Lemma 5.1.10. *The coefficient $A_{\mathsf{S},\mathsf{T}}$ is independent of the chosen path to T.*

Proof. Fix a partition $\lambda \vdash n$ and let $\mathsf{T} \in \mathcal{SYT}(\lambda)$. Let $s_{i_k} \cdots s_{i_2} s_{i_1}$ be any word for w_T, meaning $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k})$ is a path to T in Γ_λ. Then,

$$\begin{aligned} \mathbf{n}_\mathsf{T} &= \mathbf{n}_{s_{i_k} \cdots s_{i_2} s_{i_1}(\mathsf{C})} = s_{i_k} \cdots s_{i_2} s_{i_1} \cdot \mathbf{n}_\mathsf{C} \\ &= s_{i_k} \cdots s_{i_2} s_{i_1} \cdot \mathbf{v}_\mathsf{C}. \end{aligned}$$

Now, as we saw in Example 5.1.2, we compute the expansion of $\mathbf{n}_\mathsf{T}$ by acting one-by-one with the adjacent transpositions s_{i_j} on the right-hand side. For a tableau $\mathsf{S} \in \mathcal{SYT}(\lambda)$ in this expanded expression, the coefficient on $\mathbf{v}_\mathsf{S}$ will be independent of π since there is a unique way to express $\mathbf{n}_\mathsf{T}$ in the seminormal basis. We claim that this coefficient is $A_{\mathsf{S},\mathsf{T}}$ (as we defined it) regardless of the path to T we initially choose. This is because, according to the seminormal action, the only terms which contribute to $\mathbf{v}_\mathsf{S}$ are those which contain a subset of m_i coefficients which define a path to S. These terms are in bijection with the subpaths of π which terminate at S, and the weights on each of these subpaths are, by definition, computed by the product of w_i's and m_i's in the relevant term. So, $A_{\mathsf{S},\mathsf{T}}$ is independent of the particular path to T we choose. □

Before proving Theorem 5.1.7, let us provide one more illustrative example of how to calculate these coefficients in practice:

Example 5.1.11. *If we want to compute $A_{\mathsf{S},\mathsf{T}}$ in Figure 5.1, we first find a path to T. One such path is given by $\pi = (s_3, s_2, s_5, s_4, s_3)$. Then, we consider all subpaths ω of π which terminate at S. There are two such subpaths, namely*

$$\omega_1 = (s_3, e, s_5, e, e) \qquad \text{and} \qquad \omega_2 = (e, e, s_5, e, s_3).$$

To compute the weights on these subpaths, we find the coefficient c_j for each component and multiply them together:

π	s_3		s_2		s_5		s_4		s_3	
ω_1	s_3		e		s_5		e		e	
	$\begin{smallmatrix}1&4&6\\2&5&\\3&&\end{smallmatrix}$		$\begin{smallmatrix}1&3&6\\2&5&\\4&&\end{smallmatrix}$		$\begin{smallmatrix}1&3&6\\2&5&\\4&&\end{smallmatrix}$		S		S	S
$\mathsf{wt}_\pi(\omega_1)$	$\left(1+\frac{1}{3}\right)$	$\cdot$	$\frac{1}{2}$	$\cdot$	$\left(1+\frac{1}{2}\right)$	$\cdot$	$\frac{1}{4}$	$\cdot$	$-\frac{1}{3}$	$= -\frac{1}{12}$
ω_2	e		e		s_5		e		s_3	
	$\begin{smallmatrix}1&4&6\\2&5&\\3&&\end{smallmatrix}$		$\begin{smallmatrix}1&4&6\\2&5&\\3&&\end{smallmatrix}$		$\begin{smallmatrix}1&4&6\\2&5&\\3&&\end{smallmatrix}$		$\begin{smallmatrix}1&4&5\\2&6&\\3&&\end{smallmatrix}$		$\begin{smallmatrix}1&4&5\\2&6&\\3&&\end{smallmatrix}$	S
$\mathsf{wt}_\pi(\omega_2)$	$\frac{1}{3}$	$\cdot$	-1	$\cdot$	$\left(1+\frac{1}{2}\right)$	$\cdot$	1	$\cdot$	$\left(1+\frac{1}{3}\right)$	$= -\frac{2}{3}$

So,

$$A_{\mathsf{S},\mathsf{T}} = \mathsf{wt}_\pi(\omega_1) + \mathsf{wt}_\pi(\omega_2) = -\frac{1}{12} - \frac{2}{3} = -\frac{3}{4}.$$

Now, the proof of Theorem 5.1.7 will rely on three key facts:

$$\begin{aligned}
&\text{(T0)} \quad \text{For the column-reading tableau } \mathsf{C}, \mathbf{n}_\mathsf{C} = \mathbf{v}_\mathsf{C}.\\
&\text{(T1)} \quad \text{If } \sigma(\mathsf{C}) = \mathsf{T}, \text{then } \mathbf{n}_\mathsf{T} = \sigma \cdot \mathbf{n}_\mathsf{C}.\\
&\qquad\quad \text{(the defining action of the natural representation)}\\
&\text{(T2)} \quad s_i \cdot \mathbf{v}_\mathsf{T} = w_i(\mathsf{T})\mathbf{v}_\mathsf{T} + m_i(\mathsf{T})\mathbf{v}_{s_i(\mathsf{T})}.\\
&\qquad\quad \text{(the defining action of the seminormal representation)}
\end{aligned} \tag{5.1.12}$$

Proof. (of Theorem 5.1.7) We will prove this theorem by induction on the length of the path to $\mathsf{T} \in \mathcal{SYT}(\lambda)$ in Γ_λ. For the column-reading tableau C, we can pick a word of length 0, which vacuously defines a path from C to C in Γ_λ. We have $A_{\mathsf{C},\mathsf{C}} = \mathsf{wt}_\emptyset(\emptyset) = 1$, by which we correctly obtain the expression $\mathbf{n}_\mathsf{C} = \mathbf{v}_\mathsf{C}$. Assume the theorem holds when the path π has length $k-1$.

Now, consider $\mathsf{T} \in \mathcal{SYT}(\lambda)$ such that $s_{i_k} \cdots s_{i_2} s_{i_1}(\mathsf{C}) = \mathsf{T}$. Then, $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k} =: s_\ell)$ is a path to T. Let $\pi' := (s_{i_1}, s_{i_2}, \ldots, s_{i_{k-1}})$ be a path to $\mathsf{T}' = s_{i_{k-1}} \cdots s_{i_2} s_{i_1}(\mathsf{C})$. Then, by the inductive

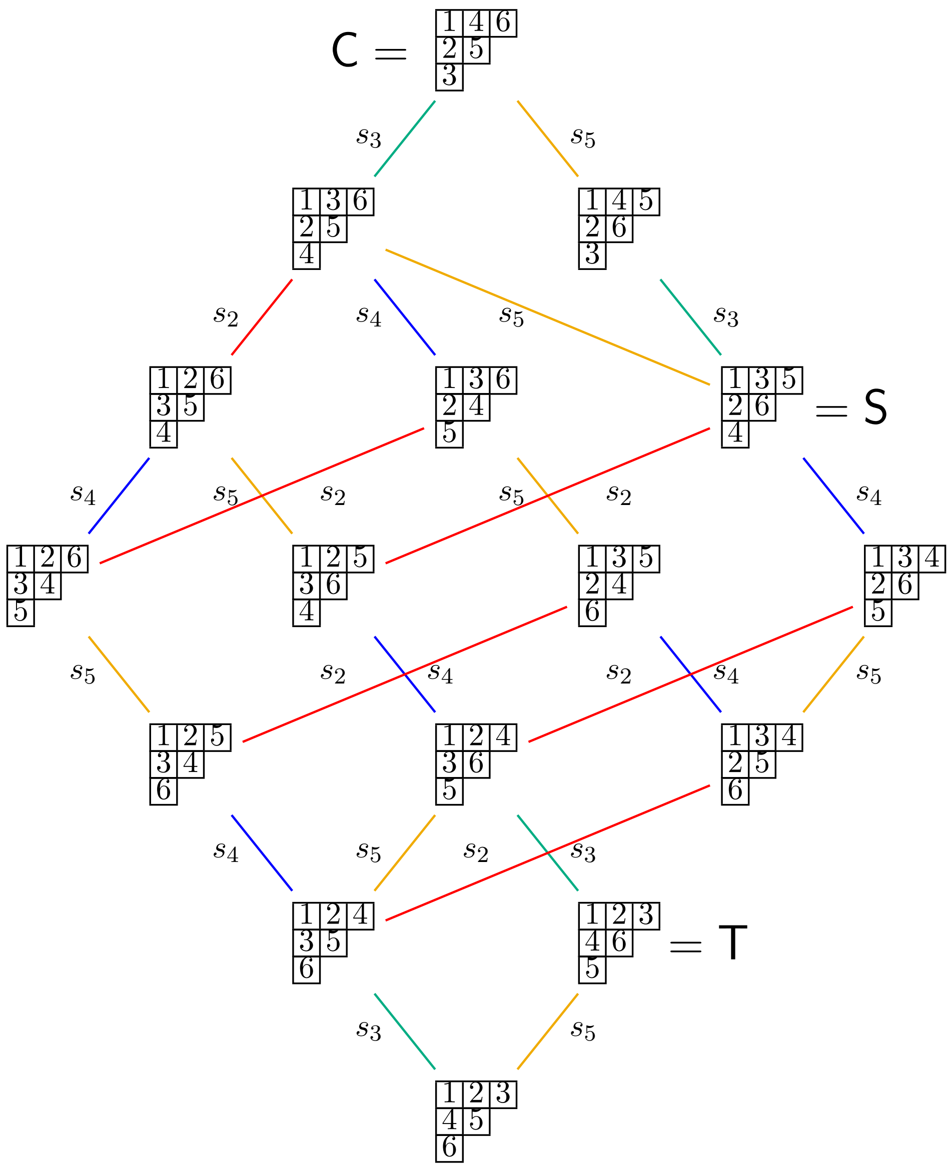


Figure 5.1: The crystal graph $\Gamma_{(3,2,1)}$, where we fix a path $\pi = (s_3, s_2, s_5, s_4, s_3)$ to T and find all subpaths of π which terminate at S.

hypothesis,

$$\begin{aligned}\mathbf{n}_{\mathsf{T}} = \mathbf{n}_{s_\ell(\mathsf{T}')} = s_\ell \cdot \mathbf{n}_{\mathsf{T}'} = s_\ell \cdot \sum_{\mathsf{S}\le\mathsf{T}'} A_{\mathsf{S},\mathsf{T}'}\mathbf{v}_{\mathsf{S}} = \sum_{\mathsf{S}\le\mathsf{T}'} A_{\mathsf{S},\mathsf{T}'}\left(s_\ell\cdot\mathbf{v}_{\mathsf{S}}\right) \\ = \sum_{\mathsf{S}\le\mathsf{T}'} A_{\mathsf{S},\mathsf{T}'}\left(w_\ell(\mathsf{S})\mathbf{v}_{\mathsf{S}} + m_\ell(\mathsf{S})\mathbf{v}_{s_\ell(\mathsf{S})}\right) = \sum_{\mathsf{S}\le\mathsf{T}'} w_\ell(\mathsf{S})A_{\mathsf{S},\mathsf{T}'}\mathbf{v}_{\mathsf{S}} + m_\ell(\mathsf{S})A_{\mathsf{S},\mathsf{T}'}\mathbf{v}_{s_\ell(\mathsf{S})},\end{aligned}$$

where $\mathbf{v}_{s_\ell(\mathsf{S})} = 0$ if $s_\ell(\mathsf{S})$ is non-standard. The terms which contribute to $A_{\mathsf{S},\mathsf{T}}$ are those that include $\mathbf{v}_{\mathsf{S}}$:

$$A_{\mathsf{S},\mathsf{T}} = \begin{cases} w_\ell(\mathsf{S})A_{\mathsf{S},\mathsf{T}'} + m_\ell(\mathsf{S}')A_{\mathsf{S}',\mathsf{T}'} & \text{if } s_\ell(\mathsf{S}) = \mathsf{S}' \text{ is standard,} \\ w_\ell(\mathsf{S})A_{\mathsf{S},\mathsf{T}'} & \text{if } s_\ell(\mathsf{S}) \text{ is non-standard.}\end{cases} \tag{5.1.13}$$

There are two types of subpaths of π which terminate at S. First, are the subpaths of π which follow π' to S and then pause in the k-th step with the weight given by s_ℓ. These give the coefficient $w_\ell(\mathsf{S})A_{\mathsf{S},\mathsf{T}'}$. Second, are the subpaths of π which follow π' to S' and then move from S' to S in the k-th step using s_ℓ. These give the coefficient $m_\ell(\mathsf{S})A_{\mathsf{S}',\mathsf{T}'}$.

