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Bases for McKay Centralizer Algebras

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Abstract

The finite subgroups of the special unitary group SU_2 have been classified to be isomorphic to one of the following groups: cyclic, binary dihedral, binary tetrahedral, binary octahedral, and binary icosahedral, of order n, $4n$, 24, 48, and 120, respectively. Associated to each group is a representation graph, which by the McKay correspondence is a Dynkin diagram of type $\hat{\mathrm{A}}_{n-1}$, $\hat{\mathrm{D}}_{n+2}$, $\hat{\mathrm{E}}_6$, $\hat{\mathrm{E}}_7$, or $\hat{\mathrm{E}}_8$. The centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$ is the algebra of transformations that commute with G acting on the k-fold tensor product of the defining representation $\mathsf{V} = \mathbb{C}^2$. The dimension of the centralizer algebra equals the number of walks on the corresponding Dynkin diagram, beginning and ending at the root node. These dimensions are generalizations of Catalan numbers, with formulas which include binomial coefficients and the Lucas numbers. In uniform ways, we find two bases for these algebras which are each in bijection with the walks on the Dynkin diagram, the first of which works for the binary tetrahedral, octahedral, and icosahedral groups, and the second of which we conjecture to work for all groups, but has only been shown to work for low values of k. This result allows us to give a presentation of generators and relations for the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$. These results answer an open question in combinatorial representation theory.

Contents

Introduction

This paper finds a novel basis for the centralizer algebras of finite subgroups of the special unitary group SU_2. The introduction motivates this discovery, and provides an overview of previous work in this field.

We begin with the special unitary group SU_2. All finite subgroup G of SU_2 must be isomorphic to one of the following:

(a) a cyclic group $\mathbf{C}_n$ of order n,

(b) a binary dihedral group $\mathbf{D}_n$ of order $4n$, or

(c) one of three exceptional groups:

- the binary tetrahedral group $\mathbf{T}$ or order 24,
- the binary octahedral group $\mathbf{O}$ or order 48, or
- the binary icosahedral group $\mathbf{I}$ or order 120.

Each of the above subgroups G of SU_2, and SU_2 itself act on $\mathsf{V} = \mathbb{C}^2$ by restriction of the action of SU_2, and so we can also investigate V as a G-module.

For any of the above subgroup $\mathsf{G} \subseteq \mathsf{SU}_2$, we will investigate the tensor product G-module described as follows. For some basis $\{\mathsf{v}_{-1}, \mathsf{v}_{+1}\}$ of V,

$$\mathsf{V}^{\otimes k} = \text{span}\{\mathsf{v}_{i_1} \otimes \mathsf{v}_{i_2} \otimes \cdots \otimes \mathsf{v}_{i_k} \mid i_j \in \{-1, +1\}\}.$$

The centralizer algebra of $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$ is the algebra of all G-module homomorphisms from $\mathsf{V}^{\otimes k}$ to itself. The inclusion $\mathsf{G} \subseteq \mathsf{SU}_2$ implies the reverse inclusion $\mathsf{Z}_k(\mathsf{SU}_2) \subseteq \mathsf{Z}_k(\mathsf{G})$. There is also a natural embedding $\mathsf{Z}_k(\mathsf{G}) \hookrightarrow \mathsf{Z}_{k+1}(\mathsf{G})$ given by $a \to a \otimes I_2$, which acts as a on the first k tensor factors and the 2×2 identity matrix I_2 on the last factor. Thus the tower of algebras

$$\mathsf{Z}_0(\mathsf{G}) \subseteq \mathsf{Z}_1(\mathsf{G}) \subseteq \mathsf{Z}_2(\mathsf{G}) \subseteq \cdots$$

can be studied using a recursive approach, which we rely on.

The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ is the graph whose nodes are indexed by the set of irreducible G modules $\Lambda(\mathsf{G})$, with $a_{\lambda,\mu}$ edges between vertices $\lambda, \mu \in \Lambda(\mathsf{G})$ if

$$\mathsf{V} \otimes \mathsf{G}^{(\lambda)} = \bigoplus_{\mu \in \Lambda(\mathsf{G})} a_{\lambda,\mu} \mathsf{G}^{(\mu)}.$$

For SU_2 and each finite subgroup ($n > 2$ if $\mathsf{G} = \mathbf{C}_n$ or $\mathbf{D}_n$), these graphs are connected and simple, both highly non-trivial properties. Further, the McKay correspondence shows that the representation graphs for SU_2, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{T}$, and $\mathbf{O}$ are the simply laced, affine Dynkin Diagrams of type $\hat{\mathrm{A}}_{+\infty}$, $\hat{\mathrm{A}}_{n-1}$, $\hat{\mathrm{D}}_{n+2}$, $\hat{\mathrm{E}}_6$, $\hat{\mathrm{E}}_7$, and $\hat{\mathrm{E}}_8$, respectively. These graphs are shown in Figures 1.1,1.2, 1.3, 1.4, 1.5, and 1.6. The Bratteli diagram $\mathcal{B}_{\mathsf{V}}(G)$ is a rooted, directed graph, with vertices at level k taken from the set $\Lambda_k(\mathsf{G}) \in \Lambda(G)$ of vertices in $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ which can be reached by a walk of k steps starting at 0, and $a_{\lambda,\mu}$ edges from $\lambda \in \Lambda_k(\mathsf{G})$ to $\mu \in \Lambda_{k+1}(\mathsf{G})$. The first few rows of the Bratteli diagrams for $\mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$ can be found in Figures 1.7, 1.8, 1.9, and 1.10.

Past work (see [BBH]) shows that the size of the set $\mathcal{W}_k^\lambda$ of walks of length k on $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ starting at the vertex $0 \in \Lambda(\mathsf{G})$ and ending at some vertex $\lambda \in \Lambda(\mathsf{G})$ is exactly the multiplicity of $\mathsf{G}^{(\lambda)}$ in the decomposition of $\mathsf{V}^{\otimes k}$ into irreducibles, which is also the size of the set $\mathcal{P}_k^\lambda$ of paths on $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ from $0 \in \Lambda_0(\mathsf{G})$ to $\lambda \in \Lambda_k(\mathsf{G})$. Double centralizer theory and general Wedderburn theory give that

$$\dim(\mathsf{Z}_k(\mathsf{G})) = \sum_{\lambda \in \Lambda_k(\mathsf{G})} \dim(\mathsf{G}^{(\lambda)})^2,$$

which by simple combinatorial argument gives that

$$\begin{aligned} \dim(\mathsf{Z}_k(\mathsf{G})) = |\mathcal{W}_{2k}^0| = &\ \#\{\text{Walks of length } 2k \text{ from } 0 \text{ to } 0 \text{ on } \mathcal{R}_{\mathsf{V}}(\mathsf{G})\}, \\ = |\mathcal{P}_{2k}^0| = &\ \#\{\text{Paths from } 0 \in \Lambda_0(\mathsf{G}) \text{ to } 0 \in \Lambda_{2k}(\mathsf{G}) \text{ on } \mathcal{B}_{\mathsf{V}}(\mathsf{G})\}. \end{aligned}$$

Given the above equations, it is natural to ask: is there a natural basis $\{w_p \mid p \in \mathcal{P}_{2k}^0\}$ for $\mathsf{Z}_k(\mathsf{G})$.

We answer the above question affirmatively in a majority of cases. Let e be a pseudo-projection ($\mathsf{e}^2 = 2\mathsf{e}$) onto the antisymmetric tensors of $\mathsf{V}^{\otimes 2}$, and let $\mathsf{f}^{(\lambda)}$ be the projection onto $\mathsf{G}^{(\lambda)}$ from the first tensor module $\mathsf{V}^{\otimes k}$ in which $\mathsf{G}^{(\lambda)}$ has nonzero multiplicity (this multiplicity will always be 1). Then, for $\mathsf{G} = \mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, or $\mathbf{I}$, we construct the set $\{w_p \mid p \in \mathcal{P}_{2k}^0\}$ for $\mathsf{Z}_k(\mathsf{G})$ as a collection of words in the generators

$$\mathsf{e}_i = \underbrace{I_2 \otimes \cdots \otimes I_2}_{i-1 \text{ factors}} \otimes\ \mathsf{e}\ \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-i-1 \text{ factors}}$$

and

$$\mathsf{f}_i^{(\lambda)} = \underbrace{I_2 \otimes \cdots \otimes I_2}_{i-1 \text{ factors}} \otimes \mathsf{f}^{(\lambda)} \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-i-|\lambda|+1 \text{ factors}},$$

where I_2 acts as the 2×2 identity matrix on each tensor factor V, and $|\lambda|$ is the number associated with the label λ. A description of the process can be found in Section 3.2, with a proof is given in Section 3.3.2.

This result also allows us to give a presentation of $\mathsf{Z}_k(\mathsf{G})$ on generators and relations, which can be found in Section 2.3.2. Thus, we find generators and relations which are sufficient to describe the centralizer algebra entirely.

The arguments of the thesis supporting the described results are organized as follows:

1. In the first chapter, we give a brief overview of the representation theory underlying this thesis before examining the specific case of SU_2 and its finite subgroups. We discuss the irreducible modules and the tensor module $\mathsf{V}^{\otimes k}$ for each subgroup. Following this, we give a detailed discussions of the McKay correspondence, and the centralizer algebra, including the double centralizer theorem, culminating in formulas for the dimensions of each centralizer algebra $\mathsf{Z}_k(\mathsf{G})$. Throughout, we provide detailed figures of representation graphs and Bratteli Diagrams.

2. The second chapter aims to develop a deeper understanding of the centralizer algebra, including the theory of the generators and the recursive construction mentioned above. We give explicit formulas for a number of the projectors $\mathsf{f}^{(\lambda)}$, and show the decomposition of exceptional subgroup modules into irreducibles for significant values of k. Finally, we give a presentation on generators and relations for $\mathsf{Z}_k(\mathsf{G})$ in certain cases, and also give a number of other interesting relations which we can derive from this presentation.

3. Chapter 3 is focused on the main result of the thesis, the basis $\{w_p \mid p \in \mathcal{P}_{2k}^0\}$ for $\mathsf{G} = \mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$. We give an algorithm for constructing this basis, and show that the result is in fact a basis. Further, we provide a large number of examples of the aforementioned algorithm in a variety of cases.

4. The fourth and final chapter describes a conjecture which we came across throughout the research process. We found an alternative bases which works with low values of k for every group, however, we were

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Chapter 1

Preliminaries

1.1 SU_2 and its Finite Subgroups

The special unitary group $\mathsf{SU}(2)$ is the group of 2×2 matrices with entries from the complex numbers $\mathbb{C}$ defined by

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \;\middle|\; \alpha\overline{\alpha} + \beta\overline{\beta} = 1,\ \alpha, \beta \in \mathbb{C} \right\} \tag{1.1}$$

where $\overline{\alpha}$ is the complex conjugate of α.

A result due to Felix Klein [Kl] in 1888 is that every finite subgroup $\mathsf{G} \subseteq \mathsf{SU}_2$ must be isomorphic to one of three families of groups, each of which manifests in the symmetries of simple geometric objects.

The classification includes the following two infinite families of groups, with one member for each integers $n \geq 2$,

$\mathbf{C}_n$: Cyclic groups of order n, presented by presented by

$$\mathbf{C}_n = \langle\, g \,|\, g^n = 1 \,\rangle.$$

$\mathbf{D}_n$: Binary dihedral groups of order $4n$, presented by

$$\mathbf{D}_n = \langle\, g, h \,|\, g^{2n} = 1, g^n = h^2, h^{-1}gh = g^{-1} \,\rangle.$$

This presentation gives that $\mathbf{D}_1 \cong \mathbf{C}_4$, hence our insistence that $n \geq 2$.

The remaining classifications are one of three so called "exceptional subgroups," which are closely related to the groups A_4, S_4, and A_5 which correspond to the rotational symmetries of the Tetrahedron, Octahedron, and Icosahedron respectively. Specifically, for an exceptional subgroup G, the center $Z(\mathsf{G}) = \{I_2, -I_2\}$ where I_2 is the identity element and $-I_2$ the additive inverse of I_2, and further $\mathsf{G}/Z(\mathsf{G})$ is isomorphic to one of the aforementioned polyhedral symmetry groups. Thus the following groups form what is known as a *two-fold cover* of A_4, S_4, and A_5

T : The binary tetrahedral group of order 24, presented by

$$\mathbf{T} = \langle\, g, h \,|\, (gh)^2 = g^3 = h^3 \,\rangle,$$

which is-a two fold cover of the alternating group A_4.

O : the binary octahedral group of order 48, presented by

$$\mathbf{O} = \langle\, g, h \,|\, (gh)^2 = g^3 = h^4 \,\rangle,$$

which is a two-fold cover of the symmetric group S_4.

I : the binary Icosahedral group of order 120, presented by

$$\mathbf{I} = \langle\, g, h \,|\, (gh)^2 = g^3 = h^5 \,\rangle,$$

which is a two-fold cover of the alternating group A_5.

For each of the above groups, a concrete (matrix) definition of the generators g and h can be found in Section 1.3.1.

While a detailed version of the proof of the classification of these subgroups can be found in [Ar, Sections 8.3 & 5.9], the essence of the proof comes from a surjective homomorphism $\phi : \mathsf{SU}_2 \to \mathsf{SO}_3$, the group of 3×3 orthogonal matrices. The group SO_3 is known to only have finite subgroups which are isomorphic to cyclic, dihedral, and regular polyhedral symmetry, and the kernel $\ker(\phi) = \{I_2, -I_2\}$, where I_2 is the 2×2 identity matrix, and thus by the isomorphism theorems, all finite subgroups of SU_2 must be twofold covers of cyclic, dihedral, and regular polyhedral symmetry groups.

We can also construct infinite analogues of $\mathbf{C}_n$ and $\mathbf{D}_n$, respectively $\mathbf{C}_\infty$ and $\mathbf{D}_\infty$, by setting $n = \infty$, i.e. there is no integer $n \in \mathbb{Z}$ for which $g^n = 1$, which we do in Chapter 4. These groups share a number of properties with their finite counterparts.

1.2 Representations and Modules

We now introduce two tightly interrelated concepts which are central to the work of this paper, those of representations and modules. Unless explicitly attributed elsewhere, all material in this section refers to [JL], which is also an excellent source for further detail and examples for introductory elements of representation theory.

For the most part, we assume that the reader has a functional knowledge of group theory and linear algebra equivalent to a standard first course in each. If need be, [JL, Chapters 1& 2] gives a brief but sufficient overview, and [Ar, Chapters 2& 3] gives a thorough overview.

1.2.1 Representations

Artin [Ar, Sec. 9.1] defines a *representation* of a group G over a finite-dimensional vector space V over a field $\mathbb{F}$ is a homomorphism $\rho : G \to \mathsf{GL}(V)$. In other words, a representation defines a G-action on V which has the properties of a linear transformation: for any $g, h \in G$, $v, w \in V$, and $\lambda \in \mathbb{F}$,

1. $\rho(gh)v = \rho(g)(\rho(h)v)$,
2. $\rho(g)(\lambda v) = \lambda(\rho(gv)$,
3. $\rho(g)(v + w) = \rho(g)v + \rho(g)w$.

From these properties, we can derive two more essential properties, first that $\rho(1_G) = I_n$, where 1_G is the identity element of G, and I_n is the $n \times n$ identity matrix in $\mathsf{GL}(V)$ (if $\dim(V) = n$), and further that $\rho(g^{-1}) = (\rho g)^{-1}$.

The *kernel* of a representation $\mathsf{ker}(p) = \{g \in G \mid p(g) = I_n\}$. A representation is *faithful* if $\mathsf{ker}(p) = \{\mathbf{1}_G\}$, and thus representation is faithful if and only if it is one-to-one. Further, by the first isomorphism theorem, $G/\mathsf{ker}(\rho) \cong \mathsf{im}(\rho)$, and so if ρ is faithful then the image $\mathsf{im}(\rho)$ is isomorphic to G.

1.2.2 Modules

Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let G be a group. Then V is an $\mathbb{F}G$*-module* if there exists a binary operation between elements $g \in G$ and $v \in V$, denoted $g \cdot v$, which satisfies the following properties for any $g, h \in G$, $v, w \in V$ and $\lambda \in \mathbb{F}$:

1. $g \cdot v \in V$,
2. $(gh) \cdot v = g \cdot (h \cdot v)$,
3. $1_G \cdot v = v$,
4. $g \cdot (\lambda v) = \lambda(g \cdot v)$,
5. $g \cdot (v + w) = g \cdot v + g \cdot w$.

When the base field $\mathbb{F}$ is clear, we may omit it and just say that V is a G-module.

This definition may seem reminiscent of a representation in some ways; in fact representations and modules share a deep connection.

Theorem 1.2. *For a group G and a vector space V with base field $\mathbb{F}$,*

1. *If $\rho : G \to \mathsf{GL}$ is a representation of G over $\mathbb{F}$, then V becomes an $\mathbb{F}G$-module if we define the multiplication $g \cdot v$ as*
$$\rho(g)v \quad \text{for } g \in G \text{ and } v \in V.$$
2. *If V is a $\mathbb{F}G$-module, and $\mathcal{B}$ is a basis for V, then the function*
$$g \to [g]_{\mathcal{B}} \quad \text{for } g \in G$$
is a representation of G over V, where $[g]_{\mathcal{B}}$ denotes the matrix of g on V with respect to the basis $\mathcal{B}$.

This result clearly follows from the definitions of representations and modules.

1.2.3 Module Homomorphisms and Isomorphisms

We now introduce a tool used to compare modules, specifically to find a sort of equivalence. A $\mathbb{F}G$*-module homomorphism* is a linear transformation $\theta : V \to W$ between two $\mathbb{F}G$-modules V and W such that for $g \in G$, $v \in V$, and $\lambda \in \mathbb{F}$,
$$\theta(g \cdot v) = g \cdot \theta(v) \quad \text{and} \quad \theta(\lambda v) = \lambda\theta(v). \tag{1.3}$$

If $\theta : V \to W$ is an invertible $\mathbb{F}G$-module homomorphism, we say that θ is an $\mathbb{F}G$-module *isomorphism*, and further, that V and W are isomorphic as modules.

1.2.4 Submodules and Irreducibility

A *submodule* of an $\mathbb{F}G$-module is a subspace W of V which is also an $\mathbb{F}G$-module, i.e. for any $g \in G$, $w \in W$, $g \cdot w \in W$. This is also known as a G-invariant subspace.

A (sub)module is *irreducible*, or *simple* if it has no submodules other than itself and the trivial module $\{0\}$ (the additive identity of the module). If a module is not irreducible, then we say that it is *reducible*. An $\mathbb{F}G$-module V is *completely reducible*, or *semisimple* if there exists an indexing set J such that

$$V = \bigoplus_{i \in J} U_i, \tag{1.4}$$

such that each U_i is an irreducible $\mathbb{F}G$-submodule of V.

We say that a representation $\rho : G \to \mathsf{GL}(V)$ is irreducible if the corresponding G-module (constructed as seen in Theorem 1.2) is irreducible. Likewise, ρ is reducible if the corresponding G-module is reducible.