Now, the sum in the statement of the theorem will be over $\mathsf{S} \le \mathsf{T}$ because by Lemma 5.1.10, we can always choose a path of minimal length to T. For a path of minimal length, observe that $\mathsf{S} \le \mathsf{T}' \le \mathsf{T}$ by induction. Furthermore, we argue that $\mathsf{S}' \le \mathsf{T}$ as follows. We know that $w_{\mathsf{T}'} = s_{i_{k-1}} \cdots s_{i_2} s_{i_1}$ is a reduced word. If $\mathsf{S} \le \mathsf{T}'$, then any reduced word for w_{S} is a subword of $s_{i_{k-1}} \cdots s_{i_2} s_{i_1}$. Thus, any reduced word for $s_\ell \cdot w_{\mathsf{S}}$ is a subword of $w_{\mathsf{T}} = s_\ell \cdot s_{i_{k-1}} \cdots s_{i_2} s_{i_1}$, which is a reduced word for w_{T} since T has depth k. Therefore, $\mathsf{S}' = s_i(\mathsf{S}) \le \mathsf{T}$, and thus we must only sum over $\mathsf{S} \le \mathsf{T}$. ☐

So, $\mathcal{A}_\lambda = [A_{\mathsf{S},\mathsf{T}}]_{\mathsf{S},\mathsf{T}\in\mathcal{SYT}(\lambda)}$. A few general structural properties immediately follow from this theorem.

Corollary 5.1.14. *Let $\mathcal{SYT}(\lambda)_k = \{\mathsf{T} \in \mathcal{SYT}(\lambda) \mid \mathrm{depth}(\mathsf{T}) = k\}$. Then, the submatrix of $\mathcal{A}_\lambda$ induced by $\mathcal{SYT}(\lambda)_k$ is diagonal.*

Proof. Denote this submatrix A_k. Consider an off-diagonal entry in row S and column T ($\mathsf{S} \ne \mathsf{T}$), where S and T both have depth k in Γ_λ. The (S,T)-entry is found by taking a path π to T and considering all subpaths of π which terminate at S. Let π be a path to T of minimal length. Then, the only subpath of π which gets to depth k in the graph is the original path π. This path terminates at T by construction, so no subpaths of π which terminate at S exist, and this off-diagonal entry is therefore 0. Thus, A_k is a diagonal matrix. ☐

Corollary 5.1.15. *The matrix $\mathcal{A}_\lambda$ is upper-triangular when the rows and columns are indexed according to depth in Γ_λ.*

Proof. Let the rows and columns of $\mathcal{A}_\lambda$ be ordered so that if the depth of S is strictly less than the depth of T in Γ_λ, row/column S must precede row/column T in the ordering. The T-th column of $\mathcal{A}_\lambda$ can be found by taking a minimal-length path π to T and considering all possible subpaths of π; the tableaux at which these subpaths terminate will be those whose rows have nonzero entries in the T-th column. Since π is of minimal length, a subpath of π could not possibly terminate at tableaux of greater depth than T, so the T-th column has no nonzero entries below the T-th row. Therefore, $\mathcal{A}_\lambda$ is upper-triangular under this ordering. ☐

$$
\left(\begin{array}{c|cc|ccc|cccc|ccc|cc|c}
1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{12} \\
\hline
0 & \frac{4}{3} & 0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{5}{12} & \frac{5}{24} & \frac{1}{6} & -\frac{7}{24} \\
0 & 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{4} \\
\hline
0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{5}{8} & -\frac{1}{2} & -\frac{5}{8} \\
0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{2} & -\frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & -\frac{3}{4} & \frac{5}{8} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{2} & 0 & \frac{3}{8} & \frac{3}{2} & \frac{3}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & \frac{3}{2} & \frac{3}{4} & 0 & \frac{3}{8} & -\frac{3}{4} & -\frac{3}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & \frac{5}{4} & 0 & \frac{5}{4} & \frac{5}{8} & 0 & -\frac{5}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & \frac{5}{4} & \frac{5}{4} & \frac{5}{8} & \frac{5}{4} & \frac{5}{8} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{4} & 0 & 0 & \frac{15}{8} & 0 & \frac{15}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{4} & 0 & \frac{15}{8} & \frac{5}{4} & \frac{5}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{4} & \frac{15}{8} & 0 & \frac{15}{8} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{45}{8} & 0 & \frac{15}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & \frac{5}{2} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{2}
\end{array}\right)
$$

Figure 5.2: The matrix $\mathcal{A}_{(3,2,1)}$ with diagonal submatrices emphasized.

We can observe these results for $\lambda = (3, 2, 1)$ in Figure 5.1 and Figure 5.2. Notice in Figure 5.2 that the matrix is upper triangular, and would remain so no matter how the rows and columns within each diagonal submatrix (corresponding to one specific depth level in Γ_λ) are ordered.

5.1.1 A Recursive Formula for $\mathcal{A}_\lambda$

From the proof of Theorem 5.1.7, we see that the T-th column in $\mathcal{A}_\lambda$ will be closely related to the $s_i(\mathsf{T})$-th, as alluded to in the inductive step in the proof. Let $\mathsf{T} < s_i(\mathsf{T})$. The T-th column is computed by finding subpaths of a path π to T which terminate at $\mathsf{S} < \mathsf{T}$. A path π' to $s_i(\mathsf{T})$ can then be created by simply appending s_i to the end of π, and as a result, subpaths of π' are closely related to subpaths of π. To quickly reiterate what was mentioned in the proof, a subpath of π' terminating at S is either

1. A subpath of π terminating at S which pauses with s_i at the end, or

2. A subpath of π terminating at $s_i(\mathsf{S})$ which acts with s_i at the end.

Using this, we can establish a recursive rule for the coefficient $A_{\mathsf{S},s_i(\mathsf{T})}$:

$$A_{\mathsf{S},s_i(\mathsf{T})} = w_i(\mathsf{S})A_{\mathsf{S},\mathsf{T}} + m_i(s_i(\mathsf{S}))A_{s_i(\mathsf{S}),\mathsf{T}}. \tag{5.1.16}$$

If $s_i(\mathsf{S})$ is non-standard, the second term vanishes. As a result, if we have computed the T-th column of $\mathcal{A}_\lambda$, then each entry in the $s_i(\mathsf{T})$-th column requires at most two steps to compute. Furthermore, $\mathcal{A}_\lambda$ is upper-triangular, so we only have to calculate i entries in the i-th column. Therefore, the overall number of computations we have to make is at most

$$\begin{aligned} 2(1 + 2 + \cdots + f_\lambda) &= 2 \cdot \binom{f_\lambda}{2} \\ &= 2 \cdot \frac{(f_\lambda + 1)f_\lambda}{2} \\ &= f_\lambda^2 + f_\lambda, \end{aligned}$$

putting the time complexity to compute $\mathcal{A}_\lambda$ on the order of $O(f_\lambda^2)$.

6. GENERALIZATIONS

6.1 The Orthogonal Representation

Young defined a third representation for S_n^λ, called the orthogonal representation, which we have so far omitted in this paper. This is mostly due to the fact that the orthogonal representation is defined almost identically to the seminormal representation (in fact, Rutherford [12] defines the orthogonal representation in terms of the seminormal representation). Young's orthogonal basis for S_n^λ is $\mathcal{W}_\lambda = \{\mathbf{w}_\mathsf{T} \mid \mathsf{T} \in \mathcal{SYT}(\lambda)\}$ and the action defined on the adjacent transpositions s_i is

$$s_i \cdot \mathbf{w}_\mathsf{T} = \frac{1}{\delta_i(\mathsf{T})}\mathbf{w}_\mathsf{T} + \sqrt{1 - \left(\frac{1}{\delta_i(\mathsf{T})}\right)^2}\,\mathbf{w}_{s_i(\mathsf{T})}. \tag{6.1.1}$$

Notice that, as one might guess, the orthogonal action produces orthogonal matrix representations. As in the seminormal representation, to compute the orthogonal representation of $\sigma \in S_n$, we must find a word $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ and compute the representations for each s_{i_j}. Also, notice that the coefficient on $\mathbf{w}_\mathsf{T}$ on the right-hand side of (6.1.1) is identical to the corresponding coefficient in the seminormal representation, and the term on $\mathbf{w}_{s_i(\mathsf{T})}$ is closely related. In fact, the seminormal and orthogonal representations differ by a diagonal matrix. To compute the entries of this matrix, Rutherford [12] defines a function ϕ_T, where we fix a tableau $\mathsf{T} \in \mathcal{SYT}(\lambda)$ and define ϕ_T piecewise as:

$$\phi_\mathsf{T}(k) := \begin{cases} 1 & \text{if } k \text{ is in the first row of } \mathsf{T}, \\ \displaystyle\prod_{1 \le j < \ell} \left(1 + \frac{1}{\delta^j(\mathsf{T})}\right) & \text{if } k \text{ is in row } \ell, \ell > 1. \end{cases} \tag{6.1.2}$$

The function $\delta^j(\mathsf{T})$ denotes the axial distance from the last entry in row j to k in the tableau T. Then, Rutherford defines

$$\psi_\mathsf{T} := \phi_\mathsf{T}(n)\phi_\mathsf{T}(n-1)\cdots\phi_\mathsf{T}(1). \tag{6.1.3}$$

Theorem 6.1.4. *([12] §27) The change-of-basis matrix $\mathcal{B}_\lambda$ which satisfies*

$$\mathcal{B}_\lambda^{-1}[\sigma]_{\mathcal{W}_\lambda}\mathcal{B}_\lambda = [\sigma]_{\mathcal{V}_\lambda}$$

for all $\sigma \in S_n$ is a diagonal matrix

$$\mathcal{B}_\lambda = \mathsf{diag}(\sqrt{\psi_{\mathsf{T}_1}}, \sqrt{\psi_{\mathsf{T}_2}}, \ldots, \sqrt{\psi_{\mathsf{T}_{f_\lambda}}}),$$

where $\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_{f_\lambda}$ is the tableau ordering given to the bases $\mathcal{V}_\lambda$ and $\mathcal{W}_\lambda$.

Notice this theorem implies that $\mathbf{v}_\mathsf{C} = \sqrt{\psi_\mathsf{C}}\mathbf{w}_\mathsf{C}$. Now, due to the similarities between the seminormal and orthogonal representations, we can find a formula for the change-of-basis matrix $\mathcal{C}_\lambda$ between the orthogonal and natural representations using weighted subpaths on Γ_λ, as before. We first define a modified weight formula:

Definition 6.1.5. *For a path $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k})$, every subpath $\omega = (z_1, z_2, \ldots, z_k)$ of π with tableau sequence $(\mathsf{S}_1 = \mathsf{C}, \mathsf{S}_2, \ldots, \mathsf{S}_{k+1})$ has an associated weight. We assign a cost to each component in the subpath, given piecewise as*

$$\mathsf{d}_j = \begin{cases} \frac{1}{\delta_{i_j}(\mathsf{S}_j)} & \text{if } z_j = e, \\ \sqrt{1 - (\frac{1}{\delta_{i_j}(\mathsf{S}_j)})^2} & \text{if } z_j = s_{i_j}. \end{cases} \tag{6.1.6}$$

The weight on the overall subpath is defined as

$$\widetilde{\mathsf{wt}}_\pi(\omega) = \sqrt{\psi_{\mathsf{C}}} \cdot \prod_{j=1}^{k} \mathsf{d}_j \tag{6.1.7}$$

By definition we let $\widetilde{\mathsf{wt}}_\emptyset(\emptyset) = \sqrt{\psi_{\mathsf{C}}}$.

Notice that the coefficient on any particular term in the subpath is derived directly from the orthogonal action. With this new terminology, we can conjecture a general formula for $\mathcal{C}_\lambda$.

Theorem 6.1.8. *For each $\lambda \vdash n$ and $\mathsf{T} \in \mathcal{SYT}(\lambda)$, the expression of the natural basis element $\mathbf{n}_\mathsf{T}$ in terms of the orthogonal basis of S_n^λ is given by*

$$\mathbf{n}_\mathsf{T} = \sum_{\mathsf{S} \leq \mathsf{T}} A_{\mathsf{S},\mathsf{T}} \mathbf{w}_\mathsf{S} \tag{6.1.9}$$

where the coefficients $A_{\mathsf{S},\mathsf{T}} \in \mathbb{R}$ are given by

$$A_{\mathsf{S},\mathsf{T}} = \sum \widetilde{\mathsf{wt}}_\pi(\omega), \tag{6.1.10}$$

where we fix a path π to T and sum over all subpaths ω of π which terminate at S.

Just as for Theorem 5.1.7, the proof of this theorem will rely on three key facts:

$$\begin{aligned} &\text{(W0)} \quad \text{For the column-reading tableau } \mathsf{C}, \mathbf{n}_\mathsf{C} = \sqrt{\psi_\mathsf{C}}\mathbf{w}_\mathsf{C}. \\ &\text{(W1)} \quad \text{If } \sigma(\mathsf{C}) = \mathsf{T}, \text{then } \mathbf{n}_\mathsf{T} = \sigma \cdot \mathbf{n}_\mathsf{C}. \\ &\qquad\quad \text{(the defining property of the natural representation)} \\ &\text{(W2)} \quad s_i \cdot \mathbf{w}_\mathsf{T} = \tfrac{1}{\delta_i(\mathsf{T})}\mathbf{w}_\mathsf{T} + \sqrt{1 - (\tfrac{1}{\delta_i(\mathsf{T})})^2}\mathbf{w}_{s_i(\mathsf{T})}. \\ &\qquad\quad \text{(the defining property of the orthogonal representation)} \end{aligned} \tag{6.1.11}$$

Proof. This proof follows directly from Theorem 5.1.7 by updating the weight formula with $\widetilde{\mathsf{wt}}_\pi(\omega)$. □

So, the matrices $\mathcal{A}_\lambda$, $\mathcal{B}_\lambda$, and $\mathcal{C}_\lambda$ allow us to move freely between our three bases (natural, seminormal, and orthogonal) for S_n^λ.

6.2 The Iwahori-Hecke Algebra $H_n(q)$

Fix $q \in \mathbb{C}^*$ which is not a root of unity. The Iwahori-Hecke algebra is a q-analog of S_n, where $H_n(q) \to \mathbb{C}S_n$ as $q \to 1$. This algebra also has irreducible modules $H_n^\lambda(q)$ indexed by $\{\lambda \mid \lambda \vdash n\}$, so that $\{H_n^\lambda(q) \mid \lambda \vdash n\}$ is a complete set of irreducible modules. This algebra is generated by $T_1, \ldots, T_{n-1}$, subject to the relations

$$\begin{aligned} &\text{(H1)} \quad T_i^2 = (q - q^{-1})T_i + 1, && 1 \le i \le n-1, \\ &\text{(H2)} \quad T_iT_j = T_jT_i, && 1 \le i, j \le n-1, |i-j| > 1, \\ &\text{(H3)} \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && 1 \le i \le n-2, \end{aligned} \tag{6.2.1}$$

where (H2) and (H3) are equivalent to the braid relations in the symmetric group. For $\sigma \in S_n$ such that $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced word, define $T_\sigma := T_{i_1}T_{i_2}\cdots T_{i_k}$. By the braid relations, the definition of T_σ is independent of the choice of reduced word for σ. Furthermore, $\{T_\sigma \mid \sigma \in S_n\}$ is a $\mathbb{C}$-basis of $H_n(q)$.

6.2.1 Seminormal Representation

The seminormal and natural representations can both be extended to $H_n(q)$. The basis for the seminormal representation is given as $\mathcal{V}_{\lambda,q} = \{\mathbf{v}_{\mathsf{T},q} \mid \mathsf{T} \in \mathcal{SYT}(\lambda)\}$, so that

$$H_n^\lambda(q) = \mathbb{C}\text{-span}\{\mathbf{v}_{\mathsf{T},q}\}.$$

The seminormal action (due to Hoefsmit [6], but derived more intuitively in [11]) defined on T_i is

$$T_i \cdot \mathbf{v}_{\mathsf{T},q} = \frac{q - q^{-1}}{1 - q^{-2\delta_i(\mathsf{T})}}\mathbf{v}_{\mathsf{T},q} + \left(q^{-1} + \frac{q - q^{-1}}{1 - q^{-2\delta_i(\mathsf{T})}}\right)\mathbf{v}_{s_i(\mathsf{T}),q} \tag{6.2.2}$$

where $\mathbf{v}_{s_i(\mathsf{T}),q} = \mathbf{0}$ if $s_i(\mathsf{T})$ is non-standard, and $\delta_i(\mathsf{T})$ denotes the axial distance from i to $i+1$ in T. We let

$$w_{i,q}(\mathsf{T}) := \frac{q - q^{-1}}{1 - q^{-2\delta_i(\mathsf{T})}} \qquad \text{and} \qquad m_{i,q}(\mathsf{T}) := q^{-1} + \frac{q - q^{-1}}{1 - q^{-2\delta_i(\mathsf{T})}}. \tag{6.2.3}$$

Observe that (by a satisfying application of l'Hopital's rule):

$$\lim_{q\to 1} \frac{q - q^{-1}}{1 - q^{-2\delta_i(\mathsf{T})}} = \frac{1}{\delta_i(\mathsf{T})} \qquad \text{and} \qquad \lim_{q\to 1}\left(q^{-1} + \frac{q - q^{-1}}{1 - q^{-2\delta_i(\mathsf{T})}}\right) = 1 + \frac{1}{\delta_i(\mathsf{T})},$$

so that the formulas in (6.2.3) are q-deformations of (3.1.3).

6.2.2 Natural Representation

In [11], Ram show that the natural representation can be extended to $H_n(q)$ using the same method we used in the symmetric group; that is, we again define the natural basis in terms of the seminormal basis. We fix a partition $\lambda \vdash n$ and consider $\mathcal{SYT}(\lambda)$ along with the new seminormal basis $\mathcal{V}_{\lambda,q}$:

(1) For the column-reading tableau C, $\mathbf{n}_{\mathsf{C},q} = \mathbf{v}_{\mathsf{C},q}$ (up to a scalar, which we choose to be 1).

(2) For all other $\mathsf{T} \in \mathcal{SYT}(\lambda)$ and the permutation σ for which $\sigma(\mathsf{C}) = \mathsf{T}$,

$$\mathbf{n}_{\mathsf{T},q} = \mathbf{n}_{\sigma(\mathsf{C}),q} =: T_\sigma \cdot \mathbf{n}_{\mathsf{C},q}. \tag{6.2.4}$$

So, we compute the natural representation of π using the action

$$T_\pi \cdot \mathbf{n}_{\mathsf{T},q} = \mathbf{n}_{\pi(\mathsf{T}),q}. \tag{6.2.5}$$

Ram shows that the Garnir relations are again satisfied under this definition in $H_n(q)$, so that $\mathcal{N}_{\lambda,q} = \{\mathbf{n}_{\mathsf{T},q} \mid \mathsf{T} \in \mathcal{SYT}(\lambda)\}$ is a natural basis for $H_n^\lambda(q)$.

6.2.3 Changing Basis

By updating the weight formula defined in (5.1.4), we can find the q-analog of the change-of-basis matrix, denoted $\mathcal{A}_\lambda(q)$.

Definition 6.2.6. *For a path $\pi = (s_{i_1}, s_{i_2}, \ldots, s_{i_k})$, every subpath $\omega = (z_1, z_2, \ldots, z_k)$ of π with tableau sequence $(\mathsf{S}_1 = \mathsf{C}, \mathsf{S}_2, \ldots, \mathsf{S}_{k+1})$ has an associated q-weight. We assign a cost to each component in the subpath, given piecewise as*

$$\mathsf{c}_j(q) = \begin{cases} w_{i,q}(\mathsf{S}_{i_j}) & \text{if } z_j = e, \\ m_{i,q}(\mathsf{S}_{i_j}) & \text{if } z_j = s_{i_j}. \end{cases} \tag{6.2.7}$$

The weight on the overall subpath is defined as

$$\mathsf{wt}_\pi(\omega, q) = \prod_{j=1}^{k} \mathsf{c}_j(q). \tag{6.2.8}$$

By definition we let $\mathsf{wt}_\emptyset(\emptyset, q) = 1$.

Theorem 6.2.9. *For each $\lambda \vdash n$ and $\mathsf{T} \in \mathcal{SYT}(\lambda)$, the expression of the natural basis element $\mathbf{n}_{\mathsf{T},q}$ in terms of the seminormal basis of $H_n^\lambda(q)$ is given by*

$$\mathbf{n}_{\mathsf{T},q} = \sum_{\mathsf{S} \leq \mathsf{T}} A_{\mathsf{S},\mathsf{T}}(q) \mathbf{v}_{\mathsf{S},q}.$$

The coefficients $A_{\mathsf{S},\mathsf{T}}(q) \in \mathbb{C}$ are given by

$$A_{\mathsf{S},\mathsf{T}}(q) = \sum \mathsf{wt}_\pi(\omega, q),$$

where we fix a path π to T and sum over all subpaths ω of π which terminate at S.