1.2.5 Tensor Products

The *tensor product* of two vector spaces V and W over the same field $\mathbb{F}$, written $V \otimes W$ is a vector space with a bilinear map from the cartesian product $V \times W$ to $V \otimes W$,

$$v \times w \to v \otimes w \quad \text{for } v \in V, w \in W, \tag{1.5}$$

such that for every linear map to a vector space U, $\beta : V \times W \to U$, there is a unique linear map $B : V \otimes W \to U$ for which $\beta = B \circ \otimes$ (With $\circ$ denoting composition). These criteria result in a unique space up to isomorphism (see [GW, Appendix B.2.2] and [FH, Appendix B.1]) which behaves in a number of desirable ways. We will give an abridged version of these, taken from [JL, Chap. 19].

First, for $v_1, v_2 \in V$, $w_1, w_2 \in W$, and $\lambda \in \mathbb{F}$, bilinearity guarantees that

1. $(v_1 \otimes w_1) + (v_2 \otimes w_1) = (v_1 + v_2) \otimes w_1$
2. $(v_1 \otimes w_1) + (v_1 \otimes w_2) = v_1 \otimes (w_1 + w_2)$, and
3. $\lambda(v_1 \otimes w_1) = (\lambda v_1) \otimes w_1 = v_1 \otimes (\lambda w_1)$.

If we choose two bases, $\mathcal{B}_V$ and $\mathcal{B}_W$ for V and W respectively, the tensor space $V \otimes W$ has a basis

$$\mathcal{B}_{V \otimes W} = \{v_i \otimes w_j \mid v_i \in \mathcal{B}_V, w_j \in \mathcal{B}_W\}. \tag{1.6}$$

Thus, $V \otimes W$ is the span of these simple tensors $v_i \otimes w_j$, and

$$\dim(V \otimes W) = \dim(V)\dim(W) \tag{1.7}$$

For a group G, if both V and W are $\mathbb{F}G$-modules, we can define a *diagonal* action for G on the tensor space; for any $g \in G$, $v \in V$, $w \in W$,

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w). \tag{1.8}$$

This action fulfills the conditions provided in the definition of a module, and so using the action $V \otimes W$ is also a $\mathbb{F}G$-module. Finally, if $W = W_1 \oplus W_2$ as a G-module, then $V \otimes W = (V \otimes W_1) \oplus (V \otimes W_2)$.

We can express $V \otimes W$ using the basis $\mathcal{B}_{V\otimes W}$ in a nice way using a lexicographic order on $\mathcal{B}_{V\otimes W}$. For $[v]_{\mathcal{B}_V} = (v_1, v_2, \ldots v_n)^T$ and $[w]_{\mathcal{B}_W} = (w_1, w_2, \ldots w_m)^T$,

$$[v \otimes w]_{\mathcal{B}_{V\otimes W}} = (v_1([w]_{\mathcal{B}_W}), v_2([w]_{\mathcal{B}_W}), \ldots v_n([w]_{\mathcal{B}_W})) \tag{1.9}$$

$$= (v_1w_1, v_1w_2, \ldots v_1w_m, v_2w_1, \ldots v_nw_m)^T. \tag{1.10}$$

We can apply the same technique to expressing group action in a $\mathbb{F}G$-module in matrix form. If we again suppose that V and W are $\mathbb{F}G$-modules then for any element $g \in G$,

$$[g]_{\mathcal{B}_V} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{21} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}, \quad [g]_{\mathcal{W}} = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{21} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n}, \end{pmatrix} \tag{1.11}$$

then

$$[g]_{\mathcal{B}_V\otimes\mathcal{B}_W} = \begin{pmatrix} a_{1,1}([g]_{\mathcal{B}_W}) & a_{1,2}([g]_{\mathcal{B}_W}) & \cdots & a_{1,n}([g]_{\mathcal{B}_W}) \\ a_{2,1}([g]_{\mathcal{B}_W}) & a_{2,2}([g]_{\mathcal{B}_W}) & \cdots & a_{2,n}([g]_{\mathcal{B}_W}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}([g]_{\mathcal{B}_W}) & a_{n,2}([g]_{\mathcal{B}_W}) & \ddots & a_{n,n}([g]_{\mathcal{B}_W}) \end{pmatrix} \tag{1.12}$$

$$= \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & \cdots & a_{1,1}b_{1,m} & a_{1,2}b_{1,1} & \cdots & a_{1,n}b_{1,m} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & \cdots & a_{1,1}b_{2,m} & a_{1,2}b_{2,1} & \cdots & a_{1,n}b_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1,1}b_{m,1} & a_{1,1}b_{m,2} & \cdots & a_{1,1}b_{m,m} & a_{1,2}b_{m,1} & \cdots & a_{1,n}b_{m,m} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & \cdots & a_{2,1}b_{1,m} & a_{2,2}b_{1,1} & \cdots & a_{2,n}b_{1,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1}b_{m,1} & a_{n,1}b_{m,2} & \cdots & a_{n,1}b_{m,m} & a_{n,2}b_{m,1} & \cdots & a_{2,n}b_{m,m} \end{pmatrix}. \tag{1.13}$$

Generally, define

$$V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}}, \tag{1.14}$$

with the convention ([FH], [BBH]) that $V^{\otimes 0} = \mathbb{F}$. Thus, $V^{\otimes k}$ has dimension $(\dim(V))^k$, and has a basis

$$\{v_{\mathbf{r}} \mid \mathbf{r} = (r_1, r_2, \ldots, r_k) \in [n]^k\} \tag{1.15}$$

where $v_{\mathbf{r}} = v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_k}$ and $[n] = \{1, 2, \ldots, n\}$.

If V is a G-module for some group G, then $V^{\otimes k}$ is also a G-module, by extension of the diagonal action discussed above. Thus, $g \in G$ acts as

$$g \cdot (v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_k}) = (g \cdot v_{r_1}) \otimes (g \cdot v_{r_2}) \otimes \cdots \otimes (g \cdot v_{r_k}). \tag{1.16}$$

1.2.6 Symmetric and Antisymmetric Tensors

The *symmetric tensors* are the subspace

$$S(V^{\otimes k}) = \left\{ \sum_{s \in \mathfrak{S}_{[n]}} v_{s(\mathbf{r})} \;\middle|\; \mathbf{r} \in [n]^k \right\} \tag{1.17}$$

where $\mathfrak{S}_{[n]}$ is the symmetric group on $[n]$, and

$$s(\mathbf{r}) = (s(r_1), s(r_2), \ldots, s(r_k)) \in [n]^k \tag{1.18}$$

is the result of applying of the permutation s to the elements of $\mathbf{r}$. Viewing $V^{\otimes k}$ as a G-module with the diagonal action, it is known ([JL, Chap. 19] and [FH, Appendix B.2]) that $S(V^{\otimes k})$ is a G-submodule, though the result should be clear from the action of G.

For the two-fold tensor power $V^{\otimes 2}$, the symmetric tensors are

$$span\,\{\{v_i \otimes v_j + v_j \otimes v_i \mid 1 \le i < j \le n\} \cup \{v_i \otimes v_i \mid 1 \le i \le n\}\} \tag{1.19}$$

Define the *antisymmetric tensors*

$$A(V \otimes V) = span\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \le i < j \le n\}. \tag{1.20}$$

If $\mathbb{F} = \mathbb{C}$, [JL] states that the antisymmetric tensors form a $\mathbb{C}G$-submodule of $V^{\otimes 2}$, and that.

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V) \tag{1.21}$$

as a $\mathbb{C}G$-module.

1.3 SU_2-modules and G-modules

Let $\rho : \mathsf{SU}_2 \to \mathsf{GL}(\mathbb{C}^2)$ be given by embedding elements of $\mathsf{SU}_2 \subseteq \mathsf{GL}(\mathbb{C}^2)$ in $\mathsf{GL}(\mathbb{C}^2)$. Defining $\mathsf{V} = \mathbb{C}^2$ with basis given by

$$\mathsf{v}_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathsf{v}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.22}$$

We can apply Theorem 1.2 to show that V is a $\mathbb{C}\mathsf{SU}_2$-module, which is known to be irreducible.

For the remainder of the paper, let G refer to a generic finite subgroup of SU_2 listed in Section 1.1. Then, for any $\mathsf{G} \neq \mathsf{SU}_2$, the restriction $\rho_{\mathsf{G}} : \mathsf{G} \to \mathsf{GL}(\mathsf{V})$ can be used to define $\mathbb{C}\mathsf{G}$-modules. Of the resulting G-modules, each save the $\mathbf{C}_n$-modules is irreducible. In this section, we use the theory of representations and modules to construct the family of G-modules which form an integral part of the motivation of this thesis, most of which is taken from [BBH].

1.3.1 Finite Subgroups of SU_2

The number of irreducible representations of a finite group is well known to be equal to the number of conjugacy classes (see [FH, Sec. 2.3], [JL, Chap. 15]), and so for each finite G, we can find every such irreducible module.

Example 1.23. For each finite subgroup $\mathsf{G} \subseteq \mathsf{SU}_2$, we list the defining representations and describe all irreducible modules.

$\mathbf{C}_n$ The irreducible modules of $\mathbf{C}_n$ are all one dimensional, as $\mathbf{C}_n$ is abelian. They are labeled by

$$\Lambda(\mathbf{C}_n) = \{0, 1, 2, \dots, n-1\}. \tag{1.24}$$

For $\ell \in \Lambda(\mathbf{C}_n)$, the irreducible module $\mathbf{C}_n^{(\ell)}$ is spanned by a single vector, $\mathsf{v}_\emptyset$, with the action of $\mathbf{C}_n$ on $\mathbf{C}_n^{(\ell)}$ given by $g \cdot \mathsf{v}_\emptyset = \omega^\ell \mathsf{v}_\emptyset$, where $\omega = e^{2\pi i/n}$. As a subgroup of SU_2, $\mathbf{C}_n$ has a natural module $\mathsf{V} = \mathbb{C}^2$ such that with respect to the basis of V in Equation 1.22, the matrix of g is

$$[g] = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{n-1} \end{pmatrix}.$$

As a $\mathbf{C}_n$-module, $\mathsf{V} = \mathbf{C}_n^{(1)} \oplus \mathbf{C}_n^{(n-1)}$, and thus is reducible.

$\mathbf{D}_n$ The irreducible modules of $\mathbf{D}_n$ are labeled by

$$\Lambda(\mathbf{D}_n) = \{0, 0', 1, 2, \ldots, n-1, n, n'\}. \tag{1.25}$$

For $\ell \in \{0, 0', n, n'\}$, the module $\mathbf{D}_n^{(\ell)}$ is 1-dimensional, while for $\ell \in \{1, 2, \ldots, n-1\}$, the module $\mathbf{D}_n^{(\ell)}$ is 2-dimensional. As a subgroup of SU_2, $\mathbf{D}_n$ has a natural module $\mathsf{V} = \mathbf{D}_n^{(1)}$. With respect to the basis of V in Equation 1.22, the generators g and h have matrices $[g]$ and $[h]$ given by

$$[g] = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{n-1} \end{pmatrix} \quad \text{and} \quad [h] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{1.26}$$

where $\omega = e^{2\pi i/n}$.

$\mathbf{T}$ The irreducible modules of $\mathbf{T}$ are labeled by

$$\Lambda(\mathbf{T}) = \{0, 1, 2, 3, 3', 4, 4'\}. \tag{1.27}$$

The dimension of each irreducible module is given below.

$\mathbf{T}^{(\ell)}$	$\mathbf{T}^{(0)}$	$\mathbf{T}^{(1)}$	$\mathbf{T}^{(2)}$	$\mathbf{T}^{(3)}$	$\mathbf{T}^{(3')}$	$\mathbf{T}^{(4)}$	$\mathbf{T}^{(4')}$
$\dim(\mathbf{T}^{(\ell)})$	1	2	3	2	2	1	1

As a subgroup of SU_2, $\mathbf{T}$ has a natural module $\mathsf{V} = \mathbf{T}^{(1)}$. With respect to the basis of V in Equation 1.22, the generators g and h have matrices $[g]$ and $[h]$ given by

$$[g] = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega & \omega \\ \omega^3 & \omega^7 \end{pmatrix} \quad \text{and} \quad [h] = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega & \omega^7 \\ \omega^5 & \omega^7 \end{pmatrix},$$

where $\omega = e^{2\pi i/8}$.

$\mathbf{O}$ The irreducible modules of $\mathbf{O}$ are labeled by

$$\Lambda(\mathbf{O}) = \{0, 1, 2, 3, 4, 4', 5, 6\}. \tag{1.28}$$

The dimension of each irreducible module is given below.

$\mathbf{O}^{(\ell)}$	$\mathbf{O}^{(0)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(3)}$	$\mathbf{O}^{(4)}$	$\mathbf{O}^{(4')}$	$\mathbf{O}^{(5)}$	$\mathbf{O}^{(6)}$
$\dim(\mathbf{O}^{(\ell)})$	1	2	3	4	3	2	2	1

As a subgroup of SU_2, $\mathbf{O}$ has a natural module $\mathsf{V} = \mathbf{O}^{(1)}$. With respect to the basis of V in Equation 1.22, the generators g and h have matrices $[g]$ and $[h]$ given by

$$[g] = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega & \omega \\ \omega^3 & \omega^7 \end{pmatrix} \quad \text{and} \quad [h] = \begin{pmatrix} \omega & 0 \\ 0 & \omega^7 \end{pmatrix},$$

where $\omega = e^{2\pi i/8}$.

I The irreducible modules of **I** are labeled by

$$\Lambda(\mathbf{I}) = \{0, 1, 2, 3, 4, 5, 6, 6', 7\}. \tag{1.29}$$

The dimension of each irreducible module is given below.

$\mathbf{I}^{(\ell)}$	$\mathbf{I}^{(0)}$	$\mathbf{I}^{(1)}$	$\mathbf{I}^{(2)}$	$\mathbf{I}^{(3)}$	$\mathbf{I}^{(4)}$	$\mathbf{I}^{(5)}$	$\mathbf{I}^{(6)}$	$\mathbf{I}^{(6')}$	$\mathbf{I}^{(7)}$
$\dim(\mathbf{I}^{(\ell)})$	1	2	3	4	5	6	4	3	2

As a subgroup of SU_2, $\mathbf{I}$ has a natural module $\mathsf{V} = \mathbf{I}^{(1)}$. With respect to the the basis of V in Equation 1.22, the generators g and h have matrices $[g]$ and $[h]$ given by

$$g = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega & \omega \\ \omega^3 & \omega^7 \end{pmatrix} \text{ and } h = \frac{1}{2} \begin{pmatrix} \phi + \frac{i}{\phi} & 1 \\ -1 & \phi - \frac{i}{\phi} \end{pmatrix},$$

where $\omega = e^{2\pi i/8}$ and $\phi = \frac{1+\sqrt{5}}{2}$.

1.3.2 Tensor Products of G-modules

Recalling the definition $\mathsf{V} = \mathbb{C}^2$ as a G-module, we consider the G-module $\mathsf{V}^{\otimes k}$ with the diagonal G-action, as shown in Equation 1.16. Let $\Lambda(\mathsf{G})$ index the complete set of irreducible G-modules (as seen in the previous section), and let $\Lambda(\mathsf{SU}_2)$ be the set of all irreducible SU_2-modules. It is known ([FH, 24-25]) that every module indexed by $\Lambda(\mathsf{G})$ will appear in the G-module decomposition of $\mathsf{V}^{\otimes k}$ for at least one k. Thus, defining $\Lambda_k(\mathsf{G})$ as the label set for the irreducible modules which appear in the decomposition of $\mathsf{V}^{\otimes k}$,

$$\Lambda(\mathsf{G}) = \bigcup_{k \geq 0} \Lambda_k(\mathsf{G}).$$

An exceptionally interesting fact about SU_2 (see [BBH]) is that

$$\Lambda(\mathsf{SU}_2) = \{0, 1, 2, \ldots\}, \tag{1.30}$$

so that for the SU_2-module $\mathsf{V}(i)$ labeled by $i \in \Lambda(\mathsf{SU}_2)$,

$$\dim(\mathsf{V}(i)) = i + 1, \tag{1.31}$$

and since we have already stated that V is irreducible, $\mathsf{V} = \mathsf{V}(1)$. For any integer $r \geq 0$, $\mathsf{V}(r)$ satisfies the Clebsh-Gordan ([GW, 339]) formula

$$\mathsf{V}(r) \otimes \mathsf{V} = \mathsf{V}(r-1) \oplus \mathsf{V}(r+1), \tag{1.32}$$

where $\mathsf{V}(-1) = 0$. This formula allows us to recursively construct a decomposition of $\mathsf{V}^{\otimes k}$ into irreducible submodules.

Generally for any G, [St] shows that if $\mathsf{G}^{(\lambda)}$ is the irreducible G module labeled by $\lambda \in \Lambda(\mathsf{G})$

$$\mathsf{G}^{\lambda} \otimes \mathsf{V} = \bigoplus_{\mu \in \Lambda(\mathsf{G})} a_{\lambda,\mu} \mathsf{G}^{\mu}, \tag{1.33}$$

where $a_{\lambda,\mu}$ is the multiplicity of G^{μ} in the decomposition, then for any $\mu, \lambda \in \Lambda(\mathsf{G})$,

1. $a_{\lambda,\mu} = a_{\mu,\lambda}$,
2. $a_{\lambda,\lambda} = 0$, (1.34)
3. $a_{\lambda,\mu} \in \{0, 1\}$ if $|\mathsf{G}| > 2$.

In fact, the above properties hold for every group discussed in this thesis, as [BBH] shows that they hold for the groups $\mathbf{C}_\infty$ and $\mathbf{D}_\infty$ as well.

1.4 McKay Correspondence

The McKay Correspondence is a central result in the study of representations of SU_2 which associates graphs with certain irreducible module sets of finite groups. In this section, we discuss the McKay Correspondence for each G with the module V, and its implications. The results discussed here are entirely from [BBH].

1.4.1 Representation Graphs

The *representation graph* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (also called the McKay graph of McKay quiver), is a graph with vertex set $\Lambda(\mathsf{G})$ with $a_{\lambda,\mu}$ edges from λ to μ, where $a_{\lambda,\mu}$ is taken from Equation 1.33. The McKay correspondence holds that the representation graphs for V as a module of the groups $\mathsf{SU}_2, \mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$ are isomorphic to the simply-laced affine Dynkin diagrams $\hat{\mathsf{A}}_{+\infty}$,

$\hat{\mathsf{A}}_{n-1}$, $\hat{\mathsf{D}}_{n+2}$, $\hat{\mathsf{E}}_6$, $\hat{\mathsf{E}}_7$, and $\hat{\mathsf{E}}_8$, respectively. This result is fascinating because these Dynkin diagrams appear as part of the classification of Lie algebras (a subject not covered here, see **??**), arising from an entirely independent set computations.

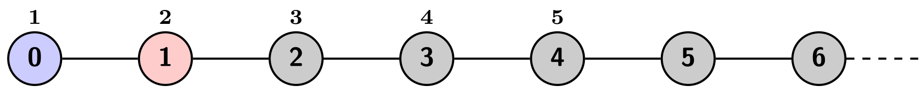


Figure 1.1 The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{SU}_2)$, which is the Dynkin diagram $\hat{\mathsf{A}}_{+\infty}$. The label on the node is the index of the SU_2-module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module V in red.