Our theorem relies on three key facts:

$$\begin{aligned}
&\text{(H0)} \quad \text{For the column-reading tableau } \mathsf{C}, \mathbf{n}_{\mathsf{C},q} = \mathbf{v}_{\mathsf{C},q}. \\
&\text{(H1)} \quad \text{If } \sigma(\mathsf{C}) = \mathsf{T}, \text{then } \mathbf{n}_{\mathsf{T},q} = T_\sigma \cdot \mathbf{n}_{\mathsf{C},q}. \\
&\qquad\quad \text{(the defining property of the natural representation)} \\
&\text{(H2)} \quad T_i \cdot \mathbf{v}_{\mathsf{T},q} = w_{i,q}(\mathsf{T})\mathbf{v}_{\mathsf{T},q} + m_{i,q}(\mathsf{T})\mathbf{v}_{s_i(\mathsf{T}),q}. \\
&\qquad\quad \text{(the defining property of the seminormal representation)}
\end{aligned} \tag{6.2.10}$$

$$
\mathcal{A}_{(3,2)}(q) = \begin{array}{c} \\ \mathbf{v}_{\substack{1\,3\,5\\2\,4}} \\ \mathbf{v}_{\substack{1\,2\,5\\3\,4}} \\ \mathbf{v}_{\substack{1\,3\,4\\2\,5}} \\ \mathbf{v}_{\substack{1\,2\,4\\3\,5}} \\ \mathbf{v}_{\substack{1\,2\,3\\4\,5}} \end{array}
\begin{array}{c} \begin{array}{ccccc} \mathbf{n}_{\substack{1\,3\,5\\2\,4}} & \mathbf{n}_{\substack{1\,2\,5\\3\,4}} & \mathbf{n}_{\substack{1\,3\,4\\2\,5}} & \mathbf{n}_{\substack{1\,2\,4\\3\,5}} & \mathbf{n}_{\substack{1\,2\,3\\4\,5}} \end{array} \\
\begin{pmatrix}
1 & \frac{q^3}{q^2+1} & \frac{q^3}{q^2+1} & \frac{q^6}{(q^2+1)^2} & -\frac{q^5}{(q^2+1)^2} \\
\cdot & \frac{q^3+q+q^{-1}}{q^2+1} & \cdot & \frac{q^6+q^4+q^2}{(q^2+1)^2} & \frac{q^7+q^5+q^3}{(q^2+1)^2} \\
\cdot & \cdot & \frac{q^3+q+q^{-1}}{q^2+1} & \frac{q^6+q^4+q^2}{(q^2+1)^2} & \frac{q^7+q^5+q^3}{(q^2+1)^2} \\
\cdot & \cdot & \cdot & \left(\frac{q^3+q+q^{-1}}{q^2+1}\right)^2 & \frac{q^5}{q^2+q+1} \cdot \left(\frac{q^3+q+q^{-1}}{q^2+1}\right)^2 \\
\cdot & \cdot & \cdot & \cdot & \frac{q^5+q+1+q^{-1}}{q^2+q+1} \cdot \left(\frac{q^3+q+q^{-1}}{q^2+1}\right)^2
\end{pmatrix} \end{array}
$$

Figure 6.1: The q-deformation of $\mathcal{A}_{(3,2)}$, denoted $\mathcal{A}_{(3,2)}(q)$.

Proof. As this theorem relies on the same facts as Theorem 5.1.7, the proof follows immediately by updating the weight formula with $\mathsf{wt}_\pi(\omega, q)$. $\square$

It follows that the (S, T)-entry of $\mathcal{A}_\lambda(q)$ is given by $A_{\mathsf{S},\mathsf{T}}(q)$, and

$$\lim_{q \to 1} \mathcal{A}_\lambda(q) = \mathcal{A}_\lambda$$

for all λ since $A_{\mathsf{S},\mathsf{T}}(q)$ is a q-deformation of $A_{\mathsf{S},\mathsf{T}}$. One can verify this by checking the matrix in Figure 6.1 against the matrix in Figure 1.1.

6.3 The Generalized Symmetric Group $G_{r,n}$

In this section we consider the wreath product group $G_{r,n} = C_r \wr S_n$, also called the generalized symmetric group, where $C_r =< \omega >$ is the cyclic group of order r generated by $\omega = e^{2\pi i/r}$. The group $G_{r,n}$ can be identified with the group of monomial matrices whose nonzero entries are chosen from $C_r = \{1, \omega, \omega^2, \ldots, \omega^{r-1}\}$. This group is generated by $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ and $\mathfrak{t}_1$ where $\mathfrak{s}_i$ and $\mathfrak{t}_i$ are identified with the matrices

$$
\mathfrak{s}_i \to \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccccccc} & & & i & i+1 & & & \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \end{array}
\quad \text{and} \quad
\mathfrak{t}_i \to \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccccccc} & & & & i & & & \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \end{array}.
$$

These generators are subject to the relations

$$
\begin{aligned}
&\text{(R1)} \quad \mathfrak{s}_i^2 = 1, && 1 \le i \le n-1, \\
&\text{(R2)} \quad \mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i, && 1 \le i, j \le n-1, |i-j| > 1, \\
&\text{(R3)} \quad \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1}, && 1 \le i \le n-2, \\
&\text{(R4)} \quad \mathfrak{s}_i\mathfrak{t}_1 = \mathfrak{t}_1\mathfrak{s}_i, && i > 1, \\
&\text{(R5)} \quad \mathfrak{t}_1\mathfrak{s}_1\mathfrak{t}_1\mathfrak{s}_1 = \mathfrak{s}_1\mathfrak{t}_1\mathfrak{s}_1\mathfrak{t}_1, && \\
&\text{(R6)} \quad \mathfrak{t}_1^r = 1. &&
\end{aligned}
\tag{6.3.1}
$$

The symmetric group S_n is identified with the subgroup of permutation matrices generated by $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$. Furthermore, observe that $\mathfrak{t}_i = \mathfrak{s}_{i-1}\cdots\mathfrak{s}_2\mathfrak{s}_1\mathfrak{t}_1\mathfrak{s}_1\mathfrak{s}_2\cdots\mathfrak{s}_{i-1}$ for $1 \le i \le n-1$.

The irreducible representations of $G_{r,n}$ are indexed by **r-partitions** $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$. The r-partition $\boldsymbol{\lambda}$ is an ordered sequence of regular integer partitions $\lambda^{(i)}$ (where $\lambda^{(i)} = \emptyset$ is allowable), and $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. For each r-partition $\boldsymbol{\lambda}$, there is an irreducible module $G_{r,n}^{\boldsymbol{\lambda}}$, and $\{G_{r,n}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \text{ is an } r\text{-partition of } n\}$ is a complete set of irreducible, pairwise nonisomorphic $G_{r,n}$-modules. (see [1])

We can naturally extend the notion of standard tableaux to fit this framework. A **standard r-tableau** of shape $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ is a filling of the boxes of $\lambda^{(1)}, \lambda^{(2)}, \ldots,$ and $\lambda^{(r)}$ with $1, 2, \ldots, n$ such that the numbers are increasing across the rows and down the columns of each $\lambda^{(i)}$. We denote the complete set of standard r-tableaux as $\mathcal{SYT}(\boldsymbol{\lambda})$.

Example 6.3.2. *A standard* 3*-tableau of shape* $\boldsymbol{\lambda} = ((2,1),(3,1),(1,1))$ *is*

$$\left(\begin{ytableau} 3 & 4 \\ 6 \end{ytableau}, \quad \begin{ytableau} 1 & 5 & 8 \\ 9 \end{ytableau}, \quad \begin{ytableau} 2 \\ 7 \end{ytableau}\right).$$

As we see in this example, $\lambda^{(i)}$ is not necessarily filled with consecutive numbers as we had in the symmetric group. There are multiple choices for the alphabet in a given component, reflected in the following definition.

Definition 6.3.3. *Given* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$*, a* ***partitioned alphabet*** *for* $\boldsymbol{\lambda}$ *is an ordered set partition of* $\{1, 2, \ldots, n\}$ *into* r *components such that the* i*-th component* $P^{(i)}$ *is the alphabet for* $\lambda^{(i)}$ *(meaning* $|P^{(i)}| = |\lambda^{(i)}|$*). We denote this partitioned alphabet as* $\mathcal{P} = (P^{(1)} \mid P^{(2)} \mid \cdots \mid P^{(r)})$.

Given an r-partition $\boldsymbol{\lambda}$, each partitioned alphabet $\mathcal{P} = (P^{(1)} \mid P^{(2)} \mid \cdots \mid P^{(r)})$ for $\boldsymbol{\lambda}$ has a corresponding column-reading tableau $\mathsf{C}_{\mathcal{P}} = (\mathsf{C}^{(1)}, \mathsf{C}^{(2)}, \ldots, \mathsf{C}^{(r)})$.

Example 6.3.4. *For the* 3*-partition* $\boldsymbol{\lambda} = ((2,1),(3,1),(1,1))$*, the partitioned alphabet* $\mathcal{P} = \{346 \mid 1589 \mid 27\}$ *has column-reading tableau*

$$\mathsf{C}_{\mathcal{P}} = \left(\begin{ytableau} 3 & 6 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 8 & 9 \\ 5 \end{ytableau}, \quad \begin{ytableau} 2 \\ 7 \end{ytableau}\right),$$

where $\mathsf{C}^{(1)} = \begin{ytableau} 3 & 6 \\ 4 \end{ytableau}, \mathsf{C}^{(2)} = \begin{ytableau} 1 & 8 & 9 \\ 5 \end{ytableau}$*, and* $\mathsf{C}^{(3)} = \begin{ytableau} 2 \\ 7 \end{ytableau}$.

Given $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ (with $|\lambda^{(i)}| = k_i$), we name the partitioned alphabet $\mathcal{P}_\alpha = \{1, \ldots, k_1 \mid k_1+1, \ldots, k_1+k_2 \mid \cdots \mid n-k_r+1, \ldots, n\}$ the **principle** partitioned alphabet for $\boldsymbol{\lambda}$. The column-reading tableau for this alphabet is called the **principle** column-reading tableau, and we denote it C_α.

Example 6.3.5. *For $\boldsymbol{\lambda} = ((2,1),(3,1),(1,1))$, the principle partitioned alphabet is $\mathcal{P}_\alpha = \{1,2,3 \mid 4,5,6,7 \mid 8,9\}$ and the principle column-reading tableau is*

$$\mathsf{C}_\alpha = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 4 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|}\hline 8 \\ \hline 9 \\ \hline\end{array} \right).$$

6.3.1 Seminormal Representation

Let $\boldsymbol{\lambda}$ be an r-partition of n. As in the symmetric group, we have $\dim\left(G_{r,n}^{\boldsymbol{\lambda}}\right) = |\mathcal{SYT}(\boldsymbol{\lambda})|$ so that a basis for $G_{r,n}^{\boldsymbol{\lambda}}$ can be indexed by $\mathcal{SYT}(\boldsymbol{\lambda})$, and Ariki and Koike (see [1]) extend Young's representations to $G_{r,n}$. The seminormal representation is given by the basis $\mathcal{V}_{\boldsymbol{\lambda}} = \{\mathbf{v}_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})\}$, so that

$$G_{r,n}^{\boldsymbol{\lambda}} = \mathbb{C}\text{-span}\{\mathbf{v}_{\mathsf{T}}\}.$$

The action on the basis is given by

$$\mathfrak{t}_i \cdot \mathbf{v}_{\mathsf{T}} = \omega^{k-1}\mathbf{v}_{\mathsf{T}}, \tag{6.3.6}$$

where the element i is in the $\lambda^{(k)}$ component of T, and

$$\mathfrak{s}_i \cdot \mathbf{v}_{\mathsf{T}} = \begin{cases} \mathbf{v}_{s_i(\mathsf{T})}, & \text{if } i \text{ and } i+1 \text{ are in different components of } \mathsf{T}, \\ w_i(\mathsf{T})\mathbf{v}_{\mathsf{T}} + m_i(\mathsf{T})\mathbf{v}_{s_i(\mathsf{T})} & \text{if } i \text{ and } i+1 \text{ are in the same component of } \mathsf{T}, \end{cases} \tag{6.3.7}$$

where $\delta_i(\mathsf{T})$ is the axial distance from i to $i+1$ in the relevant component of T, and $\mathbf{v}_{\mathfrak{s}_i(\mathsf{T})} = \mathbf{0}$ if $\mathfrak{s}_i(\mathsf{T})$ is non-standard. Furthermore, notice that the first case of (6.3.7) will always produce a standard tableau, since i and $i+1$ are in different components and will therefore never end up in the same row or column (so T standard $\Rightarrow s_i(\mathsf{T})$ standard). As in the symmetric group, the seminormal action can then be extended linearly to all of $G_{r,n}$.

Going forward, it will be helpful to define terms which allow us to write $\mathfrak{s}_i \cdot \mathbf{v}_{\mathsf{T}}$ without expressing it in cases:

$$c_i(\mathsf{T}) := \begin{cases} 0 & \text{if } i \text{ and } i+1 \text{ are in different components of } \mathsf{T}, \\ w_i(\mathsf{T}) & \text{if } i \text{ and } i+1 \text{ are in the same component of } \mathsf{T}, \end{cases} \tag{6.3.8}$$

and

$$d_i(\mathsf{T}) := \begin{cases} 1 & \text{if } i \text{ and } i+1 \text{ are in different components of } \mathsf{T}, \\ m_i(\mathsf{T}) & \text{if } i \text{ and } i+1 \text{ are in the same component of } \mathsf{T}. \end{cases} \tag{6.3.9}$$

This allows us to present the seminormal action a bit more cleanly, as

$$\mathfrak{s}_i \cdot \mathbf{v}_{\mathsf{T}} = c_i(\mathsf{T})\mathbf{v}_{\mathsf{T}} + d_i(\mathsf{T})\mathbf{v}_{s_i(\mathsf{T})}. \tag{6.3.10}$$

6.3.2 Natural Representation

Once more, we follow [11] to define the natural representation, as Ram shows that a natural basis for $G^{\boldsymbol{\lambda}}_{r,n}$ can be expressed in terms of our seminormal basis $\mathcal{V}_{\boldsymbol{\lambda}}$. For each r-partition $\boldsymbol{\lambda}$, consider the set of standard r-tableaux $\mathcal{SYT}(\boldsymbol{\lambda})$ along with $\mathcal{V}_{\boldsymbol{\lambda}}$:

(1) For the principle column-reading tableau C_α, $\mathbf{n}_{\mathsf{C}_\alpha} = \mathbf{v}_{\mathsf{C}_\alpha}$ (up to a scalar, which we choose to be 1).

(2) For all other $\mathsf{T} \in \mathcal{SYT}(\lambda)$ and the permutation σ for which $\sigma(\mathsf{C}_\alpha) = \mathsf{T}$, let a word be given by $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$. Then, for $\mathfrak{s}_{i_1}\mathfrak{s}_{i_2}\cdots\mathfrak{s}_{i_k} =: \boldsymbol{\sigma} \in G_{r,n}$,

$$\mathbf{n}_{\mathsf{T}} = \mathbf{n}_{\sigma(\mathsf{C}_\alpha)} =: \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathsf{C}_\alpha}. \tag{6.3.11}$$

So, we compute the natural representation of $\boldsymbol{\pi} \in G_{r,n}$ with corresponding permutation π using the action

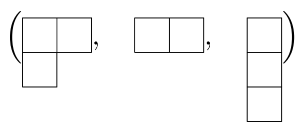
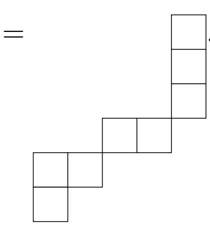
$$\boldsymbol{\pi} \cdot \mathbf{n}_{\mathsf{T}} = \mathbf{n}_{\pi(\mathsf{T})}. \tag{6.3.12}$$

Ram shows that the Garnir relations are satisfied by this action in $G_{r,n}$, so $\mathcal{N}_{\boldsymbol{\lambda}} = \{\mathbf{n}_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})\}$ is a natural basis for $G^{\boldsymbol{\lambda}}_{r,n}$.

Before continuing, it is worth noting that Ram defines the natural representation on **skew tableaux**, not on r-tableaux. Given two integer partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ such that $\nu_i \leq \mu_i$ for all i (and $k \leq n$), a skew tableau μ/ν consists of all the boxes in μ which are not boxes of ν.

We can easily use Ram's results for r-tableaux by setting up a correspondence between any r-partition $\boldsymbol{\lambda}$ and a unique skew tableau μ/ν.

Example 6.3.13. *The 3-tableau $((2,1),(2),(1,1,1))$ is associated with the skew tableau $(5,5,5,4,2,1)/(4,4,4,2)$:*

$((2,1),(2),(1,1,1)) =$  $(5,5,5,4,2,1)/(4,4,4,2) =$ .

As seen in the previous example, we can create a skew tableau by simply stack the constituent tableaux in the r-tableau $\boldsymbol{\lambda}$ such that the bottom-left box of $\lambda^{(i+1)}$ is diagonally adjacent to the upper-right box of $\lambda^{(i)}$.

6.3.3 Changing Basis

Once we generalize some existing terminology, we will be able to state the analogous change-of-basis matrix for $G^{\boldsymbol{\lambda}}_{r,n}$. We again define a graph on $\mathcal{SYT}(\boldsymbol{\lambda})$:

Definition 6.3.14. *For $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, we let $\Gamma_{\boldsymbol{\lambda}}$ be the graph with vertex set $\mathcal{SYT}(\boldsymbol{\lambda})$ and an undirected edge connecting r-tableaux $\mathsf{S}, \mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})$ if there exists $1 \leq i \leq n-1$ such that $s_i(\mathsf{S}) = \mathsf{T}$; this edge is given the label s_i. Note that $s_i(\mathsf{S}) = \mathsf{T}$ if and only if $s_i(\mathsf{T}) = \mathsf{S}$.*

We also abstract our formula for the weight on a subpath of a path to T:

Definition 6.3.15. *For a path $\pi = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$, every subpath $\omega = (z_1, z_2, \dots, z_k)$ of π with r-tableau sequence $(\mathsf{S}_1 = \mathsf{C}, \mathsf{S}_2, \dots, \mathsf{S}_{k+1})$ has an associated weight. We assign a coefficient to each component in the subpath, given piecewise as*

$$\mathsf{g}_j = \begin{cases} c_{i_j}(\mathsf{T}) & \text{if } z_j = e, \\ d_{i_j}(\mathsf{T}) & \text{if } z_j = s_{i_j}. \end{cases} \tag{6.3.16}$$

The weight on the overall subpath is defined as

$$\mathsf{wt}_\pi(\omega) = \prod_{j=1}^{k} \mathsf{g}_j. \tag{6.3.17}$$

By definition we let $\mathsf{wt}_\emptyset(\emptyset) = 1$.

Notice that any subpath which skips an alphabet-swapping adjacent transposition will necessarily have a weight of 0.

Theorem 6.3.18. *For each r-partition $\boldsymbol{\lambda}$ of n and $\mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})$, the expression of the natural basis element $\mathbf{n}_\mathsf{T}$ in terms of the seminormal basis of $G_{r,n}^{\boldsymbol{\lambda}}$ is given by*

$$\mathbf{n}_\mathsf{T} = \sum_{\mathsf{S} \leq \mathsf{T}} A_{\mathsf{S},\mathsf{T}} \mathbf{v}_\mathsf{S}.$$

The coefficients $A_{\mathsf{S},\mathsf{T}} \in \mathbb{Q}$ are given by

$$A_{\mathsf{S},\mathsf{T}} = \sum \mathsf{wt}_\pi(\omega),$$

where we fix a path π to T and sum over all subpaths ω of π which terminate at S.

As in $\mathcal{SYT}(\lambda)$, we say that for two r-tableaux $\mathsf{S}, \mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})$, $\mathsf{S} \leq \mathsf{T}$ if and only if $\ell(w_\mathsf{S}) \leq \ell(w_\mathsf{T})$. Now, the proof of this theorem relies on three key facts:

$$\begin{aligned} &\text{(G0)} && \text{For the principle column-reading tableau } \mathsf{C}_\alpha, \mathbf{n}_{\mathsf{C}_\alpha} = \mathbf{v}_{\mathsf{C}_\alpha}. \\ &\text{(G1)} && \text{If } \sigma(\mathsf{C}_\alpha) = \mathsf{T}, \text{then } \mathbf{n}_\mathsf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}_{\mathsf{C}_\alpha}. \\ & && \text{(the defining property of the natural representation)} \\ &\text{(G2)} && \mathfrak{s}_i \cdot \mathbf{v}_\mathsf{T} = c_i(\mathsf{T})\mathbf{v}_\mathsf{T} + d_i(\mathsf{T})\mathbf{v}_{s_i(\mathsf{T})}. \\ & && \text{(the defining property of the seminormal representation)} \end{aligned} \tag{6.3.19}$$

Proof. As this theorem relies on the same facts as Theorem 5.1.7, the proof follows immediately by updating the weight formula $\mathsf{wt}_\pi(\omega)$. □

However, there is further structure present in the change-of-basis matrix for the generalized symmetric group, described by the following corollary.

Corollary 6.3.20. *For the irreducible module $G^{\boldsymbol{\lambda}}_{r,n}$ corresponding to $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, the transition between the seminormal and natural bases is given by*

$$\mathcal{A}_{\boldsymbol{\lambda}} = (\mathcal{A}_{\lambda^{(1)}} \otimes \cdots \otimes \mathcal{A}_{\lambda^{(r)}})^{\oplus \binom{n}{k_1, k_2, \dots, k_r}},$$

where $k_i = |\lambda^{(i)}|$. The sum is over all possible partitioned alphabets for $\boldsymbol{\lambda}$, and $\mathcal{A}_{\lambda^{(i)}}$ is the change-of-basis matrix for $\lambda^{(i)}$ determined in Chapter 5.

The multinomial $\binom{n}{k_1, k_2, \dots, k_r}$ denotes the number of ways to pick k_1 objects from a set of size n, then $k_2, \dots,$ then k_r — this is the number of ways to create a partitioned alphabet for $\boldsymbol{\lambda}$. So, $\mathcal{A}_{\boldsymbol{\lambda}}$ is a direct sum over the tensor product of the constituent $\mathcal{A}_{\lambda^{(i)}}$, with one copy of the tensor product for each possible partitioned alphabet. We will show that the summand is independent of the particular alphabet we use.

To prove this corollary, we first make note of the fact that the $\binom{n}{k_1, k_2, \dots, k_r}$ subsets of $\mathcal{SYT}(\boldsymbol{\lambda})$ which share partitioned alphabets are all in bijection with one another. For an r-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and the principle alphabet $\mathcal{P}_\alpha = \{1, \dots k_1 \mid k_1+1, \dots, k_1+k_2 \mid \cdots \mid n-k_r+1, \dots, n\}$, consider any other partitioned alphabet $\mathcal{P} = \{a^{(1)}_1, a^{(1)}_2 \dots, a^{(1)}_{k_1} \mid a^{(2)}_1, a^{(2)}_2 \dots, a^{(2)}_{k_2} \mid \cdots \mid a^{(r)}_1, a^{(r)}_2 \dots, a^{(r)}_{k_r}\}$ where for $1 \le i \le r$, $a^{(i)}_1 < a^{(i)}_2 < \cdots < a^{(i)}_{k_i}$. The permutation

$$\beta = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \cdots & k_1 & k_1+1 & \cdots & k_1+k_2 & \cdots & n-k_r+1 & \cdots & n \\ a^{(1)}_1 & \cdots & a^{(1)}_{k_1} & a^{(2)}_1 & \cdots & a^{(2)}_{k_2} & \cdots & a^{(r)}_1 & \cdots & a^{(r)}_{k_r} \end{array} \right)$$

satisfies $\beta(\mathcal{P}_\alpha) = \mathcal{P}$. Furthermore, since the numbers are increasing within each component of $\boldsymbol{\lambda}$, this permutation β is the (unique) minimal-length permutation which generates the alphabet $\mathcal{P}$. This permutation is called the **alphabetizer** for $\mathcal{P}$.