The graph $\mathcal{R}_{\mathsf{V}}(\mathsf{SU}_2)$, show in Figure 1.1, has vertices labeled by r for $r = 0, 1, 2, \ldots$, and with $a_{i,j}$ edges as described in Equation 1.32. Thus, the graph must be simple, undirected, and connected.

Generally, the conditions in 1.34 guarantee that if $|\mathsf{G}| > 2$, $\mathcal{R}(\mathsf{G})$ is a simple undirected graph. Further, because all irreducible G modules will occur in some $\Lambda_k(\mathsf{G})$, and so $\mathcal{R}(\mathsf{G})$ must be connected. Representation graphs for each finite subgroup G can be found in Figures 1.2, 1.3, 1.4, 1.5, 1.6.

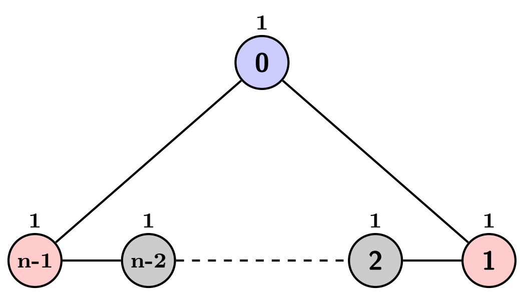


Figure 1.2 The representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{C}_n)$, which is the Dynkin diagram $\hat{\mathsf{A}}_{n-1}$, or the cycle graph on n nodes. The label on each node is the index of the $\mathbf{C}_n$-module, and the label above the node is its dimension. The trivial module is shown in blue and the two irreducible modules whose sum is V.

For $\mathsf{G} \neq \mathbf{C}_n$, the *weight* of a vertex $\lambda \in \Lambda(\mathsf{G})$ of $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$, written $|\lambda|$ is the numerical component of the label (e.g. $|4| = 4$, $|5'| = 5$).

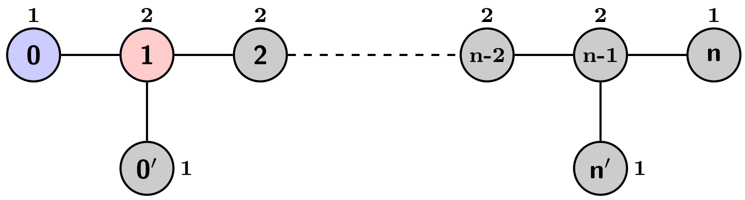


Figure 1.3 The representation graph $\mathcal{R}_\mathsf{V}(\mathbf{D}_n)$, which is the Dynkin diagram $\hat{\mathsf{D}}_{n+2}$. The label on each node is the index of the $\mathbf{D}_n$-module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module $\mathbf{D}_n^{(1)} = \mathsf{V}$ is in red.

$(\mathbf{T}, \hat{\mathsf{E}}_6)$

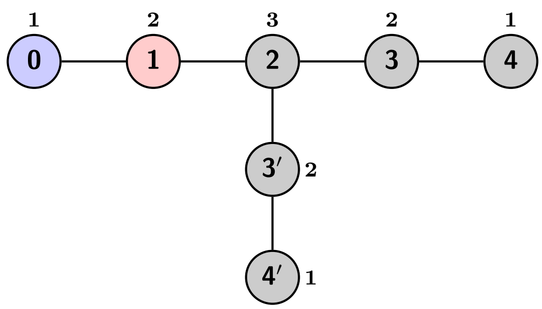


Figure 1.4 The representation graph $\mathcal{R}_\mathsf{V}(\mathbf{T})$, which is the Dynkin diagram $\hat{\mathsf{E}}_6$. The label on each node is the index of the $\mathbf{T}$-module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module $\mathbf{T}^{(1)} = \mathsf{V}$ is shown in red.

We write that that $\mu \prec \lambda$ for two vertices of $\mathcal{R}_\mathsf{V}(\mathsf{G})$, $\lambda, \mu \in \Lambda(\mathsf{G})$ if there is a path in $\mathcal{R}(G)$ $\mu = \nu_1, \nu_2, \ldots, \nu_l = \lambda$ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ such that $|\nu_1| < |\nu_2| < \cdots < |\nu_l|$.

A *branch node* in the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is any vertex of degree greater than 2, as well as the node 0 for $\mathsf{G} = \mathbf{C}_n$. Let $\mathsf{br}(\mathsf{G})$ denote the set of branch nodes in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ for $\mathsf{V} = \mathbb{C}^2$. In the special case of $\mathcal{R}_\mathsf{V}(\mathbf{C}_n)$ for $n \leq \infty$, we consider the affine node 0 to be the branch node. Let the *diameter* of $\mathcal{R}_\mathsf{V}(\mathsf{G})$, denoted by $\mathsf{di}(\mathsf{G})$, be the maximum distance between any vertex

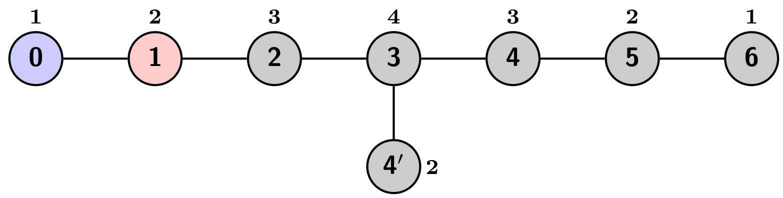


Figure 1.5 The representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{O})$, which is the Dynkin diagram $\hat{\mathsf{E}}_7$. The label on each node is the index of the $\mathbf{O}$-module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module $\mathbf{O}^{(1)} = \mathsf{V}$ is shown in red.

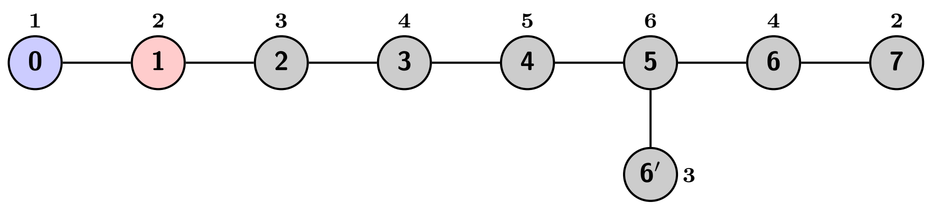


Figure 1.6 The representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{I})$, which is the Dynkin diagram $\hat{\mathsf{E}}_8$. The label on each node is the index of the $\mathbf{I}$-module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module $\mathbf{I}^{(1)} = \mathsf{V}$ is shown in red.

$\lambda \in \Lambda(\mathsf{G})$ and $0 \in \Lambda(\mathsf{G})$. For $\mathsf{G} = \mathbf{C}_n$, we let $\mathsf{di}(\mathsf{G}) = \tilde{n}$ as in Equation 1.35.

G	SU_2	$\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{T}$	$\mathbf{O}$	$\mathbf{I}$	$\mathbf{C}_\infty$	$\mathbf{D}_\infty$
$\mathsf{di}(\mathsf{G})$	∞	$\tilde{n}$	n	4	6	7	∞	∞
$\mathsf{br}(\mathsf{G})$	$\emptyset$	$\{0\}$	$\{1, n-1\}$	$\{2\}$	$\{3\}$	$\{5\}$	$\{0\}$	$\{1\}$

(1.35)

$$\tilde{n} = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases} \tag{1.36}$$

1.4.2 Bratteli Diagrams

Closely related to the representation graph, the *Bratteli diagram* $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ is an infinite rooted graph with vertices at level k labeled by the elements of $\Lambda_k(\mathsf{G})$, and $a_{\lambda,\mu}$ edges between $\lambda \in \Lambda_k(\mathsf{G})$ and $\mu \in \Lambda_{k+1}(\mathsf{G})$, with $a_{\lambda,\mu}$ as defined in Equation 1.33. The first few lines of the Bratteli diagrams for SU_2 and the exceptional subgroups are shown in Figures 1.7, 1.8, 1.9, and 1.10.

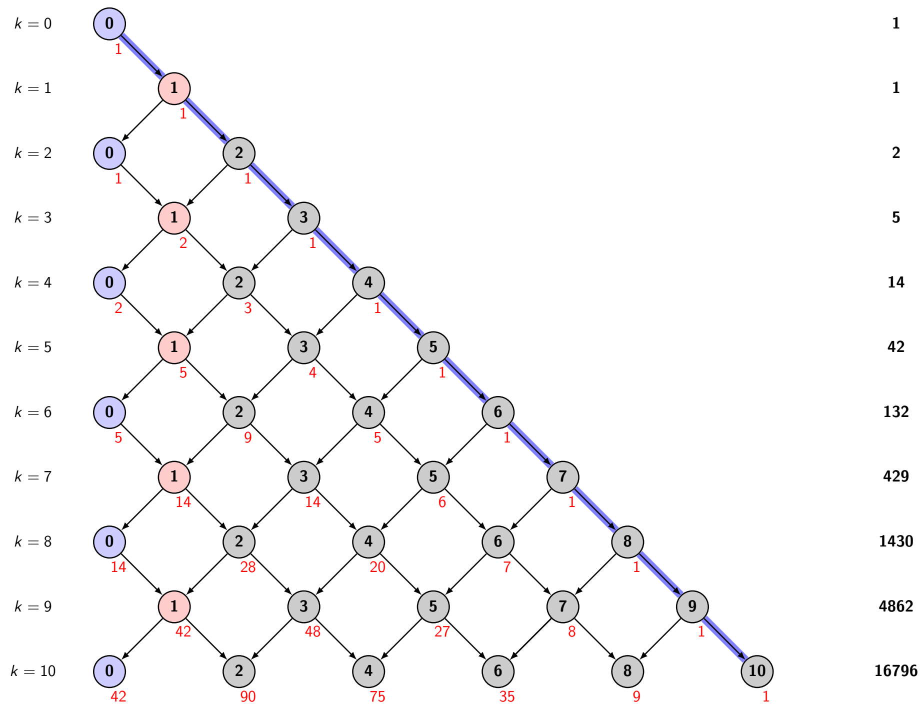


Figure 1.7 The first 10 rows of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathsf{SU}_2)$. Each vertex $\lambda \in \Lambda_k = \Lambda_k(\mathsf{SU}_2)$ (at level k) is labeled in red with the multiplicity of the corresponding irreducible module in the decomposition $\mathsf{V}^{\otimes k}$, which is also $|\mathcal{P}_k^\lambda|$. The edge (λ, μ) between two vertices is highlighted in blue if μ cannot be reached before this level in the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{SU}_2)$, meaning that the module labeled by μ is part of $\mathsf{V}_{\text{new}}^{\otimes k}$ (see Theorem 2.11).
The dimension of $\mathsf{Z}_k(\mathsf{SU}_2)$, which is also the sum of the squares of the multiplicities, or the multiplicity of $0 \in \Lambda_{2k}$ is given on the right side of the figure. For SU_2, the dimension $\mathsf{Z}_k(\mathsf{SU}_2)$ is the kth Catalan number $\mathcal{C}_k$. While there are a number of proofs of this, one which is particularly relevant to this thesis is that paths on the above Bratteli diagram from $0 \in \Lambda_0$ to $0 \in \Lambda_{2k}$ can be seen to be trivially in bijection with Dyck paths, or so called "Catalan Mountains."

Both $\mathcal{R}(\mathsf{G})$ and $\mathcal{B}(\mathsf{G})$ encode the rules of tensoring found in Equation 1.33 in their edge structure, which in fact leads to a useful connection between the two. Let $\mathcal{W}_k^\lambda$ be the set of walks on $\mathcal{R}(\mathsf{G})$ from $0 \in \Lambda(\mathsf{G})$ to $\lambda \in \Lambda(\mathsf{G})$ of length k, and let $\mathcal{P}_k^\lambda$ be the set of paths on $\mathcal{B}(\mathsf{G})$ from the root $0 \in \Lambda_0(\mathsf{G})$ to

$\lambda \in \Lambda_k(\mathsf{G})$. There is a trivial bijection between elements of $\mathcal{W}_k^\lambda$ and $\mathcal{P}_k^\lambda$ and so elements of $\Lambda_k(\mathsf{G})$ are elements which can be reached in exactly k steps from $0 \in \Lambda(\mathsf{G})$ on $\mathcal{R}(\mathsf{G})$. Thus, if

$$\mathsf{V}^{\otimes k} = \bigoplus_{\lambda \in \Lambda(\mathsf{G})} m_k^\lambda \mathsf{G}^\lambda, \tag{1.37}$$

where m_k^λ is the multiplicity of G^λ in the decomposition, induction on Equation 1.33 gives that

$$\begin{aligned} m_k^\lambda &= |\mathcal{W}_k^\lambda| &&= \#(\text{walks on } \mathcal{R}(\mathsf{G}) \text{ of length } k \text{ from at } 0 \text{ to } \lambda) \\ &= |\mathcal{P}_k^\lambda| &&= \#(\text{paths in } \mathcal{B}_{\mathsf{V}}(\mathsf{G}) \text{ from } 0 \in \Lambda_0(\mathsf{G}) \text{ to } \lambda \in \Lambda_k(\mathsf{G})). \end{aligned} \tag{1.38}$$

From this, we can calculate the decomposition of $\mathsf{V}^{\otimes k}$ for arbitrary k using graph properties.

1.5 Centralizer Algebras

In the section, we finally introduce the focus of this thesis. The *Centralizer Algebra* of G on $\mathsf{V}^{\otimes k}$ is the algebra

$$\begin{aligned} \mathsf{Z}_k(\mathsf{G}) &= \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k}) \\ &= \left\{ a \in \mathsf{End}(\mathsf{V}^{\otimes k}) \;\middle|\; a(gw) = ga(w) \text{ for all } g \in \mathsf{G}, w \in \mathsf{V}^{\otimes k} \right\}. \end{aligned} \tag{1.39}$$

If the group G is apparent from context, we write $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$.

We can naturally embed the algebra $\mathsf{Z}_k \hookrightarrow \mathsf{Z}_{k+1}$ with the embedding $a \to a \otimes I_2$, where I the 2×2 identity matrix. Thus, we can approach the study of Z_k as the study of an infinite tower of algebras $\mathsf{Z}_0 \subseteq \mathsf{Z}_1 \subseteq \mathsf{Z}_2 \subseteq \cdots$ by repeated application of this embedding. Again, the entirety of the material in this section first appeared in [BBH], save the dimension formulas in Theorem 1.49, which originate in [Ba].

1.5.1 Double Centralizer Theory

Key to our study of $\mathsf{Z}_k(\mathsf{G})$ is the well-known double centralizer theorem, which is stated in different forms in [BBH], [CR 1981, 3B], and [GW, 4.1.5]. The statement here most closely resembles that found in the first source given.

Theorem 1.40 (Double Centralizer Thoerem)**.** *Let G be a group and V be a semisimple $\mathbb{C}G$-module such that*

$$V^{\otimes k} = \bigoplus_{\lambda \in \Lambda_k(G)} m_k^{\lambda} G^{(\lambda)}$$

is a decomposition of $V^{\otimes k}$ into irreducible $\mathbb{C}G$-submodules, and let $\mathsf{Z}_k(G) = \mathsf{End}_G(V^{\otimes k})$. Then

1. $\mathsf{Z}_k(G)$ *is completely reducible.*
2. *The pairwise non-isomorphic irreducible modules of $\mathsf{Z}_k(G)$ are labeled by elements of $\Lambda_k(G)$, with each such module denoted by $\mathsf{Z}_k^{(\lambda)}$.*
3. $\dim(\mathsf{Z}_k^{(\lambda)}) = m_k^{\lambda}$.
4. *As a $\mathbb{C}\mathsf{Z}_k$-module,*

$$V^{\otimes k} = \bigoplus_{\lambda \in \Lambda_k} \dim(G^{(\lambda)}) \mathsf{Z}_k^{(\lambda)}. \tag{1.41}$$

Applying this result to the centralizer algebra of $\mathsf{V}^{\otimes k}$ for SU_2 and its subgroups, general Wedderburn theory ([CR 1966, Sec. 26.1], [Ram, Sec. 4]) gives that $\mathsf{Z}_k(\mathsf{G})$ decomposes as a $\mathsf{Z}_k(\mathsf{G})$-module as

$$\mathsf{Z}_k(\mathsf{G}) = \bigoplus_{\lambda \in \Lambda} m_k^{\lambda} \mathsf{Z}_k^{(\lambda)}, \tag{1.42}$$

and so $\dim(\mathsf{Z}_k(\mathsf{G}))$ is the sum of the squares of the dimensions of its irreducible modules. Thus the double centralizer theorem gives that

$$\begin{aligned} \dim(\mathsf{Z}_k(\mathsf{G})) &= \sum_{\lambda \in \Lambda_k(\mathsf{G})} (\dim(\mathsf{Z}_k^{\lambda}))^2 && (1.43) \\ &= \sum_{\lambda \in \Lambda_k(\mathsf{G})} (m_k^{\lambda})^2 && (1.44) \\ &= \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{W}_0^{\lambda}|^2. && (1.45) \end{aligned}$$

Now, consider a walk $w \in \mathcal{W}_{2k}^0$. As there are $2k$ steps, we can choose a middle element λ so that w can be subdivided into a walk of k steps from 0 to λ, and a walk of k steps from λ to 0. Reversing the second walk, we see that w uniquely determines an ordered pair of elements of $\mathcal{W}_k^{\lambda}$, and vice versa. With this, we come to the culminating result of the section:

Theorem 1.46. *For* $\mathsf{G} = \mathsf{SU}_2, \mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}$, *and* $\mathbf{I}$,

$$\dim(\mathsf{Z}_K(\mathsf{G})) = \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{W}^0_{2k}| \tag{1.47}$$

$$= \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{P}^0_{2k}|. \tag{1.48}$$

1.5.2 Dimension Formulas

Using the results presented in this section, [BBH] provides formulas for the dimension of $\mathsf{Z}_k(\mathsf{G})$ for each $\mathsf{G} \subseteq \mathsf{SU}_2$ given in Section 1.1, which is also the multiplicity of Z^0_{2k} in $\mathsf{V}^{\otimes 2k}$.

Theorem 1.49 ([BBH], [Ba] Dimension Formulas)**.** *For* $k \geq 1$, *the following formulas give the dimension* $\dim \mathsf{Z}_k(\mathsf{G})$ *of the McKay centralizer algebra, which also equals the number of* $2k$*-walks on the representation graph* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ *from* 0 *to* 0*.*

(a) $\dim(\mathsf{Z}_k(\mathbf{C}_n)) = 2\ \dim(\mathsf{Z}_k(\mathbf{D}_n)) = \sum_{\substack{0 \leq a,b \leq k \\ a \equiv b \bmod \tilde{n}}} \binom{k}{a}\binom{k}{b}$, *the* $2k$*-*k *coefficient in Pascal's triangle on a cylinder of "diameter"* $\tilde{n}$ (*Fig.* 1.35).

(b) $\dim(\mathsf{Z}_k(\mathbf{T}) = \dfrac{4^k + 8}{12}$ ([OEIS] OEIS sequence A047849).