Proposition 6.3.21. *Fix an r-partition $\boldsymbol{\lambda}$ and its principle alphabet, $\mathcal{P}_\alpha$. Consider another partitioned alphabet $\mathcal{P}$, and its alphabetizer $\beta \in S_n$. Furthermore, let $\mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})$ have the principle alphabet $\mathcal{P}_\alpha$, so that $\beta(\mathsf{T})$ has the alphabet $\mathcal{P}$. Then,*

$$\beta \cdot \mathbf{v}_{\mathsf{T}} = \mathbf{v}_{\beta(\mathsf{T})}.$$

This proposition relies on a well-known fact from geometric group theory called the Exchange Condition (see, for instance, [7]):

Lemma 6.3.22. *(Exchange Condition, [7] 1.7) Let $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a word for $\sigma \in S_n$ and consider some other adjacent transposition s_m. If $\ell(s_m\sigma) < \ell(\sigma)$, then there exists some $\sigma' \in S_n$ such that $\sigma = s_m\sigma'$ with $\ell(\sigma') < \ell(\sigma)$.*

With this, we can prove the preceding proposition.

Proof. (of Proposition 6.3.21) Fix an r-tableau $\mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})$ with the principle alphabet. We will prove this proposition by induction on the length of the alphabetizer β. For $\ell(\beta) = 0$, we have $\beta = e$. Thus $e(\mathsf{T}) = \mathsf{T}$, and $e \cdot \mathbf{v}_{\mathsf{T}} = \mathbf{v}_{e(\mathsf{T})} = \mathbf{v}_{\mathsf{T}}$.

Now, if $\ell(\beta) \ge 1$, then the tableau $\beta(\mathsf{T})$ (which has partitioned alphabet $\beta(\mathcal{P}_\alpha) = \mathcal{P}, \mathcal{P} \ne \mathcal{P}_\alpha$) contains a pair $(i, i+1)$ such that $i+1$ is in a component that is to the left of the component of

i. If not, then the entries of the tableau would strictly increase across components, which would give $\beta(\mathsf{T})$ the principle partitioned alphabet. Assume that if $\beta \in S_n$ is an alphabetizer such that $\ell(\beta) = k-1$, then $\beta \cdot \mathbf{v}_{\mathsf{T}} = \mathbf{v}_{\beta(\mathsf{T})}$.

Now, consider an alphabetizer β such that $\ell(\beta) = k$, along with its corresponding partitioned alphabet $\mathcal{P}$. The r-tableau $\beta(\mathsf{T})$ contains a pair $(i, i+1)$ such that $i+1$ is in a component to the left of i. Then, $\ell(s_i(\beta)) < \ell(\beta)$ because acting with s_i decreases the number of inversions. Therefore, $\beta = s_i\beta'$ for some $\beta' \in S_n$ such that $\ell(\beta) = \ell(\beta') + 1$ by the Exchange Condition. Furthermore, notice that β' is also an alphabetizer since β is — when we swap i and $i+1$ in β to get β', numbers will still be strictly increasing within each component, since i and $i+1$ are adjacent and are in different components. Then, by the inductive hypothesis, $\beta' \cdot \mathbf{v}_{\mathsf{T}} = \mathbf{v}_{\beta'(\mathsf{T})}$. Thus,

$$\beta \cdot \mathbf{v}_{\mathsf{T}} = s_i\beta' \cdot \mathbf{v}_{\mathsf{T}} = s_i \cdot \mathbf{v}_{\beta'(\mathsf{T})} = \mathbf{v}_{s_i\beta'(\mathsf{T})} = \mathbf{v}_{\beta(\mathsf{T})}.$$

The second-to-last equality holds by the action in (6.3.7) since i and $i+1$ are in different components. This proves the proposition. □

Now, to prove our corollary, we will again appeal to the structure of $\Gamma_{\boldsymbol{\lambda}}$ and relate the entries of $\mathcal{A}_{\boldsymbol{\lambda}}$ to weighted paths on the graph.

Proof. (of Corollary 6.3.20) The crux of this proof is that a subpath ω in $\Gamma_{\boldsymbol{\lambda}}$ is closely related to a sequence of subpaths $\omega^{(1)}, \dots, \omega^{(r)}$ in each $\Gamma_{\lambda^{(i)}}$, and that $\mathsf{wt}(\omega) = \mathsf{wt}(\omega^{(1)}) \cdot \ldots \cdot \mathsf{wt}(\omega^{(r)})$.

Fix an r-partition $\boldsymbol{\lambda}$ and consider r-tableaux $\mathsf{S} = (\mathsf{S}^{(1)}, \dots, \mathsf{S}^{(r)})$ and $\mathsf{T} = (\mathsf{T}^{(1)}, \dots, \mathsf{T}^{(r)})$ which have the principle partitioned alphabet. By Theorem 6.3.18, the (S, T)-entry of $\mathcal{A}_{\boldsymbol{\lambda}}$ is found by summing the weights on subpaths of a path to T which terminate at S. Pick a path π to T in this way:

(1) Start with a path π_1 from $\mathsf{C}_\alpha = (\mathsf{C}^{(1)}, \mathsf{C}^{(2)}, \dots, \mathsf{C}^{(r)})$ to $(\mathsf{T}^{(1)}, \mathsf{C}^{(2)}, \dots, \mathsf{C}^{(r)})$.

(2) Append a path π_2 from $(\mathsf{T}^{(1)}, \mathsf{C}^{(2)}, \dots, \mathsf{C}^{(r)})$ to $(\mathsf{T}^{(1)}, \mathsf{T}^{(2)}, \mathsf{C}^{(3)}, \dots, \mathsf{C}^{(r)})$.

⋮

(r+1) Append a path π_r from $(\mathsf{T}^{(1)}, \dots, \mathsf{T}^{(r-1)}, \mathsf{C}^{(r)})$ to $(\mathsf{T}^{(1)}, \dots, \mathsf{T}^{(r-1)}, \mathsf{T}^{(r)})$.

Notice that each π_i only acts on the i-th component of the r-tableau, so it can be taken to define a path from $\mathsf{C}^{(i)}$ to $\mathsf{T}^{(i)}$ in $\Gamma_{\lambda^{(i)}}$ by shifting the indices of the adjacent transpositions in the path to reflect the changed entries. Notice that this will not affect the way we compute weights on subpaths, though, as the axial distances between the elements being swapped is unchanged. Let $\omega_1, \dots, \omega_r$ be subpaths such that ω_i is a subpath of π_i terminating at $\mathsf{S}^{(i)}$ in $\Gamma_{\lambda^{(i)}}$. Now, consider a subpath ω following π which terminates at S in $\Gamma_{\boldsymbol{\lambda}}$. Such a subpath will

(1) Follow a subsequence of π_1 from C_α to $(\mathsf{T}^{(1)}, \mathsf{C}^{(2)}, \dots, \mathsf{C}^{(r)})$ which terminates at $(\mathsf{S}^{(1)}, \mathsf{C}^{(2)}, \dots, \mathsf{C}^{(r)})$. The weight on this segment of ω is $\mathsf{wt}_{\pi_1}(\omega_1)$.

(2) Then, follow a subsequence of π_2 from $(\mathsf{T}^{(1)}, \mathsf{C}^{(2)}, \dots, \mathsf{C}^{(r)})$ to $(\mathsf{T}^{(1)}, \mathsf{T}^{(2)}, \mathsf{C}^{(3)}, \dots, \mathsf{C}^{(r)})$ which terminates at $(\mathsf{S}^{(1)}, \mathsf{S}^{(2)}, \mathsf{C}^{(3)}, \dots, \mathsf{C}^{(r)})$. The weight on this segment of ω is $\mathsf{wt}(\omega_2)$.

⋮

(r+1) Finally, follow a subsequence of π_r from $(\mathsf{T}^{(1)}, \ldots, \mathsf{T}^{(r-1)}, \mathsf{C}^{(r)})$ to $(\mathsf{T}^{(1)}, \ldots, \mathsf{T}^{(r-1)}, \mathsf{T}^{(r)})$ which terminates at $(\mathsf{S}^{(1)}, \mathsf{S}^{(2)}, \ldots, \mathsf{S}^{(r)})$. The weight on this segment of ω is $\mathsf{wt}(\omega_r)$.

The subpath ω must be constructed in this way, because the only time we make changes in the i-th component is in the i-th step.

Multiplying together the weights along our entire subpath ω, we get that

$$\mathsf{wt}_\pi(\omega) = \prod_{i=1}^{r} \mathsf{wt}_{\pi_i}(\omega_i). \tag{6.3.23}$$

It follows that

$$A_{\mathsf{S},\mathsf{T}} = \prod_{i=1}^{r} A_{\mathsf{S}^{(i)},\mathsf{T}^{(i)}}. \tag{6.3.24}$$

Therefore, for the principle partitioned alphabet $\mathcal{P}_\alpha$,

$$\mathcal{A}_{\boldsymbol{\lambda}}|_{\mathcal{P}_\alpha} = \mathcal{A}_{\lambda^{(1)}} \otimes \cdots \otimes \mathcal{A}_{\lambda^{(r)}}. \tag{6.3.25}$$

Now, we argue that (6.3.25) holds for all possible partitioned alphabets as follows. Given a tableau $\mathsf{T} \in \mathcal{SYT}(\boldsymbol{\lambda})$ with the principle alphabet $\mathcal{P}_\alpha$, consider any other partitioned alphabet $\mathcal{P}$ and its alphabetizer β. Then, by Proposition 6.3.21,

$$\begin{aligned} \beta \cdot \mathbf{n}_{\mathsf{T}} &= \beta \cdot \sum A_{\mathsf{S},\mathsf{T}} \mathbf{v}_{\mathsf{S}} = \sum A_{\mathsf{S},\mathsf{T}} \beta \cdot \mathbf{v}_{\mathsf{S}} \\ &= \sum A_{\mathsf{S},\mathsf{T}} \mathbf{v}_{\beta(\mathsf{S})}. \end{aligned}$$

So, we obtain an identical expression for $\mathcal{A}_{\boldsymbol{\lambda}}|_{\mathcal{P}}$ for each partitioned alphabet.

Furthermore, we have noted that if S and T have different partitioned alphabets, then the (S, T)-entry of $\mathcal{A}_{\boldsymbol{\lambda}}$ is 0 since a subpath to S will pause at an alphabet-swapping step and therefore zero out the weight. So, the only nonzero entries in $\mathcal{A}_{\boldsymbol{\lambda}}$ correspond to tableaux with identical partitioned alphabets. It follows that

$$\mathcal{A}_{\boldsymbol{\lambda}} = (\mathcal{A}_{\lambda^{(1)}} \otimes \cdots \otimes \mathcal{A}_{\lambda^{(r)}})^{\oplus \binom{n}{k_1, k_2, \ldots, k_r}}$$

since there are $\binom{n}{k_1, k_2, \ldots, k_r}$ partitioned alphabets for $\boldsymbol{\lambda}$. □

Although our original theorem in the symmetric group may have seemed overly intricate and technical, we see that it is easily generalizable to more abstract algebraic structures on which the seminormal and natural representations are defined.

7. APPENDIX

7.1 Γ_λ and $\mathcal{A}_\lambda$ for various $\lambda \vdash n$

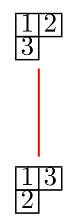
We omit all $\lambda \vdash n$ of the form $\lambda = (n)$ and $\lambda = (1)^n$, as in this case, $\mathcal{SYT}(\lambda)$ consists of a single tableau (hence the seminormal and natural representation are equivalent).

Note — the rows and columns of $\mathcal{A}_\lambda$ are indexed according to permutation length (i.e. depth in Γ_λ). Within each row, the index moves from left to right.

7.1.1 $n = 3$

$\lambda = (2, 1)$

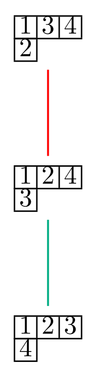
$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} \\ \cdot & \frac{3}{2} \end{pmatrix}$$



7.1.2 $n = 4$

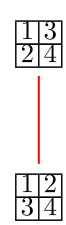
$\lambda = (3, 1)$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{3}{2} & \frac{1}{2} \\ \cdot & \cdot & 2 \end{pmatrix}$$



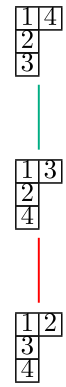
$\lambda = (2, 2)$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} \\ \cdot & \frac{3}{2} \end{pmatrix}$$



$\underline{\lambda = (2,1,1)}$

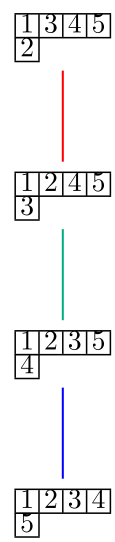
$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ \cdot & \frac{4}{3} & \frac{2}{3} \\ \cdot & \cdot & 2 \end{pmatrix}$$



7.1.3 $n = 5$

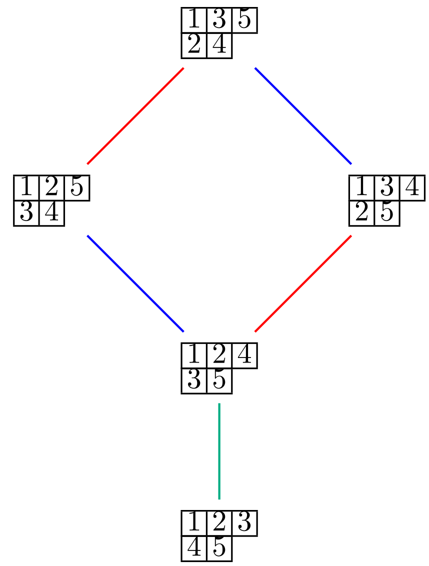
$\underline{\lambda = (4,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & 2 & \frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{5}{2} \end{pmatrix}$$



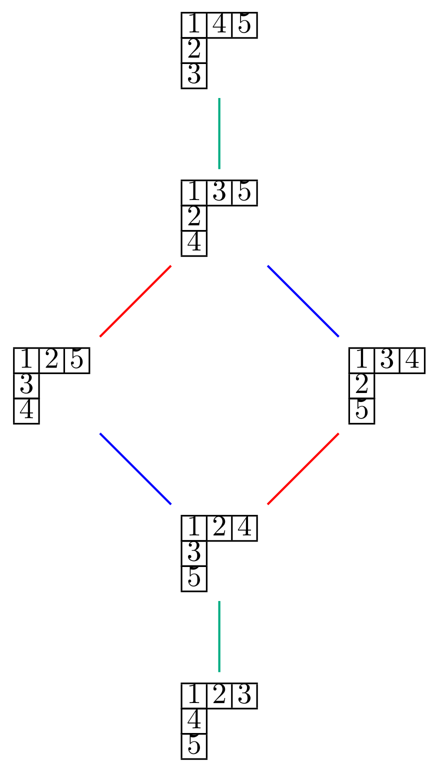
$\underline{\lambda = (3,2)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \cdot & \frac{3}{2} & \cdot & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \frac{9}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix}$$



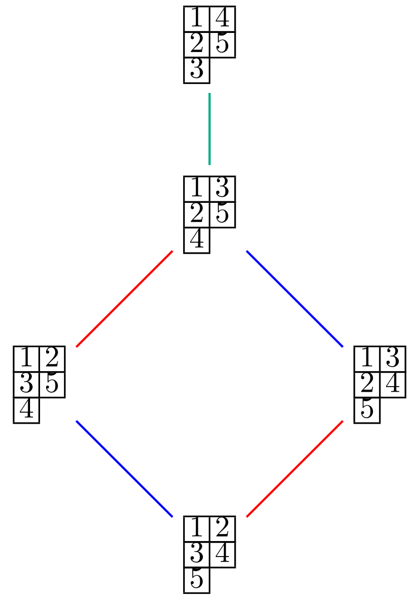
$\underline{\lambda = (3,1,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \cdot \\ \cdot & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} \\ \cdot & \cdot & 2 & \cdot & \frac{1}{2} & -\frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{5}{3} & \frac{5}{6} & \frac{5}{6} \\ \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \frac{5}{6} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{10}{3} \end{pmatrix}$$



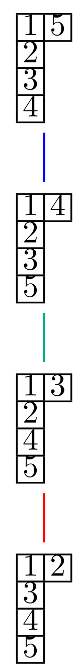
$\underline{\lambda = (2,2,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \cdot & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \cdot & \cdot & 2 & \cdot & 1 \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix}$$



$\underline{\lambda = (2,1,1,1)}$

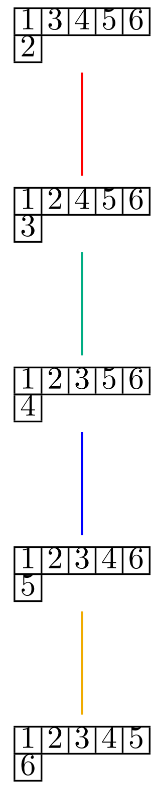
$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \cdot & \frac{5}{4} & \frac{5}{12} & -\frac{5}{12} \\ \cdot & \cdot & \frac{5}{3} & \frac{5}{6} \\ \cdot & \cdot & \cdot & \frac{5}{2} \end{pmatrix}$$



7.1.4 $n = 6$

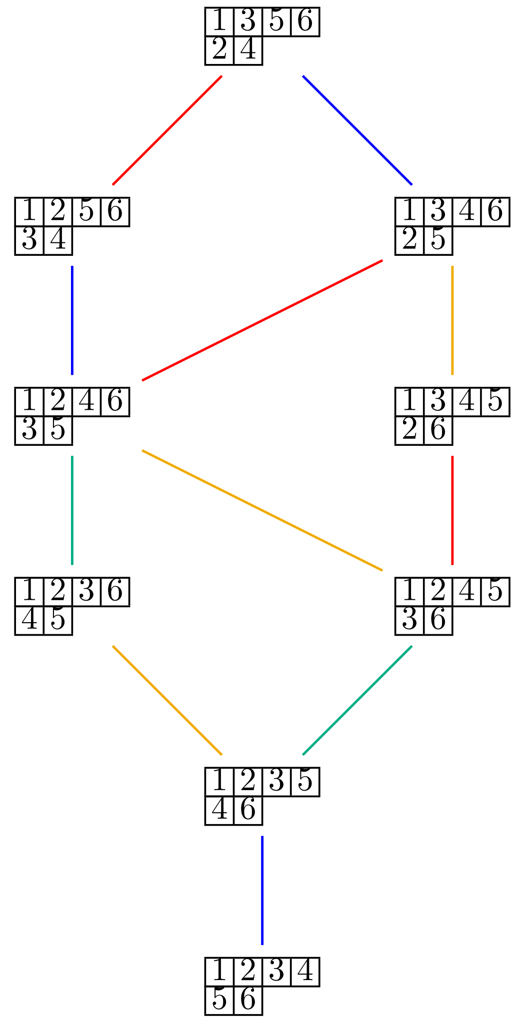
$\underline{\lambda = (5, 1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \cdot & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & 2 & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{5}{2} & \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix}$$



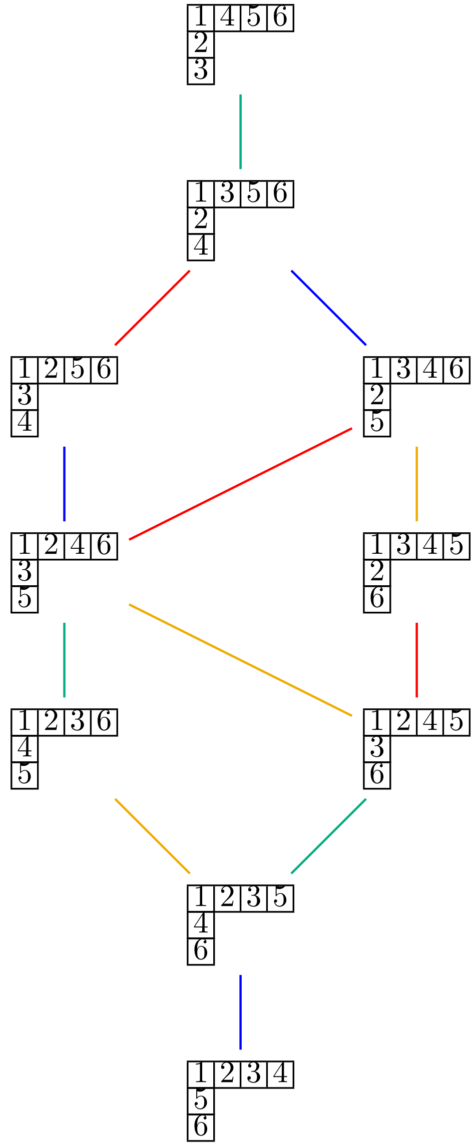
$\underline{\lambda = (4,2)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \cdot \\ \cdot & \frac{3}{2} & \cdot & \frac{3}{4} & \cdot & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{2} \\ \cdot & \cdot & \frac{3}{2} & \frac{3}{4} & \frac{1}{2} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{9}{4} & \cdot & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & 1 \\ \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \end{pmatrix}$$



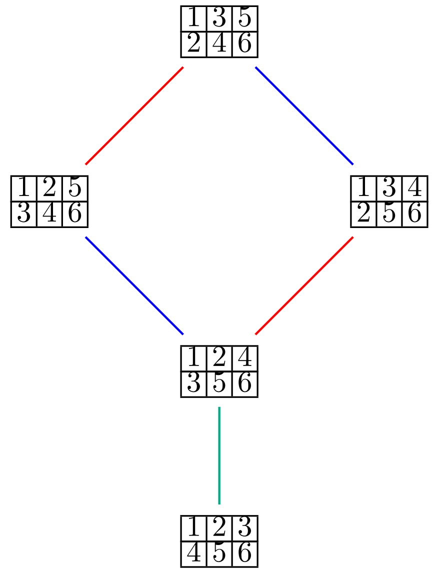
$\underline{\lambda = (4,1,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \cdot & -\frac{1}{3} & \cdot & \cdot \\ \cdot & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & 2 & \cdot & \frac{1}{2} & \cdot & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & \frac{5}{3} & \frac{5}{6} & \frac{1}{3} & \frac{5}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} \\ \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \cdot & \frac{5}{6} & \frac{1}{2} & \frac{1}{6} & -\frac{2}{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{10}{3} & \cdot & \frac{2}{3} & -\frac{2}{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \end{pmatrix}$$