(c) $\dim(\mathsf{Z}_k(\mathbf{O})) = \dfrac{4^k + 6 \cdot 2^k + 8}{24}$ ([OEIS] OEIS sequence A007581).

(d) $\dim(\mathsf{Z}_k(\mathbf{I})) = \dfrac{4^k + 12L_{2k} + 20}{60}$, *where* L_n *is the Lucas number defined by* $L_0 = 2, L_1 = 1,$ *and* $L_{n+2} = L_{n+1} + L_n$*.*

Further provided are formulas for the dimensions of each irreducible module of $\mathsf{Z}_k(\mathsf{G})$, which are also the multiplicities of the irreducible G-modules in the decomposition of $\mathsf{V}^{\otimes k}$. These results, however, are not included here due to their limited relevance.

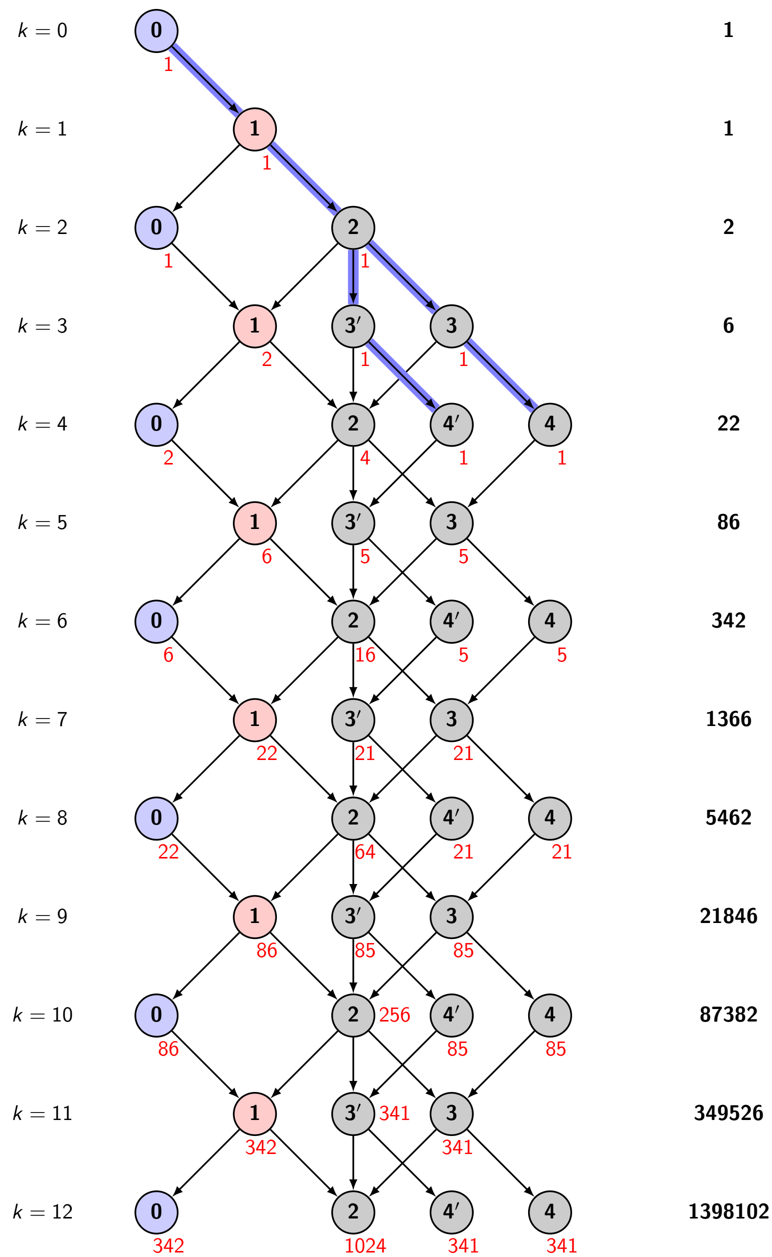


Figure 1.8 The first 10 rows of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathbf{T})$. Each vertex $\lambda \in \Lambda_k = \Lambda_k(\mathbf{T})$ (at level k) is labeled in red with the multiplicity of the corresponding irreducible module in the decomposition $\mathsf{V}^{\otimes k}$, which is also $|\mathcal{P}_k^\lambda|$. The edge (λ, μ) between two vertices is highlighted in blue if μ cannot be reached before this level in the representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{T})$, meaning that the module labeled by μ is part of $\mathsf{V}_{\text{new}}^{\otimes k}$ (see Theorem 2.11).
The dimension of $\mathsf{Z}_k(\mathbf{T})$, which is also the sum of the squares of the multiplicities, or the multiplicity of $0 \in \Lambda_{2k}$ is given on the right side of the figure, corresponding the the formula given in Theorem 1.49. Note the interesting phenomena that pairs of modules labeled by 3 and $3'$, and by 4 and $4'$ have the same dimension.

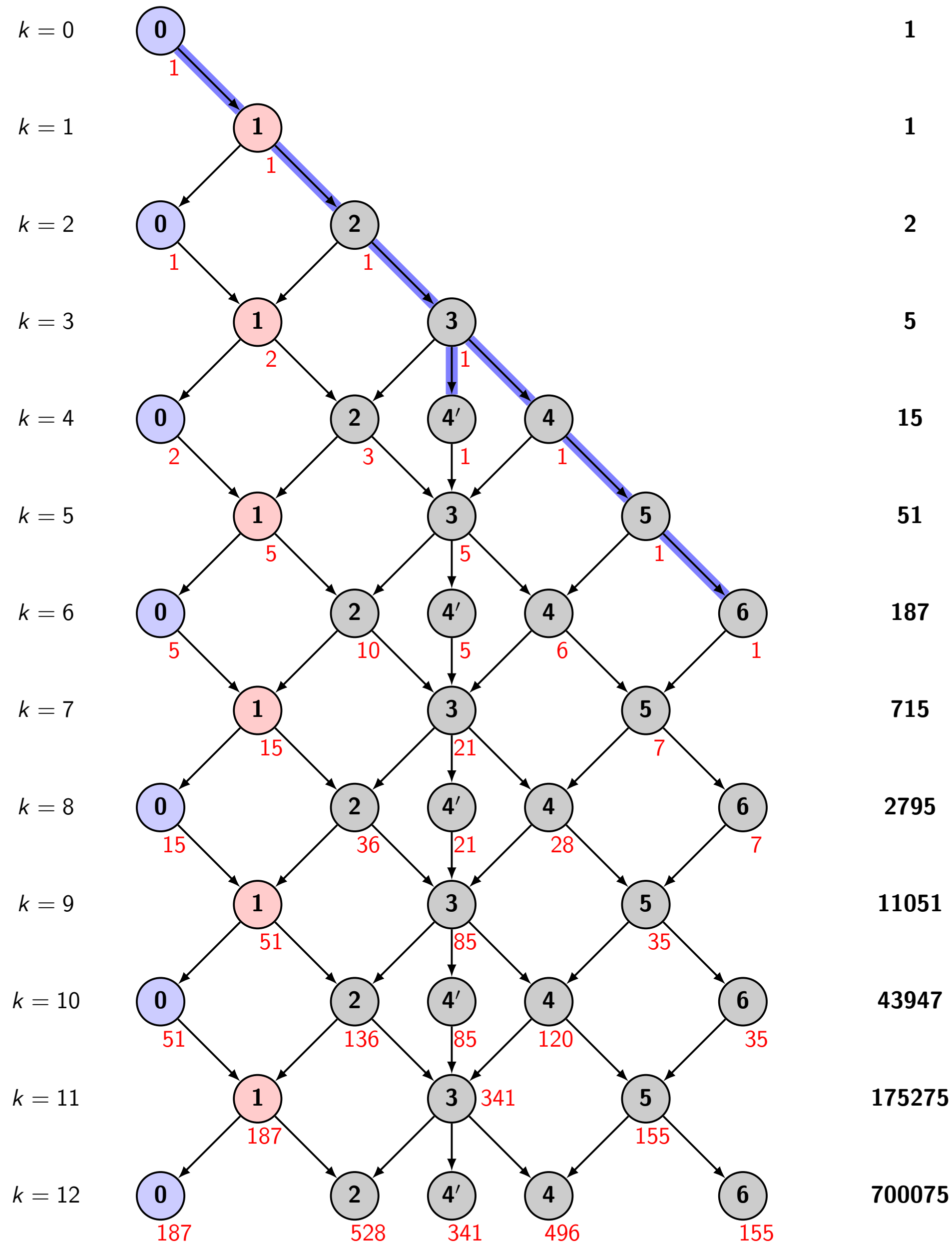


Figure 1.9 The first 10 rows of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathbf{O})$. Each vertex $\lambda \in \Lambda_k = \Lambda_k(\mathbf{O})$ (at level k) is labeled in red with the multiplicity of the corresponding irreducible module in the decomposition $\mathsf{V}^{\otimes k}$, which is also $|\mathcal{P}_k^{\lambda}|$. The edge (λ, μ) between two vertices is highlighted in blue if μ cannot be reached before this level in the representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{O})$, meaning that the module labeled by μ is part of $\mathsf{V}_{\mathsf{new}}^{\otimes k}$ (see Theorem 2.11).
The dimension of $\mathsf{Z}_k(\mathbf{O})$, which is also the sum of the squares of the multiplicities, or the multiplicity of $0 \in \Lambda_{2k}$ is given on the right side of the figure, corresponding the the formula given in Theorem 1.49.

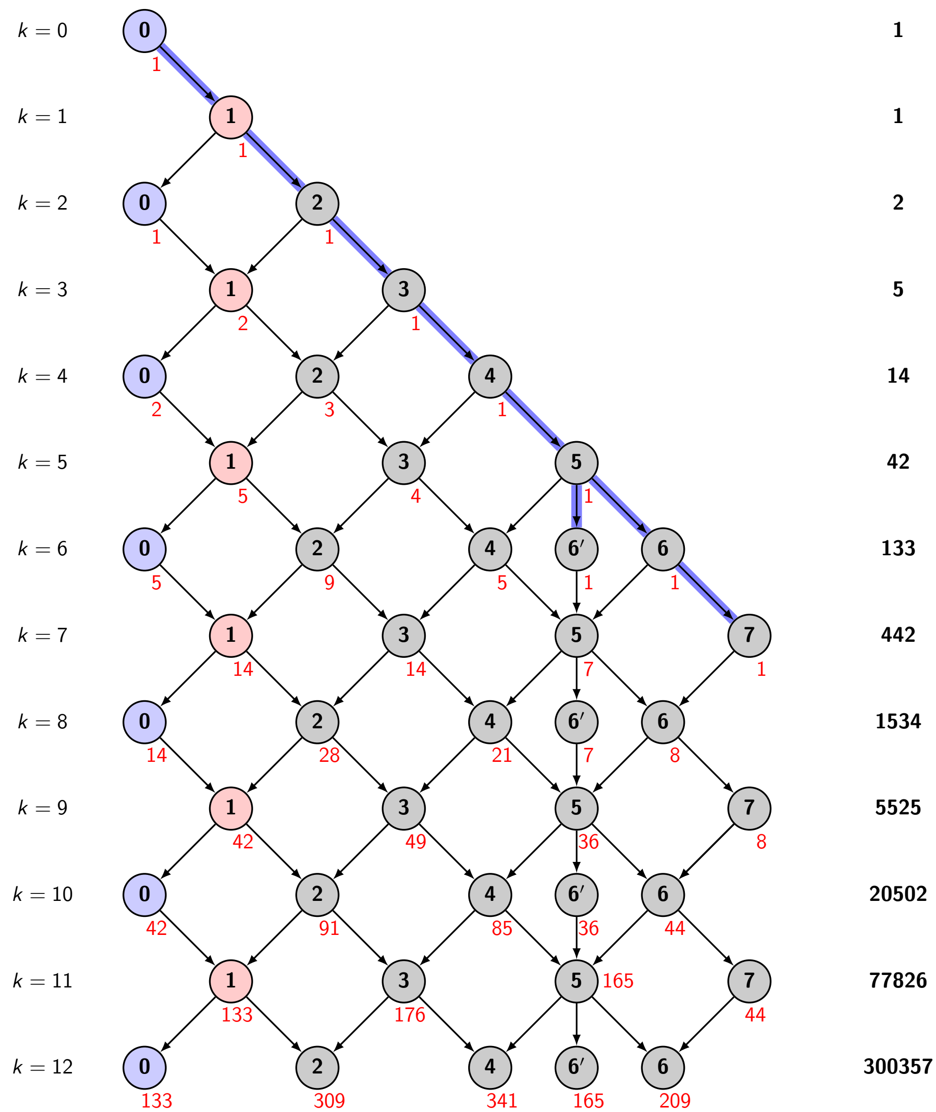


Figure 1.10 The first 10 rows of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathbf{I})$. Each vertex $\lambda \in \Lambda_k = \Lambda_k(\mathbf{I})$ (at level k) is labeled in red with the multiplicity of the corresponding irreducible module in the decomposition $\mathsf{V}^{\otimes k}$, which is also $|\mathcal{P}_k^\lambda|$. The edge (λ, μ) between two vertices is highlighted in blue if μ cannot be reached before this level in the representation graph $\mathcal{R}_{\mathsf{V}}(\mathbf{I})$, meaning that the module labeled by μ is part of $\mathsf{V}_{\text{new}}^{\otimes k}$ (see Theorem 2.11).
The dimension of $\mathsf{Z}_k(\mathbf{I})$, which is also the sum of the squares of the multiplicities, or the multiplicity of $0 \in \Lambda_{2k}$ is given on the right side of the figure, corresponding the the formula given in Theorem 1.49, which involves the Lucas numbers.

Chapter 2

Presentation of $\mathsf{Z}_k(\mathsf{G})$

In Section 1.5, we introduced the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$, and provided a basic understanding of it, including the dimension. In this chapter, we further this investigate by examining concrete elements of $\mathsf{Z}_k(\mathsf{G})$, including providing a presentation on generators and relations for $\mathsf{Z}_k(\mathsf{G})$ for each subgroup $\mathsf{G} = \mathsf{SU}_2$ save the finite groups $\mathbf{C}_n$ and $\mathbf{D}_n$.

In the first part of the chapter, we derive generators for $\mathsf{Z}_k(\mathsf{G})$, first accomplished in [BBH], with the novel addition of a functional description for the generators of $\mathsf{Z}_k(\mathbf{T})$, $\mathsf{Z}_k(\mathbf{O})$, and $\mathsf{Z}_k(\mathbf{I})$. Following this description, we provide the aforementioned presentation. Finally, we use this presentation to derive a number of interesting and useful relations and properties.

2.1 The Temperley-Lieb Algebras $\mathsf{TL}_k(2)$

This section shows that $\mathsf{Z}_k(\mathsf{SU}_2)$ is isomorphic to the *Temperley-Lieb Algebra* $\mathsf{TL}_k(2)$, and is intended to summarize results of [BBH, Sec. 1].

2.1.1 The Temperley-Lieb Algebra $\mathsf{TL}_k(2)$

Consider the action and element $s \in \mathfrak{S}_k$ of the symmetric group of k objects on $\mathsf{V}^{\otimes k}$ given by

$$s \cdot (\mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k}) = \mathsf{v}_{r_{s(1)}} \otimes \mathsf{v}_{r_{s(2)}} \otimes \cdots \otimes \mathsf{v}_{r_{s(k)}}. \tag{2.1}$$

It is clear that SU_2 (and so its subgroups) commute with this action, and so we can define a representation $\Phi_k : \mathfrak{S}_k \to \mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k})$. This map, however, is only injective while $k \leq 2$, (see [BBH]), and has an image isomorphic to the well-known *Temperley-Lieb* algebra, which will be discussed later.

Let $\sigma_i = \Phi_k((i, i+1)) \in \mathsf{Z}_k(\mathsf{SU}_2)$ for $1 \leq i \leq k-1$. The following symmetric group relations hold trivially:

$$\begin{array}{lll} \text{(S1)} & \sigma_i^2 = \mathbf{1}, & \\ \text{(S2)} & \sigma_i\sigma_{i\pm1}\sigma_i = \sigma_{i\pm1}\sigma_i\sigma_{i\pm1}, & \\ \text{(S3)} & \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{for } |i-j| > 1, \end{array} \tag{2.2}$$

where $\mathbf{1}$ is the identity element of $\mathsf{Z}_k(\mathsf{SU}_2)$.

For $\mathsf{V}^{\otimes 2}$, define $\mathsf{e} = \mathbf{1} - \sigma_1$. Thus for $i, j \in \{+1, -1\}$,

$$\mathsf{e}(v_i \otimes v_j) = v_i \otimes v_j - v_j \otimes v_i, \tag{2.3}$$

and so if $v_{\mathbf{r}} \in S(\mathsf{V}^{\otimes 2})$, then $\mathsf{e}v_{\mathbf{r}} = 0$. Further, if $v_{\mathbf{r}} \in A(\mathsf{V}^{\otimes 2})$, $\mathsf{e}v_{\mathbf{r}} = 2v_{\mathbf{r}}$. Define

$$\mathsf{e}_i = \mathbf{1} - \sigma_i, \tag{2.4}$$

so that as an element of $\mathsf{Z}_k(\mathsf{SU}_2)$, the action of e_i is given by

$$\underbrace{I_2 \otimes \cdots \otimes I_2}_{i-1 \text{ factors}} \otimes \mathsf{e} \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-i-1 \text{ factors}} \tag{2.5}$$

where I_2 is the 2×2 identity matrix.

The image $\mathsf{im}(\Phi_k)$ forms a subalgebra $\langle \mathsf{e}_i \mid 1 \leq i \leq k-1 \rangle \subseteq \mathsf{Z}_k(\mathsf{SU}_2)$ isomorphic to the Temperley-Lieb algebra $\mathsf{TL}_k(2)$, which is subject to the relations:

$$\begin{array}{lll} \text{(TL1)} & \mathsf{e}_i^2 = 2\mathsf{e}_i, & \\ \text{(TL2)} & \mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i, & \\ \text{(TL3)} & \mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i, & \text{for } |i-j| > 1. \end{array} \tag{2.6}$$

These relations can also be derived from Equation 2.2. See [TL] and [GHJ] for further detail.

As subalgebras of $\mathsf{End}(\mathsf{V}^{\otimes k})$, the actions of SU_2 and $\mathsf{TL}_k(2)$ generate the full centralizer of the other, and so $\mathsf{TL}_k(2) \cong \mathsf{Z}_k(\mathsf{SU}_2)$. Further, by reverse-inclusion, if $\mathsf{G} \subseteq \mathsf{SU}_2$, it follows that $\mathsf{Z}_k(\mathsf{SU}_2) \subseteq \mathsf{Z}_k(\mathsf{G})$, and so $\mathsf{TL}_k(2) \subseteq \mathsf{Z}_k(\mathsf{G})$ for every G.

2.1.2 Catalan Numbers and $\mathsf{TL}_k(2)$

A *Catalan family* is a family of sets $\{A_k\}_{k\geq 0}$ for which $|A_k| = \mathcal{C}_k$, the kth Catalan number, given by

$$\mathcal{C}_k = \sum_{i=0}^{k-1} \mathcal{C}_i \mathcal{C}_{k-1-i}, \quad \text{for } \mathcal{C}_0 = 1 \tag{2.7}$$

$$= \frac{1}{k+1}\binom{2k}{k}. \tag{2.8}$$

Catalan families and numbers appear with surprising frequency in the answers to mathematical questions. Further, many interesting bijections have been found between distinct Catalan families.

A basis of the Temperley-Lieb algebra is one of the many known Catalan families, which can be shown ([GHJ]) with a bijection with Dyck paths. The main result of this paper includes two novel bijections between bases of $\mathsf{TL}_k(2)$ and Dyck paths. Dyck paths are discussed further in Section 3.1.1, and the two bijections are introduced in Sections 3.2 and 4.2.

2.2 Construction of $\mathsf{Z}_k(\mathsf{G})$

The containment relation $\mathsf{TL}_k(2) \subseteq \mathsf{Z}_k(\mathsf{G})$ provides a wealth of information about the centralizer algebras of proper subgroups of SU_2. In fact, for all G but the cyclic groups, the algebras $\mathsf{Z}_k(\mathsf{G})$ and $\mathsf{Z}_k(\mathsf{SU}_2)$ are isomorphic for sufficiently low values of k. Ultimately, each subgroup has a distinct centralizer for large k, but retains certain properties of the Temperley-Lieb algebras, so that we may think of $\mathsf{Z}_k(\mathsf{G})$ as a generalization of $\mathsf{TL}_k(2)$. In this section, we investigate how each centralizer $\mathsf{Z}_k(\mathsf{G})$ differs from $\mathsf{Z}_k(\mathsf{SU}_2)$.

In [BBH], this question is answered thoroughly, constructing each centralizer algebra out of $\mathsf{TL}_k(\mathsf{G})$, and the group-specific theory of irreducible modules. For each $k \geq 1$,

$$\mathsf{V}^{\otimes k} = \mathsf{V}^{\otimes k}_{old} \otimes \mathsf{V}^{\otimes k}_{new} \tag{2.9}$$

as a G-module, where $\mathsf{V}^{\otimes k}_{old}$ is the sum of irreducible modules which appear in the decomposition of $\mathsf{V}^{\otimes j}$ for some $j < k$, and $\mathsf{V}^{\otimes k}_{new}$ is the sum of irreducible G-modules do not appear in the decomposition for any lower tensor power of V.

It is easy to see that

$$\mathsf{V}^{\otimes k+1}_{new} \subseteq \mathsf{V} \otimes \mathsf{V}^{\otimes k}_{new}, \tag{2.10}$$

as each irreducible of $\mathsf{V}^{\otimes k+1}$ must be contained in either $\mathsf{V}\otimes\mathsf{V}^{\otimes k}_{new}$ or $\mathsf{V}\otimes\mathsf{V}^{\otimes k}_{old}$, and the latter would imply that the irreducible appears in the decomposition of $\mathsf{V}^{\otimes j+1}$, using the same j as in the previous paragraph.

This above definitions are contextualized by Theorem 2.11.

Theorem 2.11 ([BBH, Theorem 1.46]). *Let* $\mathsf{br}(\mathsf{G}), \mathsf{di}(\mathsf{G})$*, and* $\tilde{n}$ *be as in Equation* 1.35*, and let* $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$*. Then,* $\mathsf{Z}_1 = \mathbb{C}\mathbf{1} = \mathsf{Z}_0$*. Moreover,*

(*a*) *if* $1 \leq k < \mathsf{di}(G)$*, and* $k \neq \tilde{n} - 1$ *in the case* $G = C_n$*, then* $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k \oplus \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k}_{new})$*, where* $\mathsf{End}(\mathsf{V}^{\otimes k}_{new})$ *is a commutative subalgebra of dimension equal to the number of nodes in* $\mathcal{R}_\mathsf{V}(\mathsf{G})$ *a distance* k *from the trivial node* (0.)*;*

(*b*) *if* $k \geq \mathsf{di}(\mathsf{G})$*, then* $\mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$*;*

(*c*) *if* $1 \geq k < \mathsf{di}(\mathsf{G})$*,* $k \notin \mathsf{br}(\mathsf{G})$*, then* $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k\rangle$*;*

(*d*) *if* $k \in \mathsf{br}(\mathsf{G})$ *and* $\mathsf{G} \neq \mathbf{D}_2$*, then* $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k, \mathsf{f}^{(\mu)}\rangle$*, where* μ *is either of the two elements in* $\Lambda_{k+1} \setminus \Lambda_{k-1}$*, and* $\mathsf{f}^{(\mu)}$ *is the projection from* $\mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k}_{new})$ *onto* $\mathsf{G}^{(\mu)}$*;*

(*e*) *if* $\mathsf{G} = \mathbf{C}_n$*,* $n < \infty$*, then* $\mathsf{Z}_{\tilde{n}} = \langle \mathsf{Z}_{\tilde{n}-1}, \mathsf{e}_{\tilde{n}-1}, \mathsf{E}^p_q \text{ for } p, q \in \{+1, -1\}\rangle$*, where* E^p_q *is a map from the subspace spanned by* $\mathsf{v}_{p(+1,+1,\ldots,+1)}$ *to the subspace spanned by* $\mathsf{v}_{q(+1,+1,\ldots,+1)}$*;*

(*f*) *if* $\mathsf{G} = \mathbf{D}_2$*, then* $\mathsf{Z}_2 = \langle \mathsf{Z}_1, \mathsf{e}_1, \mathsf{f}^{(\mu_1)}, \mathsf{f}^{(\mu_2)}\rangle$ *where* $\mu_1, \mu_2 \in \{0, 0', n, n'\}$ *as introduced in Example* 1.23*, and* $\mu_1 \neq \mu_2$*.*

A projection onto an irreducible module is a member of the centralizer algebra, and the opposite direction of the proof comes from the fact that $\mathsf{Z}_k(\mathsf{G})$ is spanned by projections onto irreducible G-modules in $\mathsf{V}^{\otimes k}$.

For the remainder of the section, we derive formulas for the projectors $\mathsf{f}^{(\mu)}$ using the symmetric submodule $S(\mathsf{V}^{\otimes k})$. While the larger results are taken from [BBH], the specific descriptions given in the second subsection is original to this thesis.

2.2.1 SU_2 Projection Mappings

Let the *weight* of a tuple $\mathbf{r} \in \{+1, -1\}^k$ to be the number of 1s which appear in $\mathbf{r}$, written $|\mathbf{r}|$. Then, following the definition provided in Equation 1.17,

$$S(\mathsf{V}^{\otimes k}) = span\left\{ \frac{1}{\binom{k}{t}} \sum_{|\mathbf{r}|=t} \mathsf{v}_\mathbf{r} \;\middle|\; 0 \leq t \leq k \right\}. \tag{2.12}$$

It is a well-known result that $S(\mathsf{V}^{\otimes k})$ is an irreducible SU_2-module, which is isomorphic to $\mathsf{V}(k)$.

Recall from Theorem 2.11 that for $\mathsf{Z}_k(\mathsf{SU}_2)$, $\mathsf{f}^{(i)} \in \mathsf{End}(\mathsf{V}^{\otimes k})$ is a projection onto $\mathsf{V}(i)$. In fact, this is exactly the action of the well-known *Jones-Wenzl idempotent* $\mathsf{J}_i \in \mathsf{TL}_k(2)$ (see [Wz], [Jo, Sec. 3], [FK]), which can be defined recursively by setting $\mathsf{J}_1 = \mathbf{1}$, and

$$\mathsf{J}_{i+1} = \mathsf{J}_i - \frac{i}{i+1}\mathsf{J}_i\mathsf{e}_i\mathsf{J}_i \tag{2.13}$$

$$= \mathsf{J}_i\left(\mathbf{1} - \frac{i}{i+1}\mathsf{e}_i\right)\mathsf{J}_i. \tag{2.14}$$

The Jones-Wenzl idempotents obey the following relations,

$$\begin{array}{lll}
\text{(JW1)} & (\mathsf{J}_i)^2 = \mathsf{J}_i, & 1 \le i \le k, \\
\text{(JW2)} & \mathsf{J}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{J}_i = 0, & 1 \le j \le i \le k, \\
\text{(JW3)} & \mathsf{J}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{J}_i, & 1 \le i < j \le k-1, \\
\text{(JW4)} & \mathsf{e}_i\mathsf{J}_i\mathsf{e}_i = \frac{i+1}{i}\mathsf{J}_{i-1}\mathsf{e}_{i-1}, & 1 \le i \le k, \\
\text{(JW5)} & \mathbf{1} - \mathsf{J}_i \in \langle \mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_{i-1}\rangle, & 1 \le i \le k, \\
\text{(JW6)} & \mathsf{J}_i\mathsf{J}_j = \mathsf{J}_j\mathsf{J}_i = \mathsf{J}_i, & 1 \le j \le i \le k,
\end{array} \tag{2.15}$$

where $\langle \mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_{|\lambda|}\rangle$ is the subalgebra generated by the set $\{\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_i\}$, which is also isomorphic to $\mathsf{TL}_i(2)$. These relations are not difficult to verify directly, and proofs can be found in a number of sources, such as [Wz], [CJ]. Further, an explicit formula for each J_i in a basis consisting of products of elements from $\{\mathsf{e}_1, \ldots, \mathsf{e}_{i-1}\}$ is given in [FK] and [Mo].

2.2.2 Branch Projection Mappings

The Jones-Wenzl idempotents provide the projections onto $\mathsf{End}_{\mathsf{SU}_2}(\mathsf{V}^{\otimes k}_{\text{new}})$ as seen in Theorem 2.11. We now endeavor to find analogous definitions of each $\mathsf{f}^{(\lambda)}$ for other groups, specifically for the subgroups of $\mathbf{T}$, $\mathbf{O}$, and $\mathbf{I}$ (the cases for $\mathbf{C}_n$ and $\mathbf{D}_n$ are not relevant to the work of this thesis). Perhaps unsurprisingly, the Jones-Wenzl idempotents play an essential role in the construction of such projection maps. Because we wish to use the same notation (i.e. f) for all projections while also being able to discuss Jones-Wenzl elements, define $\mathsf{J}^{(i)} \in \mathsf{Z}_k(\mathsf{G})$ to be the element $\mathsf{f}^{(i)} \in \mathsf{Z}_k(\mathsf{SU}_2) \subseteq \mathsf{Z}_k(\mathsf{G})$. Thus, each $\mathsf{J}^{(i)}$ fulfills relations (JW1-6), and projects $\mathsf{V}^{\otimes i}$ onto $S(\mathsf{V}^{\otimes i})$ as an element of $\mathsf{Z}_i(\mathsf{G})$.

Following [BBH], each case relies on finding a decomposition of

$$S(\mathsf{V}^{\otimes k+1}) = \mathsf{V}^{\otimes k+1}_{\text{new}}$$

for $k \in \mathsf{br}(\mathsf{G})$ into irreducible G-modules, and then finding projections onto these modules. In the remainder of the section, we will frequently refer to the basis of $S(\mathsf{V}^{\otimes k+1})$ given in Equation 2.12.

T

Recall $\mathsf{br}(\mathbf{T}) = \{2\}$, and so we decompose $S(\mathsf{V}^{\otimes 3})$. Define

$$\mathbf{u}_1^+ = \frac{i}{\sqrt{3}}\mathbf{w}_3 + \mathbf{w}_1, \text{ and } \qquad \mathbf{u}_2^+ = -i\sqrt{3}\mathbf{w}_2 + \mathbf{w}_0, \tag{2.16}$$

and let $\mathbf{u}_1^- = \bar{\mathbf{u}}_1^+$ and $\mathbf{u}_2^- = \bar{\mathbf{u}}_2^+$ be the complex conjugates of the above vectors. It is easy to verify that $\mathcal{U} = \{\mathbf{u}_1^+, \mathbf{u}_2^+, \mathbf{u}_1^-, \mathbf{u}_2^-\}$ is a basis for $S(\mathsf{V}^{\otimes 3})$. With $g, h \in \mathbf{T}$ as defined in Section 1.1,

$$[g]_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^5 & \sqrt{3}\zeta^{23} & 0 & 0 \\ \frac{\zeta^5}{\sqrt{3}} & \zeta^{11} & 0 & 0 \\ 0 & 0 & \zeta^{13} & \sqrt{3}\zeta^{19} \\ 0 & 0 & \frac{\zeta}{\sqrt{3}} & \zeta^{19} \end{pmatrix}, \tag{2.17}$$

and

$$[h]_{\mathcal{U}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^{13} & \sqrt{3}\zeta^{13} & 0 & 0 \\ \frac{\zeta^7}{\sqrt{3}} & \zeta^{19} & 0 & 0 \\ 0 & 0 & \zeta^{5} & \sqrt{3}\zeta^{17} \\ 0 & 0 & \frac{\zeta^{11}}{\sqrt{3}} & \zeta^{11} \end{pmatrix}, \tag{2.18}$$

where $\zeta = e^{2\pi i/24}$. Defining

$$\mathbf{T}^{(3)} = \text{span}\{\mathbf{u}_1^+, \mathbf{u}_2^+\}, \text{ and } \mathbf{T}^{(3')} = \text{span}\{\mathbf{u}_1^-, \mathbf{u}_2^-\}, \tag{2.19}$$

is is clear that $S(\mathsf{V}^{\otimes 3}) = \mathbf{T}^{(3)} \oplus \mathbf{T}^{(3')}$ as a $\mathbf{T}$-module.

Thus, the actions of $\mathsf{f}_1^{(3)}$ and $\mathsf{f}_1^{(3')}$ on a basis vector $\mathsf{v}_\mathbf{r}$ of $\mathsf{V}^{\otimes 3}$ are

$$\mathsf{f}_1^{(3)}(\mathsf{v}_\mathbf{r}) = \begin{cases} \frac{-i\sqrt{3}}{2}\mathbf{u}_1^+ & \text{if } |\mathbf{r}| = 3 \\ \frac{1}{2}\mathbf{u}_1^+ & \text{if } |\mathbf{r}| = 1 \\ \frac{i}{2\sqrt{3}}\mathbf{u}_2^+ & \text{if } |\mathbf{r}| = 2 \\ \frac{1}{2}\mathbf{u}_2^+ & \text{if } |\mathbf{r}| = 0 \end{cases}, \tag{2.20}$$

$$\mathsf{f}_1^{(3')}(\mathsf{v}_\mathbf{r}) = \begin{cases} \frac{i\sqrt{3}}{2}\mathbf{u}_1^- & \text{if } |\mathbf{r}| = 3 \\ \frac{1}{2}\mathbf{u}_1^- & \text{if } |\mathbf{r}| = 1 \\ \frac{-i}{2\sqrt{3}}\mathbf{u}_2^- & \text{if } |\mathbf{r}| = 2 \\ \frac{1}{2}\mathbf{u}_2^- & \text{if } |\mathbf{r}| = 0 \end{cases}. \tag{2.21}$$

It is a straightforward task to verify that $f_1^{(3)}$ and $f_1^{(3')}$ are projections onto $\mathbf{T}^{(3)}$ and $\mathbf{T}^{(3')}$, respectively.

O

Recall $\mathsf{br}(\mathbf{O}) = \{3\}$, and so we decompose $S(\mathsf{V}^{\otimes 4})$. Define

$$\mathbf{u}^+ = \frac{1}{2}\left(\mathbf{w}_4 - \mathbf{w}_0\right), \text{ and } \mathbf{u}^- = \frac{1}{2}\left(\mathbf{w}_4 + \mathbf{w}_0\right). \tag{2.22}$$

It is easy to verify that $\mathcal{U} = \{\mathbf{u}^+, \mathbf{w}_3, \mathbf{w}_1, \mathbf{u}^-, \mathbf{w}_2\}$ is a basis for $S(\mathsf{V}^{\otimes 4})$. With $g, h \in \mathbf{O}$ as defined in Section 1.1, the action on this basis is given by:

$$[g]_{\mathcal{U}} = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ -2i & -i & i & 0 & 0 \\ 2i & -i & i & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 & -1 \end{pmatrix}, \tag{2.23}$$

$$[h]_{\mathcal{U}} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.24}$$

Defining

$$\mathbf{O}^{(4)} = \operatorname{span}\{\mathbf{u}^+, \mathbf{w}_3, \mathbf{w}_1\}, \text{ and } \quad \mathbf{O}^{(4')} = \operatorname{span}\{\mathbf{u}^-, \mathbf{w}_2\}. \tag{2.25}$$

is is clear that $S(\mathsf{V}^{\otimes 4}) = \mathbf{O}^{(4)} \oplus \mathbf{O}^{(4')}$ as an $\mathbf{O}$-module.

Thus, $f_1^{(4)}$ and $f_1^{(4')}$ act on a basis vector $\mathsf{v}_{\mathbf{r}}$ of $\mathsf{V}^{\otimes 4}$ as

$$f_1^{(4)}(\mathsf{v}_{\mathbf{r}}) = \begin{cases} \mathbf{u}^+ & \text{if } |\mathbf{r}| = 4 \\ -\mathbf{u}^+ & \text{if } |\mathbf{r}| = 0 \\ \mathbf{w}_3 & \text{if } |\mathbf{r}| = 3 \\ \mathbf{w}_1 & \text{if } |\mathbf{r}| = 1 \\ 0 & \text{otherwise} \end{cases}, \text{ and} \tag{2.26}$$

$$f_1^{(4')}(\mathsf{v}_{\mathbf{r}}) = \begin{cases} \mathbf{u}^- & \text{if } |\mathbf{r}| = 4, 0 \\ \mathbf{w}_2 & \text{if } |\mathbf{r}| = 2 \\ 0 & \text{otherwise} \end{cases}. \tag{2.27}$$

It is a straightforward task to verify that $f_1^{(4)}$ and $f_1^{(4')}$ are projections onto $\mathbf{O}^{(4)}$ and $\mathbf{O}^{(4')}$, respectively.