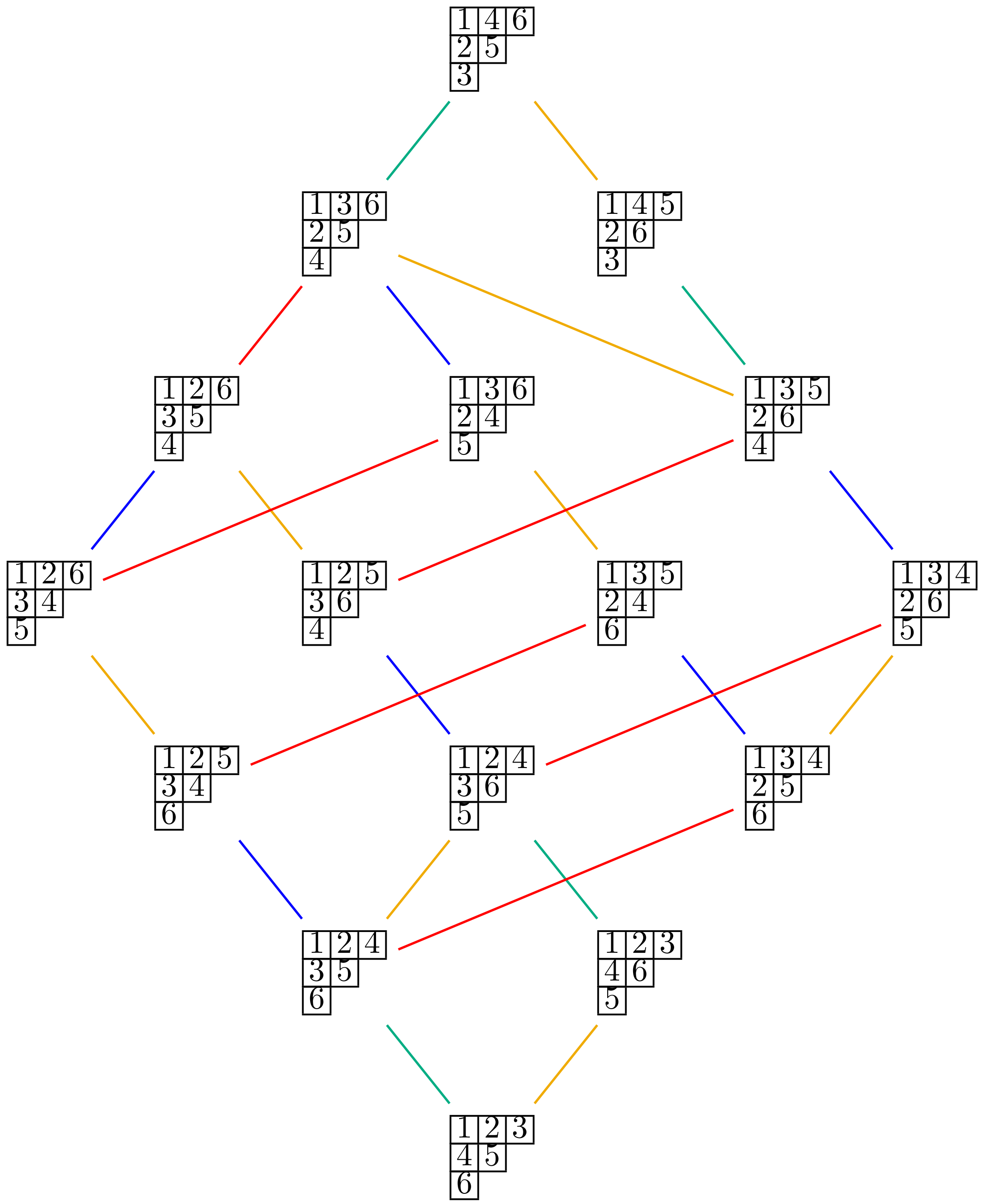
$\underline{\lambda = (3,3)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \cdot & \frac{3}{2} & \cdot & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \frac{9}{4} & \frac{3}{4} \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix}$$



$\underline{\lambda = (3,2,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix}
1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{12} \\
\cdot & \frac{4}{3} & \cdot & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{5}{12} & \frac{5}{24} & \frac{1}{6} & -\frac{7}{24} \\
\cdot & \cdot & \frac{3}{2} & \cdot & \cdot & \frac{1}{2} & \cdot & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \cdot & \frac{1}{4} \\
\cdot & \cdot & \cdot & 2 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \frac{1}{2} & \frac{1}{2} & \cdot & \frac{5}{8} & -\frac{1}{2} & -\frac{5}{8} \\
\cdot & \cdot & \cdot & \cdot & 2 & \cdot & 1 & \cdot & \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{2} & -\frac{1}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & -\frac{3}{4} & \frac{5}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \frac{3}{4} & \frac{3}{2} & \cdot & \frac{3}{8} & \frac{3}{2} & \frac{3}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \frac{3}{2} & \frac{3}{4} & \cdot & \frac{3}{8} & -\frac{3}{4} & -\frac{3}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \cdot & \frac{5}{4} & \cdot & \frac{5}{4} & \frac{5}{8} & \cdot & -\frac{5}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \cdot & \frac{5}{4} & \frac{5}{4} & \frac{5}{8} & \frac{5}{4} & \frac{5}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{15}{4} & \cdot & \cdot & \frac{15}{8} & \cdot & \frac{15}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{15}{4} & \cdot & \frac{15}{8} & \frac{5}{4} & \frac{5}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{15}{4} & \frac{15}{8} & \cdot & \frac{15}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{45}{8} & \cdot & \frac{15}{8} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & \frac{5}{2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{15}{2}
\end{pmatrix}$$



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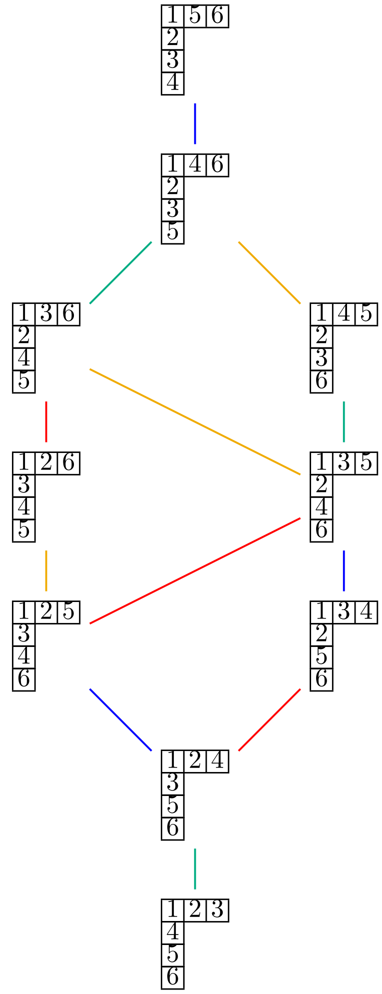
6

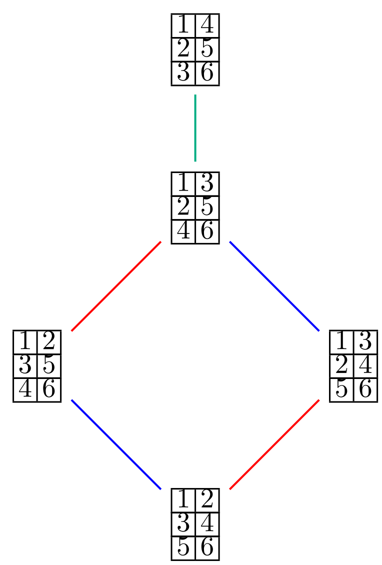
$\underline{\lambda = (3,1,1,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \cdot & \cdot & \cdot \\ \cdot & \frac{5}{4} & \frac{5}{12} & \frac{1}{4} & -\frac{5}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{3} & \frac{1}{3} & \cdot \\ \cdot & \cdot & \frac{5}{3} & \cdot & \frac{5}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{3}{2} & \cdot & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \cdot & \frac{1}{2} & \cdot & -\frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 & \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \frac{3}{4} & -\frac{3}{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \frac{5}{4} & \frac{5}{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{15}{4} & \frac{5}{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \end{pmatrix}$$



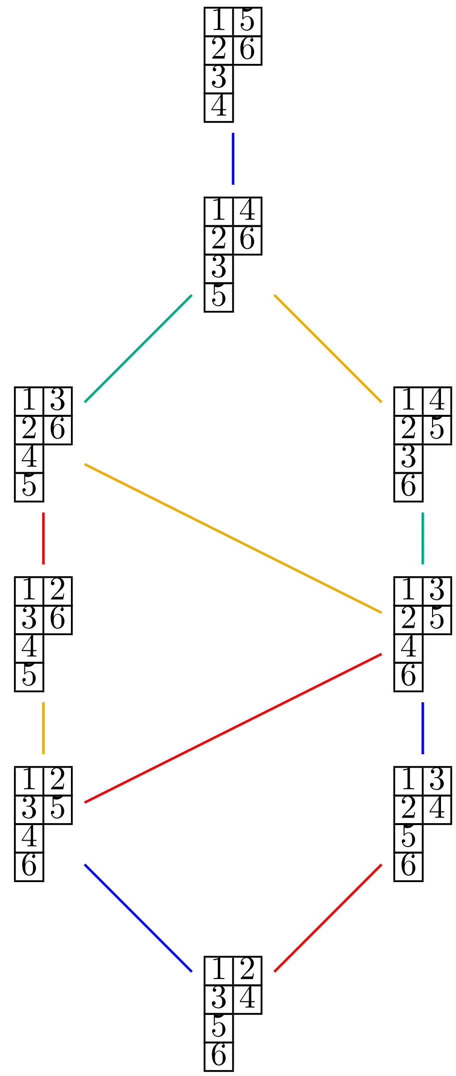
$\underline{\lambda = (2,2,2)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \cdot & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \cdot & \cdot & 2 & \cdot & 1 \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix}$$



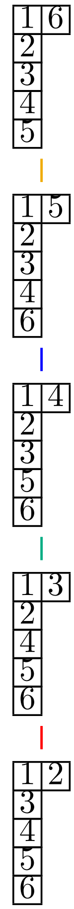
$\underline{\lambda = (2,2,1,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{6} & -\frac{1}{6} \\ \cdot & \frac{5}{4} & \frac{5}{12} & \frac{5}{12} & -\frac{5}{12} & \frac{5}{36} & -\frac{5}{36} & \frac{5}{18} & -\frac{5}{18} \\ \cdot & \cdot & \frac{5}{3} & \cdot & \frac{5}{6} & \frac{5}{9} & \frac{5}{18} & -\frac{5}{9} & -\frac{5}{18} \\ \cdot & \cdot & \cdot & \frac{5}{3} & \cdot & \frac{5}{9} & -\frac{5}{9} & -\frac{5}{9} & \frac{5}{9} \\ \cdot & \cdot & \cdot & \cdot & \frac{5}{2} & \cdot & \frac{5}{6} & \cdot & -\frac{5}{6} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{20}{9} & \frac{10}{9} & \frac{10}{9} & \frac{5}{9} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{10}{3} & \cdot & \frac{5}{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{10}{3} & \frac{5}{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \end{pmatrix}$$



$\underline{\lambda = (2,1,1,1,1)}$

$$\mathcal{A}_\lambda = \begin{pmatrix} 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ \cdot & \frac{6}{5} & \frac{3}{10} & -\frac{3}{10} & \frac{3}{10} \\ \cdot & \cdot & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 3 \end{pmatrix}$$



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