I

Recall that $\mathsf{br}(\mathbf{I}) = \{5\}$, and so we decompose $S(\mathsf{V}^{\otimes 6})$. Define

$$\mathbf{u}_1^+ = \frac{9}{16}\mathbf{w}_6 + \frac{3\sqrt{5}}{16}\mathbf{w}_4 + \frac{15}{16}\mathbf{w}_2 - \frac{3\sqrt{5}}{16}\mathbf{w}_0, \tag{2.28}$$

$$\mathbf{u}_2^+ = \frac{5}{8}\mathbf{w}_5 - \frac{2\sqrt{5}}{8}\mathbf{w}_3 - \frac{3}{8}\mathbf{w}_1, \tag{2.29}$$

$$\mathbf{u}_3^+ = \frac{1}{16\sqrt{5}}\mathbf{w}_6 + \frac{7}{16}\mathbf{w}_4 + \frac{3\sqrt{5}}{16}\mathbf{w}_2 + \frac{1}{16}\mathbf{w}_0, \text{ and} \tag{2.30}$$

$$\mathbf{u}_4^+ = -\frac{3}{8\sqrt{5}}\mathbf{w}_5 + \frac{6}{8}\mathbf{w}_3 - \frac{3}{8\sqrt{5}}\mathbf{w}_1, \tag{2.31}$$

as well as

$$\mathbf{u}_1^- = \frac{7}{16}\mathbf{w}_6 - \frac{3\sqrt{5}}{16}\mathbf{w}_4 - \frac{15}{16}\mathbf{w}_2 + \frac{3\sqrt{5}}{16}\mathbf{w}_0, \tag{2.32}$$

$$\mathbf{u}_2^- = \frac{3}{8}\mathbf{w}_5 + \frac{2\sqrt{5}}{8}\mathbf{w}_3 + \frac{3}{8}\mathbf{w}_1, \text{ and} \tag{2.33}$$

$$\mathbf{u}_3^- = -\frac{1}{16\sqrt{5}}\mathbf{w}_6 + \frac{9}{16}\mathbf{w}_4 - \frac{3\sqrt{5}}{16}\mathbf{w}_2 - \frac{1}{16}\mathbf{w}_0. \tag{2.34}$$

It is easy to verify that $\mathcal{U} = \{\mathbf{u}_1^+, \mathbf{u}_2^+, \mathbf{u}_3^+, \mathbf{u}_4^+, \mathbf{u}_1^-, \mathbf{u}_2^-, \mathbf{u}_3^-\}$ is a basis for $S(\mathsf{V}^{\otimes 6})$. The action of the generators $g, h \in \mathbf{I}$ as defined in section 1.1 on this basis is given by

$$[g]_{\mathcal{U}} = \begin{pmatrix} -\frac{i(\sqrt{5}+5)}{20} & \frac{i(3\sqrt{5}-5)}{60} & -\frac{i(3\sqrt{5}+7))}{60} & \frac{i(\sqrt{5}-3)}{20} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3(\sqrt{5}-3)}{4} & \frac{i(\sqrt{5}+1)}{4} & \frac{i(\sqrt{5}+5)}{20} & \frac{-3i(\sqrt{5}+5)}{20} & 0 & 0 & 0 \\ \frac{-\sqrt{5}-5}{2} & \frac{\sqrt{5}}{3} & \frac{3-\sqrt{5}}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i(\sqrt{5}+5)}{20} & -\frac{i(\sqrt{5}+5)}{20} & \frac{i(\sqrt{5}-3)}{20} \\ 0 & 0 & 0 & 0 & \frac{3-\sqrt{5}}{2} & 0 & \frac{\sqrt{5}+5}{10} \\ 0 & 0 & 0 & 0 & \frac{i(3\sqrt{5}+7)}{4} & -\frac{(\sqrt{5}-3)}{4} & -\frac{i(\sqrt{5}+5)}{20} \end{pmatrix}, \tag{2.35}$$

and

$$[h]_{\mathcal{U}} = A + Bi, \tag{2.36}$$

for the imaginary number $i = \sqrt{-1}$ and the matrices

$$A = \begin{pmatrix} \frac{4\sqrt{5}-5}{20} & \frac{45-\sqrt{5}}{240} & -\frac{1}{30} & \frac{-3(\sqrt{5}+3)}{80} & 0 & 0 & 0 \\ \frac{3(\sqrt{5}-5)}{16} & -\frac{1}{4} & \frac{(-\sqrt{5}-3}{16} & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & \frac{13-\sqrt{5}}{16} & -\frac{4\sqrt{5}+5}{20} & \frac{-9(\sqrt{5}-5)}{80} & 0 & 0 & 0 \\ \frac{\sqrt{5}+35}{16} & 0 & \frac{3\sqrt{5}-23}{48} & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{8}-\frac{3}{8\sqrt{5}} & \frac{\sqrt{5}-5}{40} & \frac{-\sqrt{5}-1}{40} \\ 0 & 0 & 0 & 0 & \frac{3-\sqrt{5}}{4} & \frac{-\sqrt{5}-1}{4} & \frac{\sqrt{5}+5}{20} \\ 0 & 0 & 0 & 0 & \frac{-\sqrt{5}-1}{8} & \frac{-\sqrt{5}-3}{8} & \frac{5-7\sqrt{5}}{40} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1}{8\sqrt{5}} & \frac{13\sqrt{5}+25}{240} & \frac{7}{120} & \frac{\sqrt{5}-3}{80} & 0 & 0 & 0 \\ \frac{3(\sqrt{5}+5)}{16} & -\frac{\sqrt{5}}{8} & \frac{\sqrt{5}-3}{16} & \frac{3}{8} & 0 & 0 & 0 \\ -\frac{9}{8} & \frac{3(\sqrt{5}-3)}{16} & -\frac{1}{8\sqrt{5}} & \frac{-3(\sqrt{5}+5)}{80} & 0 & 0 & 0 \\ \frac{7\sqrt{5}-5}{16} & -\frac{25}{24} & \frac{-5\sqrt{5}-1}{48} & \frac{\sqrt{5}}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-\sqrt{5}-5}{40} & \frac{1}{4\sqrt{5}} & \frac{3(\sqrt{5}+1)}{40} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}+1}{4} & 0 & \frac{5-3\sqrt{5}}{20} \\ 0 & 0 & 0 & 0 & \frac{-7(\sqrt{5}+1)}{8} & \frac{2-\sqrt{5}}{4} & \frac{\sqrt{5}+5}{40} \end{pmatrix}.$$

An open question coming out of this thesis is whether there is a more concise way to express these matrices.

Defining

$$\mathbf{I}^{(6)} = \text{span}\{\mathbf{u}_1^+, \mathbf{u}_2^+, \mathbf{w}_4, \mathbf{w}_2\}, \text{ and } \quad \mathbf{I}^{(6')} = \text{span}\{\mathbf{u}_1^-, \mathbf{u}_2^-, \mathbf{w}_3\}, \qquad (2.37)$$

it is clear that $S(V^{\otimes 6}) = \mathbf{I}^{(6)} \oplus \mathbf{I}^{(6')}$ as an $\mathbf{I}$-module.

Thus, $\mathsf{f}_1^{(6)}$ and $\mathsf{f}_1^{(6')}$ act on a basis vector $\mathsf{v}_{\mathbf{r}}$ of $\mathsf{V}^{\otimes 6}$ as

$$\mathsf{f}_1^{(6)}(\mathsf{v}_{\mathbf{r}}) = \begin{cases} \mathbf{u}_1^+ & \text{if } |\mathbf{r}| = 6 \\ \frac{-1}{3}\mathbf{u}_1^+ & \text{if } |\mathbf{r}| = 1 \\ \mathbf{u}_2^+ & \text{if } |\mathbf{r}| = 0 \\ \frac{1}{3}\mathbf{u}_2^+ & \text{if } |\mathbf{r}| = 5 \\ \mathbf{w}_4 & \text{if } |\mathbf{r}| = 4 \\ \mathbf{w}_2 & \text{if } |\mathbf{r}| = 2 \\ 0 & \text{otherwise} \end{cases}, \text{ and } \quad \mathsf{f}_1^{(6')}(\mathsf{v}_{\mathbf{r}}) = \begin{cases} \mathbf{u}_1^- & \text{if } |\mathbf{r}| = 6 \\ \frac{1}{2}\mathbf{u}_1^- & \text{if } |\mathbf{r}| = 1 \\ \mathbf{u}_2^- & \text{if } |\mathbf{r}| = 0 \\ \frac{-1}{2}\mathbf{u}_2^- & \text{if } |\mathbf{r}| = 5 \\ \mathbf{w}_3 & \text{if } |\mathbf{r}| = 3 \\ 0 & \text{otherwise} \end{cases}. \tag{2.38}$$

It is a straightforward task to verify that $\mathsf{f}^{(6)}$ and $\mathsf{f}^{(6')}$ are projections onto $\mathbf{I}^{(6)}$ and $\mathbf{I}^{(6')}$, respectively.

2.2.3 Generalized Branch Projections

Given the projections onto the irreducible modules in $\mathsf{V}_{new}^{\otimes k+1}$ for $k \in \mathsf{br}(\mathsf{G})$, [BBH] shows that we can recursively generate projections for $\mathsf{V}_{new}^{\otimes k+2}$, $\mathsf{V}_{new}^{\otimes k+3}$, etc. recursively, leading to a general statement. For any $\lambda, \mu \in \Lambda(\mathsf{SU}_2)$ or $\Lambda(\mathsf{G})$, write $d_{(\lambda)}$ for the dimension of the irreducible module labeled by λ. Further, we say that $\mu = \lambda + 1$ if $|\mu| = |\lambda| + 1$, and $\mu \prec \lambda$, and likewise, if these conditions hold, we say that $\lambda = \mu - 1$ (this relationship uniquely determines λ). Then, for $\lambda \neq 0$,

$$\sum_{\mu = \lambda + 1} \mathsf{f}^{(\mu)} = \mathsf{f}^{(\lambda)} - \frac{d_{(\lambda - 1)}}{d_{(\lambda)}} \mathsf{f}^{(\lambda)} \mathsf{e}_{|\lambda|} \mathsf{f}^{(\lambda)}. \tag{2.39}$$

This leads to an interesting property.

Example 2.40. Each pair of projections defined in the previous section sums to a Jones-Wenzl element.

1. For $\mathbf{T}$,
$$\mathsf{f}^{(3)} + \mathsf{f}^{(3')} = \mathsf{J}_2 - \frac{2}{3}\mathsf{J}_2\mathsf{e}_2\mathsf{J}_2 = \mathsf{J}_3. \tag{2.41}$$
2. For $\mathbf{O}$,
$$\mathsf{f}^{(4)} + \mathsf{f}^{(4')} = \mathsf{J}_3 - \frac{3}{4}\mathsf{J}_3\mathsf{e}_3\mathsf{J}_3 = \mathsf{J}_4. \tag{2.42}$$
3. For $\mathbf{I}$,
$$\mathsf{f}^{(6)} + \mathsf{f}^{(6')} = \mathsf{J}_5 - \frac{5}{6}\mathsf{J}_5\mathsf{e}_5\mathsf{J}_5 = \mathsf{J}_6. \tag{2.43}$$

A number of other relations on these projectors which are generalizations of Equation 2.15 are found in [BBH]. For labels λ, μ in $\Lambda(\mathsf{SU}_2)$ or $\lambda(\mathsf{G})$,

$$
\begin{array}{lll}
\text{(F1)} & (\mathsf{f}^{(\lambda)})^2 = \mathsf{f}^{(\lambda)}, & \\
\text{(F2)} & \mathsf{f}^{(\lambda)}\mathsf{e}_j = \mathsf{e}_j\mathsf{f}^{(\lambda)} = 0, & 1 \leq j \leq |\lambda, \\
\text{(F3)} & \mathsf{f}^{(\lambda)}\mathsf{e}_j = \mathsf{e}_j\mathsf{f}^{(\lambda)}, & j > |\lambda|, \\
\text{(F4)} & \mathsf{e}_{|\lambda|}\mathsf{f}_1^{(\lambda)}\mathsf{e}_{|\lambda|} = \frac{d_{(\lambda)}}{d_{(\lambda-1)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{|\lambda|}, & \\
\text{(F5)} & \mathsf{f}^{(\lambda)} = \mathsf{J}_{|\lambda|}, & |\lambda| \leq \min(\mathsf{br}(\mathsf{G})), \\
\text{(F6.1)} & \mathsf{f}^{(\lambda)}\mathsf{f}^{(\mu)} = \mathsf{f}^{(\mu)}\mathsf{f}^{(\lambda)} = \mathsf{f}^{(\lambda)}, & \lambda \preceq \mu, \\
\text{(F6.2)} & \mathsf{f}^{(\lambda)}\mathsf{f}^{(\mu)} = \mathsf{f}^{(\mu)}\mathsf{f}^{(\lambda)} = 0, & \lambda \not\preceq \mu \text{ and } \mu \not\preceq \lambda.
\end{array}
\tag{2.44}
$$

Note that each of the above relations correspond roughly to a relation on the Jones-Wenzl idempotents (perhaps with the exception of (F5)).

2.3 Relations and Presentation

Now that we have introduced the generators, we can give a presentation of Z_k with all but two exceptions (ironically the two non-exceptional groups). First, however, we introduce a set of elements which are essential to our main result, and the aforementioned theorem.

2.3.1 Shifting Action of σ_i

Recall that $\sigma_i \in \mathsf{Z}_k(\mathsf{G})$ acts on the basis $\{\mathsf{v}_{\mathbf{r}} \mid \mathbf{r} \in \{-1, +1\}^k\}$ of $\mathsf{V}^{\otimes k}$ as

$$
\sigma_i(\mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_i} \otimes \mathsf{v}_{r_{i+1}} \otimes \cdots \otimes \mathsf{v}_{r_k}) = \mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_{i+1}} \otimes \mathsf{v}_{r_i} \otimes \cdots \otimes \mathsf{v}_{r_k}, \tag{2.45}
$$

and so the product $\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1$ acts as

$$
\sigma_1\sigma_2\cdots\sigma_k(\mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k}) = \mathsf{v}_{r_k} \otimes \mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_{k-1}}, \tag{2.46}
$$

and its inverse $\sigma_1\sigma_2\cdots\sigma_{k-1}$ acts as

$$
\sigma_1\sigma_2\cdots\sigma_{k-1}(\mathsf{v}_{r_1} \otimes \mathsf{v}_{r_2} \otimes \cdots \otimes \mathsf{v}_{r_k}) = \mathsf{v}_{r_2} \otimes \mathsf{v}_{r_3} \otimes \cdots \otimes \mathsf{v}_{r_{k-1}} \otimes \mathsf{v}_{r_1}. \tag{2.47}
$$

With a small amount of contemplation, this leads to a very interesting result. For any element $a \in \mathsf{Z}_{k-1}(\mathsf{G})$ which we embed in $\mathsf{Z}_k(\mathsf{G})$ as described at the beginning of Section 1.5, so that $a \in \mathsf{Z}_k(\mathsf{G})$ acts on $\mathsf{V}^{\otimes k}$ as $a \in \mathsf{Z}_k(\mathsf{G})$ on the first $k-1$ tensor factors, and as the identity on the last tensor factor. Then, however, with the above description, the element

$$(\sigma_1\sigma_2\cdots\sigma_{k-1})a(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)$$

acts on $\mathsf{V}^{\otimes k}$ as $a \in \mathsf{Z}_k(\mathsf{G})$ on the last $k-1$ tensor factors, and as the identity on the first tensor factors.

We can iterate this process. Thus, if we recall the description of the action of e_i given in Equation 2.5, we can see that e_i and

$$(\sigma_1\sigma_2\cdots\sigma_{k-1})^{i-1}\mathsf{e}_1(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)^{i-1}$$

act equivalently. In representation theoretic terminology, the action of elements of $\mathsf{Z}_k(\mathsf{G})$ on $\mathsf{V}^{\otimes k}$ corresponds to the defining representation of $\mathsf{Z}_k(\mathsf{G})$, which must be faithful, and so any two elements with the same action are equivalent. Thus, defining

$$\tau = \sigma_1\sigma_2\cdots\sigma_{k-1}, \tag{2.48}$$

it follows that $\mathsf{e}_i = \tau^{i-1}\mathsf{e}_1\tau^{1-i}$. A similar construction exists for each projector $\mathsf{f}^{(\lambda)}$, achieved by defining a collection of projections acting on various tensor factors as follows. For $1 \leq i \leq k-|\lambda|+1$, let

$$\mathsf{f}_i^{(\lambda)} = \tau^{i-1}\mathsf{f}^{(\lambda)}\tau^{i-1}, \tag{2.49}$$

so that $\mathsf{f}_i^{(\lambda)}$ acts as

$$\underbrace{I_2 \otimes \cdots \otimes I_2}_{i-1 \text{ factors}} \otimes \mathsf{f}^{(\lambda)} \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{k-i-|\lambda|+1 \text{ factors}} \tag{2.50}$$

on $\mathsf{V}^{\otimes k}$, where I_2 is the 2×2 identity matrix in $\mathsf{GL}(\mathsf{V})$. We will call these elements *shifted idempotents*, as they correspond to the shifting of an idempotent's action on tensor space.

2.3.2 Presentation Theorem

Using the shifted branch generators described above, we can give the long awaited presentation the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$.

Theorem 2.51 (Presentation on Generators and Relations)**.** *The centralizer algebra $\mathsf{Z}_k(\mathsf{G})$, for $k \geq 1$, with $k < \tilde{n}$ if $\mathsf{G} = \mathbf{C}_n$, and $k < n$ if $\mathsf{G} = \mathbf{D}_n$ is generated by*

$$\{1, \mathsf{e}_1, \ldots, \mathsf{e}_{k-1}\} \cup \{\, \mathsf{f}_i^{\nu} \mid \nu \in \Lambda(\mathsf{G}), |\nu| \leq k, i + |\lambda| \leq k+1\}$$

subject to the following relations:

1. *Temperley-Lieb relations, as seen in Equation* 2.6:

$$\begin{array}{lll} \text{(TL1)} & \mathsf{e}_i^2 = 2\mathsf{e}_i, & \\ \text{(TL2)} & \mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i, & \\ \text{(TL3)} & \mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i, & \textit{for } |i-j| > 1. \end{array}$$

2. *Projection interaction relations, generalized from Equation* 2.44:

$$\begin{array}{lll} \text{(FS1)} & (\mathsf{f}_i^{(\lambda)})^2, & \\ \text{(FS2)} & \mathsf{f}_i^{(\lambda)}\mathsf{e}_j = \mathsf{e}_j\mathsf{f}_i^{(\lambda)} = 0, & i \leq j \leq i + |\lambda| - 1, \\ \text{(FS3)} & \mathsf{f}_i^{(\lambda)}\mathsf{e}_j = \mathsf{e}_j\mathsf{f}_i^{(\lambda)}, & j < i-1 \textit{ or } j > i + |\lambda|, \\ \text{(FS4)} & \mathsf{e}_j\mathsf{f}_i^{(\lambda)}\mathsf{e}_j = \frac{d_{(\lambda)}}{d_{(\lambda-1)}}\mathsf{f}_i^{(\lambda-1)}\mathsf{e}_j, & j = i + |\lambda| - 1. \end{array}$$

3. *The induction relation, generalized from the definition in Equation* 2.39:

$$\text{(FS5)} \qquad \sum_{\mu=\lambda+1} \mathsf{f}_i^{(\mu)} = \mathsf{f}_i^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_i^{(\lambda)}\mathsf{e}_{i+|\lambda|-1}\mathsf{f}_i^{(\lambda)}.$$

4. *Shifted projection relations, generalized from Equation* 2.44: *for $i \leq j$ and $|\lambda| + i \geq |\mu| + j$,*

$$\begin{array}{ll} \text{(FS6.1)} & \mathsf{f}_i^{(\lambda)}\mathsf{f}_j^{(\mu)} = \mathsf{f}_j^{(\mu)}\mathsf{f}_j^{(\lambda)} = \mathsf{f}_i^{(\lambda)},\ \lambda \preceq \mu, \\ \text{(FS6.2)} & \mathsf{f}_i^{(\lambda)}\mathsf{f}_j^{(\mu)} = \mathsf{f}_j^{(\mu)}\mathsf{f}_i^{(\lambda)} = 0, \quad \lambda \not\preceq \mu \textit{ and } \mu \not\preceq \lambda, \\ \text{(FS6.3)} & \mathsf{f}_i^{(\lambda)}\mathsf{f}_j^{(\mu)} = \mathsf{f}_j^{(\mu)}\mathsf{f}_i^{(\lambda)}, \qquad j > i + |\lambda| - 1. \end{array}$$

The cases of $k \geq \tilde{n}$ for $\mathsf{G} = \mathbf{C}_n$ and $k \geq n$ for $\mathsf{G} = \mathbf{D}_n$ are incomplete, see Chapter 4 *for further details.*

Remark 2.52. *In fact, due to the definition of each generator set, we can refine our generator set to simply*

$$\{1, \mathsf{e}_1, \ldots, \mathsf{e}_{k-1}\} \cup \{\, \mathsf{f}_1^{\nu} \mid \nu \in \mathsf{br}(\mathsf{G}), |\nu| \leq k\},$$

or if $\mathsf{G} = \mathsf{SU}_2$, $\mathbf{T}$, $\mathbf{O}$*, or* $\mathbf{I}$

$$\{\, \mathsf{f}_i^{\nu} \mid \nu \in \Lambda(\mathsf{G}), |\nu| \leq k, i + |\lambda| \leq k+1\}.$$

For the sake of convenience, and a simpler way to convey relations, we give the larger generator set in the above theorem.

Proof. It is a simple exercise to verify these relations using the defining representation of $\mathsf{Z}_k(\mathsf{G})$ and simple algebraic facts, primarily that for $a, b \in \mathsf{Z}_k(G)$

$$\left(\tau a \tau^{-1}\right)\left(\tau b \tau^{-1}\right) = \left(\tau ab\, \tau^{-1}\right),$$

so that the generalizations of Equation 2.44 and Equation 2.39 follow immediately from the original results.

Showing that these generators and relations form a presentation of Z_k is slightly more subtle, as we must show that the freest algebra with the above generators subject to the above relations is isomorphic to $\mathsf{Z}_k(\mathsf{G})$. Let $Z_k(\mathsf{G})$ be this freest algebra.

Theorem 2.11 shows that the above generators generate all of $\mathsf{Z}_k(\mathsf{G})$, so we can define a surjective algebra homomorphism $\Psi : Z_k(\mathsf{G}) \to \mathsf{Z}_k(\mathsf{G})$ given by mapping each generator in $Z_k(\mathsf{G})$ to is counterpart in $\mathsf{Z}_k(\mathsf{G})$. Thus $|Z_k(\mathsf{G})| \geq |\mathsf{Z}_k(\mathsf{G})|$. Then, using only the above relations we find a spanning set for the above generators which is in bijection with $\mathcal{P}_{2k}^0$ in Theorem 3.19 and Theorem **??**. Thus, $|Z_k(\mathsf{G})| \leq |\mathcal{P}_{2k}^0| = |\mathsf{Z}_k(\mathsf{G})|$. Thus, $|Z_k(\mathsf{G})| = |\mathsf{Z}_k(\mathsf{G})|$, and so Ψ is an algebra isomorphism. □

Chapter 3

Basis Construction

In this chapter we give the main result of this thesis, a basis of $\mathsf{Z}_k(\mathsf{G})$ which is in bijection with paths on the Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{G})$. First, we discuss several preliminary concepts generally focused on establishing several key properties of these paths, which involves a discussion of Dyck paths, a well-known Catalan Family. Then, we discuss the basis algorithm, which give instructions on creating a basis element using a path. Finally, we prove that this algorithm generates a spanning set for $\mathsf{Z}_k(\mathsf{G})$, and so a basis.

3.1 Paths on the Bratteli Diagram

3.1.1 Dyck Paths

We now turn to a topic briefly mentioned in Section 2.1.2. A *Dyck path* of length $2k$ is a path on $\mathbb{Z}^2$ from $(0,0)$ to $(2k,0)$, such that steps must be either $(1,1)$ or $(1,-1)$, and the path never goes below the x-axis, i.e. the second coordinate is always nonnegative. Alternatively, a Dyck path can be thought of as a walk of $2k$ steps from 0 to 0 on a graph of the nonnegative integers with an undirected edge between i and j if $|i-j|=1$, which is exactly the description of $\mathcal{R}_\mathsf{V}(\mathsf{SU}_2)$ (see figure 1.1). Trivially, a third definition is a path on the Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{SU}_2)$ from $0 \in \Lambda_0(\mathsf{SU}_2)$ to $0 \in \Lambda_{2k}(\mathsf{SU}_2)$.

Let $\mathcal{D}_k$ be the set of all Dyck paths of length $2k$. It is well known that $\mathcal{D}_k$ is a Catalan family, and further, there is a previously existing bijection with a basis for $\mathsf{TL}_k(2) = \mathsf{Z}_k(\mathsf{SU}_2)$, which can be found in [GHJ]. In this thesis, we provide two novel bijections, one of which functions for the aforementioned basis, and the other of which uses a new basis.

Using the third interpretation of Dyck paths above, any path $p \in \mathcal{D}_k$ can be described as the series of vertices visited,

$$p = (0 = p_0, p_1, \ldots, p_{2k-1}, p_{2k} = 0), \tag{3.1}$$

where p_i is an element of $\Lambda_i(\mathsf{SU}_2)$. A *peak* of such a path is an entry p_i such that $|p_{i-1}| < |p_i|$ and $|p_i| > |p_{i+1}|$. By examining the peaks of a path, we can in fact determine to entire path, as we will show.

Proposition 3.2. *Any path $p \in \mathcal{D}_k$ is determined uniquely by the set*

$$\{p_{i_1}, p_{i_2}, \ldots, p_{i_\ell}\}$$

of its peaks.

Proof. Assume that we are given a set of peaks $\{p_{i_1}, p_{i_2}, \ldots, p_{i_\ell}\}$ which is a valid peak set for at least one path in $\mathcal{D}_k$. We will show that there is exactly one path $p \in \mathcal{D}_k$ which has this peak set, by showing that we can construct all of p using the set.

First, the steps in p between $p_0 = 0$ and p_{i_1} and the steps between p_{i_ℓ} and $p_{2k} = 0$ are uniquely determined; the steps of a path before the first peak can only be increasing steps (to a vertex of higher value) ending at p_{i_1}, and the steps after the last peak can only be decreasing steps (to vertices of lower value) ending at $p_{2k} = 0$.

Then, for two peaks p_{i_j} and $p_{i_{j+1}}$ in our set, there is also only one possible connecting path, which can be found by joining of the paths

$$(p_{i_j}, (|p_{i_j}| - 1)_{(i_j)+1}, \ldots, 0) \text{ and } (0, 1, \ldots, p_{i_{(j+1)}})$$

at their intersection, so that the resulting path begins like the first path, and ends like the second. Note that if the above two paths do not intersect, there is no possible path between the p_{i_j} and $p_{i_{j+1}}$ which does not include a third peak, violating out initial assumption. □

As Dyck paths are uniquely determined by their peak sets, we can deduce a set of conditions on peak sets equivalent to the conditions on $\mathcal{D}_k$. Namely, the set of peak set for $\mathcal{D}_k$ has elements of the form $\{p_{i_1}, p_{i_2}, \ldots, p_{i_\ell}\}$, the following hold:

$$\begin{array}{ll} 1. & 1 < i_1 < i_2 < \cdots < i_\ell < 2k \\ 2. & |p_{i_j}| \geq 0 \ \text{ for all } 1 \leq j \leq \ell \\ 3. & 1 = \frac{i_1 - |p_{i_1}|}{2} + 1 < \frac{i_2 - |p_{i_2}|}{2} + 1 < \cdots < \frac{i_\ell - |p_{i_\ell}|}{2} + 1 < 2k, \\ 4. & 1 < \frac{i_1 + |p_{i_1}|}{2} + 1 < \frac{i_2 + |p_{i_2}|}{2} + 1 < \cdots < \frac{i_\ell + |p_{i_\ell}|}{2} + 1 = 2k + 1. \end{array} \tag{3.3}$$

The first two conditions simply specify that our peak set must contain legitimate peaks. Condition 3. specifies a quantity which remains constant under upward steps, but increases for downward steps, while condition 4. specifies a quantity which remains constant under downward steps, but increases for upward steps (again both have well-definedness included in the bounding parameters). Thus, together, conditions 3. and 4. specify that there must be exactly one downward step and one upward step between each peak, which is the only consistent condition on peaks of Dyck paths. Thus, the conditions in Equation 3.3 describe exactly the set of peaks of paths in $\mathcal{D}_k$.

3.1.2 Paths on the Bratteli Diagram for Finite Subgroups

We now attempt to perform a similar analysis for paths on other Bratteli diagrams. While these paths are not Dyck paths, they share certain commonalities. Let a path $p \in \mathcal{P}^0_{2k}$ be a sequence

$$p = (0 = p_0, p_1, \dots, p_{2k-1}, p_{2k} = 0). \tag{3.4}$$

If an element p_i is marked with a prime symbol (e.g. $4'_8$ or $6'_6$), we refer to this element as *primed*. We can define peaks exactly as we did for Dyck paths, and a similar result holds about the peak set.

Proposition 3.5. *Let* G *be as defined throughout the paper, with the added condition that* $\mathsf{G} \neq \mathbf{C}_n$*. Then, elements of* $\mathcal{P}^0_{2k}$ *on* $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ *are defined uniquely by their peak set.*

Proof. It is known from the previous propositions that Dyck paths are determined uniquely by their peaks. With this in mind, consider the mapping f from $\mathcal{P}^0_{2k}$ to Dyck paths $\mathcal{D}_k$ given by $p \to (|p_1|, |p_2|, \dots, |p_{2k}|)$. For paths on $\mathcal{B}_{\mathsf{V}}(\mathsf{SU}_2)$, this mapping is simply a bijection. However, for other groups the mapping is neither injective nor surjective. To illustrate this, consider paths on $\mathcal{B}_{\mathsf{V}}(\mathbf{O})$ which differ only by having 4 and $4'$ as a peak, and the fact that there can be no path with a peak higher than 6.

Construct a new mapping $\psi : \mathcal{P}^0_{2k} \to \mathcal{D}_k \times P$, where P is the set of all possible sets of primed peaks, given by $\psi(p) = (f(p), \{p_i \in P \mid p_i \text{ is primed in } p\})$. By our earlier remarks, ψ is clearly injective, and so the sets $\mathcal{P}^0_{2k}$ and $\mathsf{im}(\psi)$ are in bijection. Then, however, elements of $\mathsf{im}(\psi) \subseteq \mathcal{D}_k \times P$ are also in bijection with the peak set of a Dyck path, with some peaks marked as prime. This is exactly how we have defined the peak set of p is, and so ψ is a bijection. □

We can now characterize peak sets for elements of $\mathcal{P}^0_{2k}$ as we did for elements of $\mathcal{D}_k$. If $p \in \mathcal{P}^0_{2k}$ has a peak set $\{p_{i_1}, p_{i_2}, \ldots, p_{i_\ell}\}$, the following hold:

$$\begin{array}{ll} 1. & 1 < i_1 < i_2 < \cdots < i_\ell < 2k \\ 2. & p_{i_j} \in \Lambda(\mathsf{G}) \ \text{ for all } 1 \le j \le \ell \\ 3. & 1 = \frac{i_1 - |p_{i_1}|}{2} + 1 < \frac{i_2 - |p_{i_2}|}{2} + 1 < \cdots < \frac{i_\ell - |p_{i_\ell}|}{2} + 1 < 2k, \\ 4. & 1 < \frac{i_1 + |p_{i_1}|}{2} + 1 < \frac{i_2 + |p_{i_2}|}{2} + 1 < \cdots < \frac{i_\ell + |p_{i_\ell}|}{2} + 1 = k + 1, \\ 5. & \frac{i_j + |p_{i_j}|}{2} \le \min(\mathsf{br}(\mathsf{G})) + \frac{i_{j-1} - |p_{i_{j-1}}|}{2} \ \text{ for } j > 1, \mathsf{G} = \mathbf{T}. \end{array} \tag{3.6}$$

Note that these are nearly identical to the conditions on peak sets of Dyck paths, save the final one which applies only to the case of $\mathsf{G} = \mathbf{T}$, and is the result of $\mathcal{R}_\mathsf{V}(\mathbf{T})$ having a branch node with no children who are leaves.

3.2 Path to Words Correspondence

In this section, we describe an association of elements of $\mathsf{Z}_k(\mathsf{G})$ with paths $p \in \mathcal{P}^0_{2k}$ on the Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{G})$ from $0 \in \Lambda_0$ to $0 \in \Lambda_{2k}$. Specifically, for each path p we construct a *word* w_p in the generators of $\mathsf{Z}_k(\mathsf{G})$ for $\mathsf{G} = \mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$ such that $\{w_p \mid p \in \mathcal{P}^0_{2k}\}$ is a basis for $\mathsf{Z}_k(\mathsf{G})$. For the remainder of the chapter, let $\beta = \min(\mathsf{br}(\mathsf{G}))$, and assume that G refers only to $\mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$.

3.2.1 Algorithm

For a path $p \in \mathcal{P}^0_{2k}$ with peaks $\{p_{i_j} \mid 1 \le j \le \ell\}$, w_p takes the form of a word in the projection generators introduced in Section 2.3. Explicitly,

$$w_p = \mathsf{f}^{(p_{i_1})}_{\alpha_1} \mathsf{f}^{(p_{i_2})}_{\alpha_2} \cdots \mathsf{f}^{(p_{i_\ell})}_{\alpha_\ell} \qquad \text{for } \alpha_j = \frac{i - |p_{i_j}|}{2} + 1 \tag{3.7}$$

If $\mathsf{f}^{(p_{i_j})}_{\alpha_j} = \mathbf{1}$ (and so $p_{i_j} = 1_{i_j}$), then we say that p_{i_j} is an *identity peak*.

Equation 3.6 allows us to characterize words with only a few conditions. For any word w_p as defined above,

$$\begin{array}{llll} (1) & \text{(Well-Definedness)} & |p_{i_x}| + \alpha_x \le k + 1, & \text{for all } x, \\ (2) & (\mathsf{SU}_2 \text{ conditions}) & \alpha_x < \alpha_y & \\ & & |p_{i_x}| + \alpha_x < |p_{i_y}| + \alpha_y, & \\ (3) & (\mathbf{T} \text{ condition}) & |p_{i_x}| + \alpha_x \le \beta + \alpha_y, & |p_{i_x}| > \beta, \text{ and} \\ & & & p_{i_x} \not\prec p_{i_y}. \end{array} \tag{3.8}$$

In fact, $\{w_p \mid p \in \mathcal{P}^0_{2k}\}$ is every product in the form given in Equation 3.7 that also fulfills Equation 3.8.

3.2.2 Examples

Example 3.9 (Paths and their Corresponding Words)**.** The following are examples of paths and the corresponding words, with peaks circled.

In the case of $\mathsf{Z}_9(\mathsf{SU}_2) \cong \mathsf{TL}_{10}(2)$ the paths are Dyck paths.

$$p = (0_0, 1_1, 2_2, 3_3, 4_4, \enclose{circle}{5_5}, 4_6, 3_7, 4_8, 5_9, \enclose{circle}{6_{10}}, 5_{11}, 4_{12}, 3_{13}, 2_{14}, \enclose{circle}{3_{15}}, 2_{16}, 1_{17}, 0_{18})$$

$$w_p = \left(\mathsf{f}_1^{(5)}\right)\left(\mathsf{f}_3^{6}\right)\left(\mathsf{f}_7^{(3)}\right)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4_4}, 3_5, 4_6, 5_7, \enclose{circle}{6_8}, 5_9, 4_{10}, 3_{11}, 2_{12}, 1_{13}, 0_{14}, \enclose{circle}{1_{15}}, 0_{16}, 1_{17}, 0_{18})$$

$$w_p = \left(\mathsf{f}_1^{(4)}\right)\left(\mathsf{f}_2^{(6)}\right)\left(\mathsf{f}_8^{(1)}\right)\left(\mathsf{f}_9^{(1)}\right)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4_4}, 3_5, 4_6, 5_7, \enclose{circle}{6_8}, 5_9, 4_{10}, 3_{11}, \enclose{circle}{4_{12}}, 3_{13}, 2_{14}, 1_{15}, \enclose{circle}{2_{16}}, 1_{17}, 0_{18})$$

$$w_p = \left(\mathsf{f}_1^{(4)}\right)\left(\mathsf{f}_2^{(6)}\right)\left(\mathsf{f}_5^{(4)}\right)\left(\mathsf{f}_8^{(2)}\right)$$

$$p = (0_0, \enclose{circle}{1_1}, 0_2, \enclose{circle}{1_3}, 0_4, \enclose{circle}{1_5}, 0_6, \enclose{circle}{1_7}, 0_8, \enclose{circle}{1_9}, 0_{10}, \enclose{circle}{1_{11}}, 0_{12}, \enclose{circle}{1_{13}}, 0_{14}, \enclose{circle}{1_{15}}, 0_{16}, \enclose{circle}{1_{17}}, 0_{18})$$

$$w_p = (\mathsf{f}_1^{(1)})(\mathsf{f}_2^{(1)})(\mathsf{f}_3^{(1)})(\mathsf{f}_4^{(1)})(\mathsf{f}_5^{(1)})(\mathsf{f}_6^{(1)})(\mathsf{f}_7^{(1)})(\mathsf{f}_8^{(1)})(\mathsf{f}_9^{(1)}) = \mathbf{1}$$

$$p = (0_0, 1_1, 2_2, 3_3, 4_4, 5_5, 6_6, 7_7, 8_8, \enclose{circle}{9_9}, 8_{10}, 7_{11}, 6_{12}, 5_{13}, 4_{14}, 3_{15}, 2_{16}, 1_{17}, 0_{18})$$

$$w_p = \mathsf{f}_1^{(9)}$$

In $\mathsf{Z}_9(\mathbf{O})$

$$p = (0_0, 1_1, 2_2, 3_3, 4_4, \enclose{circle}{5_5}, 4_6, 3_7, 4_8, 5_9, \enclose{circle}{6_{10}}, 5_{11}, 4_{12}, 3_{13}, 2_{14}, 1_{15}, \enclose{circle}{2_{16}}, 1_{17}, 0_{18})$$

$$w_p = \left(\mathsf{f}_1^{(5)}\right)\left(\mathsf{f}_3^{6}\right)\left(\mathsf{f}_8^{(2)}\right)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4'_4}, 3_5, 4_6, 5_7, \enclose{circle}{6_8}, 5_9, 4_{10}, 3_{11}, 2_{12}, 1_{13}, 0_{14}, \enclose{circle}{1_{15}}, 0_{16}, \enclose{circle}{1_{17}}, 0_{18})$$

$$w_p = \left(\mathsf{f}_1^{(4')}\right)\left(\mathsf{f}_2^{(6)}\right)\left(\mathsf{f}_8^{(1)}\right)\left(\mathsf{f}_9^{(1)}\right)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4_4}, 3_5, 4_6, 5_7, \enclose{circle}{6_8}, 5_9, 4_{10}, 3_{11}, \enclose{circle}{4'_{12}}, 3_{13}, \enclose{circle}{4_{14}}, 3_{15}, 2_{16}, 1_{17}, 0_{18})$$

$$w_p = \left(\mathsf{f}_1^{(4)}\right)\left(\mathsf{f}_2^{(6)}\right)\left(\mathsf{f}_5^{(4')}\right)\left(\mathsf{f}_6^{(4)}\right)$$

In $\mathsf{Z}_9(\mathbf{I})$:

$$p = (0_0, 1_1, 2_2, 3_3, 4_4, 5_5, 6_6, \textcircled{7_7}, 6_8, 5_9, \textcircled{6'_{10}}, 5_{11}, \textcircled{6_{12}}, 5_{13}, 4_{14}, 3_{15}, 2_{16}, 1_{17}, 0_{18})$$
$$w_p = \left(\mathsf{f}_1^{(7)}\right)\left(\mathsf{f}_3^{(6')}\right)\left(\mathsf{f}_4^{(6)}\right)$$

In $\mathsf{Z}_9(\mathbf{T})$

$$p = (0_0, 1_1, 2_2, \textcircled{3_3}, 2_4, \textcircled{3_5}, 2_6, 1_7, 2_8, 3'_9, \textcircled{4'_{10}}, 3'_{11}, 2_{12}, 3_{13}, \textcircled{4_{14}}, 3_{15}, 2_{16}, 1_{17}, 0_{18})$$
$$w_p = \left(\mathsf{f}_1^{(3)}\right)\left(\mathsf{f}_2^{(3)}\right)\left(\mathsf{f}_4^{(4')}\right)\left(\mathsf{f}_6^{(4)}\right)$$
$$p = (0_0, 1_1, 2_2, \textcircled{3_3}, 2_4, \textcircled{3_5}, 2_6, 1_7, 2_8, 3'_9, \textcircled{4'_{10}}, 3'_{11}, \textcircled{4'_{12}}, 3'_{13}, \textcircled{4_{14}}, 3_{15}, 2_{16}, 1_{17}, 0_{18})$$
$$w_p = \left(\mathsf{f}_1^{(3)}\right)\left(\mathsf{f}_2^{(3)}\right)\left(\mathsf{f}_4^{(4')}\right)\left(\mathsf{f}_5^{(4)}\right)\left(\mathsf{f}_6^{(4)}\right)$$
$$p = (0_0, 1_1, 2_2, 3_3, \textcircled{4_4}, 3_5, \textcircled{4_6}, 3_7, \textcircled{4_8}, 3_9, 2_{10}, 1_{11}, 0_{12}, 1_{13}, 2_{14}, \textcircled{3'_{15}}, 2_{16}, 1_{17}, 0_{18})$$
$$w_p = \left(\mathsf{f}_1^{(4)}\right)\left(\mathsf{f}_2^{(4)}\right)\left(\mathsf{f}_3^{(4)}\right)\left(\mathsf{f}_7^{(3')}\right)$$

3.2.3 More Examples

Here, we list a complete basis (with corresponding paths) for $\mathsf{Z}_4(\mathbf{T})$, $\mathsf{Z}_5(\mathsf{SU}_2)$, and $\mathsf{Z}_5(\mathbf{O})$. Due to the number of examples, we do not circle peaks, because $\mathsf{f}_i^{(1)} = \mathbf{1}$ for any i, we omit these elements, save for the identity element of our basis.

Example 3.10. For $\mathsf{G} = \mathbf{T}$, we list all 22 elements $p \in \mathcal{P}_8^0$ and the corresponding basis element w_p.

Basis Element	Path	Basis Element	Path
$\mathsf{f}_1^{(4)}$	$(0,1,2,3,4,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3')}$	$(0,1,2,1,2,3',2,1,0)$
$\mathsf{f}_1^{(4')}$	$(0,1,2,3',4',3',2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}\mathsf{f}_3^{(2)}$	$(0,1,2,1,2,1,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)}$	$(0,1,2,3,2,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}$	$(0,1,2,1,2,1,0,1,0)$
$\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3)}$	$(0,1,2,3',2,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_3^{(2)}$	$(0,1,2,1,0,1,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3')}$	$(0,1,2,3,2,3',2,1,0)$	$\mathsf{f}_1^{(2)}$	$(0,1,2,1,0,1,0,1,0)$
$\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3')}$	$(0,1,2,3',2,3',2,1,0)$	$\mathsf{f}_2^{(3)}$	$(0,1,0,1,2,3,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_3^{(2)}$	$(0,1,2,3,2,1,2,1,0)$	$\mathsf{f}_2^{(3')}$	$(0,1,0,1,2,3',2,1,0)$
$\mathsf{f}_1^{(3')}\mathsf{f}_3^{(2)}$	$(0,1,2,3',2,1,2,1,0)$	$\mathsf{f}_2^{(2)}\mathsf{f}_3^{(2)}$	$(0,1,0,1,2,1,2,1,0)$
$\mathsf{f}_1^{(3)}$	$(0,1,2,3,2,1,0,1,0)$	$\mathsf{f}_2^{(2)}$	$(0,1,0,1,2,1,0,1,0)$
$\mathsf{f}_1^{(3')}$	$(0,1,2,3',2,1,0,1,0)$	$\mathsf{f}_3^{(2)}$	$(0,1,0,1,0,1,2,1,0)$
$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3)}$	$(0,1,2,1,2,3,2,1,0)$	$\mathbf{1}$	$(0,1,0,1,0,1,0,1,0)$

Example 3.11. For $\mathsf{G} = \mathsf{SU}_2$, we list all 42 elements $p \in \mathcal{P}^0_{10}$ and the corresponding basis element w_p.

Basis Element	Path	Basis Element	Path
$f_1^{(5)}$	$(0,1,2,3,4,5,4,3,2,1,0)$	$f_1^{(2)}f_2^{(2)}f_3^{(2)}$	$(0,1,2,1,2,1,2,1,0,1,0)$
$f_1^{(4)}f_2^{(4)}$	$(0,1,2,3,4,3,4,3,2,1,0)$	$f_1^{(2)}f_2^{(2)}$	$(0,1,2,1,2,1,0,1,0,1,0)$
$f_1^{(4)}f_3^{(3)}$	$(0,1,2,3,4,3,2,3,2,1,0)$	$f_1^{(2)}f_3^{(3)}$	$(0,1,2,1,2,1,2,3,2,1,0)$
$f_1^{(4)}f_4^{(2)}$	$(0,1,2,3,4,3,2,1,2,1,0)$	$f_1^{(2)}f_3^{(2)}f_4^{(2)}$	$(0,1,2,1,0,1,2,1,2,1,0)$
$f_1^{(4)}$	$(0,1,2,3,4,3,2,1,0,1,0)$	$f_1^{(2)}f_4^{(2)}$	$(0,1,2,1,0,1,0,1,2,1,0)$
$f_1^{(3)}f_2^{(4)}$	$(0,1,2,3,2,3,4,3,2,1,0)$	$f_1^{(2)}f_3^{(2)}$	$(0,1,2,1,0,1,2,1,0,1,0)$
$f_1^{(3)}f_2^{(3)}f_3^{(3)}$	$(0,1,2,3,2,3,2,3,2,1,0)$	$f_1^{(2)}$	$(0,1,2,1,0,1,0,1,0,1,0)$
$f_1^{(3)}f_2^{(3)}f_4^{(2)}$	$(0,1,2,3,2,3,2,1,2,1,0)$	$f_2^{(4)}$	$(0,1,0,1,2,3,4,3,2,1,0)$
$f_1^{(3)}f_2^{(3)}$	$(0,1,2,3,2,3,2,1,0,1,0)$	$f_2^{(3)}f_3^{(3)}$	$(0,1,0,1,2,3,2,3,2,1,0)$
$f_1^{(3)}f_3^{(3)}$	$(0,1,2,3,2,1,2,3,2,1,0)$	$f_2^{(3)}f_4^{(2)}$	$(0,1,0,1,2,3,2,1,2,1,0)$
$f_1^{(3)}f_3^{(2)}f_4^{(2)}$	$(0,1,2,3,2,1,2,1,2,1,0)$	$f_2^{(3)}$	$(0,1,0,1,2,3,2,1,0,1,0)$
$f_1^{(3)}f_4^{(2)}$	$(0,1,2,3,2,1,0,1,2,1,0)$	$f_2^{(2)}f_3^{(3)}$	$(0,1,0,1,2,1,2,3,2,1,0)$
$f_1^{(3)}f_3^{(2)}$	$(0,1,2,3,2,1,2,1,0,1,0)$	$f_2^{(2)}f_3^{(2)}f_4^{(2)}$	$(0,1,0,1,2,1,2,1,2,1,0)$
$f_1^{(3)}$	$(0,1,2,3,2,1,0,1,0,1,0)$	$f_2^{(2)}f_4^{(2)}$	$(0,1,0,1,2,1,0,1,2,1,0)$
$f_1^{(2)}f_2^{(4)}$	$(0,1,2,1,2,3,4,3,2,1,0)$	$f_2^{(2)}f_3^{(2)}$	$(0,1,0,1,2,1,2,1,0,1,0)$
$f_1^{(2)}f_2^{(3)}f_3^{(3)}$	$(0,1,2,1,2,3,2,3,2,1,0)$	$f_2^{(2)}$	$(0,1,0,1,2,1,0,1,0,1,0)$
$f_1^{(2)}f_2^{(3)}f_4^{(2)}$	$(0,1,2,1,2,3,2,1,2,1,0)$	$f_3^{(3)}$	$(0,1,0,1,0,1,2,1,0,1,0)$
$f_1^{(2)}f_2^{(3)}$	$(0,1,2,1,2,3,2,1,0,1,0)$	$f_3^{(2)}f_4^{(2)}$	$(0,1,0,1,0,1,2,1,2,1,0)$
$f_1^{(2)}f_2^{(2)}f_3^{(3)}$	$(0,1,2,1,2,1,2,1,0,1,0)$	$f_4^{(2)}$	$(0,1,0,1,0,1,0,1,2,1,0)$
$f_1^{(2)}f_2^{(2)}f_3^{(2)}f_4^{(2)}$	$(0,1,2,1,2,1,2,1,2,1,0)$	$f_3^{(2)}$	$(0,1,0,1,0,1,2,1,0,1,0)$
$f_1^{(2)}f_2^{(2)}f_4^{(2)}$	$(0,1,2,1,2,1,0,1,2,1,0)$	$\mathbf{1}$	$(0,1,0,1,0,1,0,1,0,1,0)$

Example 3.12. For $\mathsf{G} = \mathbf{O}$, we list all 51 elements $p \in \mathcal{P}_{10}^0$ and the corresponding basis element w_p. Note that each of the 42 paths in the previous example appear here, as well as additional paths which contain the vertex $4'$.

Basis Element	Path	Basis Element	Path
$\mathsf{f}_1^{(5)}$	$(0,1,2,3,4,5,4,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}\mathsf{f}_3^{(3)}$	$(0,1,2,1,2,1,2,1,0,1,0)$
$\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4)}$	$(0,1,2,3,4,3,4,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}\mathsf{f}_3^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,2,1,2,1,2,1,2,1,0)$
$\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4')}$	$(0,1,2,3,4,3,4',3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,2,1,2,1,0,1,2,1,0)$
$\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4)}$	$(0,1,2,3,4',3,4,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}\mathsf{f}_3^{(2)}$	$(0,1,2,1,2,1,2,1,0,1,0)$
$\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4')}$	$(0,1,2,3,4',3,4',3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(2)}$	$(0,1,2,1,2,1,0,1,0,1,0)$
$\mathsf{f}_1^{(4)}\mathsf{f}_3^{(3)}$	$(0,1,2,3,4,3,2,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_3^{(3)}$	$(0,1,2,1,2,1,2,3,2,1,0)$
$\mathsf{f}_1^{(4')}\mathsf{f}_3^{(3)}$	$(0,1,2,3,4',3,2,3,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_3^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,2,1,0,1,2,1,2,1,0)$
$\mathsf{f}_1^{(4)}\mathsf{f}_4^{(2)}$	$(0,1,2,3,4,3,2,1,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,2,1,0,1,0,1,2,1,0)$
$\mathsf{f}_1^{(4')}\mathsf{f}_4^{(2)}$	$(0,1,2,3,4',3,2,1,2,1,0)$	$\mathsf{f}_1^{(2)}\mathsf{f}_3^{(2)}$	$(0,1,2,1,0,1,2,1,0,1,0)$
$\mathsf{f}_1^{(4)}$	$(0,1,2,3,4,3,2,1,0,1,0)$	$\mathsf{f}_1^{(2)}$	$(0,1,2,1,0,1,0,1,0,1,0)$
$\mathsf{f}_1^{(4')}$	$(0,1,2,3,4',3,2,1,0,1,0)$	$\mathsf{f}_2^{(4)}$	$(0,1,0,1,2,3,4,3,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(4)}$	$(0,1,2,3,2,3,4,3,2,1,0)$	$\mathsf{f}_2^{(4')}$	$(0,1,0,1,2,3,4',3,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(4')}$	$(0,1,2,3,2,3,4',3,2,1,0)$	$\mathsf{f}_2^{(3)}\mathsf{f}_3^{(3)}$	$(0,1,0,1,2,3,2,3,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)}\mathsf{f}_3^{(3)}$	$(0,1,2,3,2,3,2,3,2,1,0)$	$\mathsf{f}_2^{(3)}\mathsf{f}_4^{(2)}$	$(0,1,0,1,2,3,2,1,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)}\mathsf{f}_4^{(2)}$	$(0,1,2,3,2,3,2,1,2,1,0)$	$\mathsf{f}_2^{(3)}$	$(0,1,0,1,2,3,2,1,0,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)}$	$(0,1,2,3,2,3,2,1,0,1,0)$	$\mathsf{f}_2^{(2)}\mathsf{f}_3^{(3)}$	$(0,1,0,1,2,1,2,3,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_3^{(3)}$	$(0,1,2,3,2,1,2,3,2,1,0)$	$\mathsf{f}_2^{(2)}\mathsf{f}_3^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,0,1,2,1,2,1,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_3^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,2,3,2,1,2,1,2,1,0)$	$\mathsf{f}_2^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,0,1,2,1,0,1,2,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_4^{(2)}$	$(0,1,2,3,2,1,0,1,2,1,0)$	$\mathsf{f}_2^{(2)}\mathsf{f}_3^{(2)}$	$(0,1,0,1,2,1,2,1,0,1,0)$
$\mathsf{f}_1^{(3)}\mathsf{f}_3^{(2)}$	$(0,1,2,3,2,1,2,1,0,1,0)$	$\mathsf{f}_2^{(2)}$	$(0,1,0,1,2,1,0,1,0,1,0)$
$\mathsf{f}_1^{(3)}$	$(0,1,2,3,2,1,0,1,0,1,0)$	$\mathsf{f}_3^{(3)}$	$(0,1,0,1,0,1,2,1,0,1,0)$
$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(4)}$	$(0,1,2,1,2,3,4,3,2,1,0)$	$\mathsf{f}_3^{(2)}\mathsf{f}_4^{(2)}$	$(0,1,0,1,0,1,2,1,2,1,0)$
$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(4')}$	$(0,1,2,1,2,3,4',3,2,1,0)$	$\mathsf{f}_4^{(2)}$	$(0,1,0,1,0,1,0,1,2,1,0)$
$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3)}\mathsf{f}_3^{(3)}$	$(0,1,2,1,2,3,2,3,2,1,0)$	$\mathsf{f}_3^{(2)}$	$(0,1,0,1,0,1,2,1,0,1,0)$
$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3)}\mathsf{f}_4^{(2)}$	$(0,1,2,1,2,3,2,1,2,1,0)$	**1**	$(0,1,0,1,0,1,0,1,0,1,0)$
$\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3)}$	$(0,1,2,1,2,3,2,1,0,1,0)$		

3.2.4 Proof of Algorithm

While the main proof that $\{w_p \mid p \in \mathcal{P}^0_{2k}\}$ is a basis is reserved for the next section, we show the no less essential result that the mapping $p \to w_p$ is a bijection between paths and words.

Proposition 3.13. *The map $\psi : \mathcal{P}^0_{2k} \to \{w_p \mid p \in \mathcal{P}^0_{2k}\}$ which takes p to w_p is a bijection, up to distinctness of words.*

Proof. First, note that ψ is by definition surjective. Further, as w_p is determined uniquely by the peak set, as the tuple $((p_{i_j}), \alpha_j)$ is unique to the peak p_{i_j}, and so as long as no two paths have the same peak set, ψ is injective. Proposition 3.5 shows exactly this, and so ψ is a bijection.

□

3.3 Basis Theorem

In this section we give our main result. First, however, we must give several preliminary results. These center around the interaction of elements of $\{w_p \mid p \in \mathcal{P}^0_{2k}\}$ and the generating elements

$$\mathsf{gen}(\mathsf{Z}_k(\mathsf{G})) = \{\mathbf{1}\} \cup \{\mathsf{f}_i^{(2)} \mid 1 \leq i \leq k-1\} \cup \{\mathsf{f}_1^{(\lambda)} \mid |\lambda| \leq k, \lambda \in \mathsf{br}(\mathsf{G})\},$$

which Theorem 2.51, Remark 2.52 and the induction in from Theorem 2.11 show can be used to recursively generate $\mathsf{Z}_k(\mathsf{G})$.

3.3.1 Factoring Lemmas

Lemma 3.14 (Factoring Lemma)**.** *For $k > 0$, if w_p is a path basis word in $\mathsf{Z}_{k+1}(\mathsf{G})$, for $\mathsf{G} = \mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$, then w_p factors as $w_p = w_{p'}\mathsf{f}_{\alpha_\ell}^{(p_\ell)}$, with $p' \in \mathcal{P}^0_{2k}$ and $|p_\ell| + \ell = 2(k+1)$.*

Proof. This should be clear from the definition of w_p. As a product of generators $\mathsf{f}_i^{(\lambda)}$, only the last factor of w_p is in $\mathsf{gen}(\mathsf{Z}_{k+1}(\mathsf{G})) \setminus \mathsf{gen}(\mathsf{Z}_k(\mathsf{G}))$. Example 3.15 shows this process in several cases, and the reader is encouraged to examine Example 3.9 if more assurance is desired. □

Example 3.15. Here we give several applications of Lemma 3.14. These examples are also visualized in Figure 3.1.

(a) For $\mathsf{G} = \mathbf{O}$ with $k = 7$, let

$$p = (0, 1, 2, 1, 0, 1, 2, 3, 4', 3, 4, 3, 2, 1, 0),$$
$$w_p = \left(\mathsf{f}_1^{(2)}\right)\left(\mathsf{f}_3^{(4')}\right)\left(\mathsf{f}_4^{(4)}\right).$$

The factoring lemma states that we can find a path $p' \in \mathcal{P}_{12}^0(\mathbf{O})$ such that $w_{p'} = \left(\mathsf{f}_1^{(2)}\right)\left(\mathsf{f}_3^{(4')}\right)$, which is satisfied by

$$p' = (0, 1, 2, 1, 0, 1, 2, 3, 4', 3, 2, 1, 0).$$

(b) For $\mathsf{G} = \mathbf{I}$ with $k = 7$, let

$$p = (0, 1, 2, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0),$$
$$w_p = \left(\mathsf{f}_1^{(3)}\right)\left(\mathsf{f}_3^{(5)}\right).$$

The factoring lemma claims that we can find a path $p' \in \mathcal{P}_{12}^0(\mathbf{I})$ such that $w_{p'} = \mathsf{f}_1^{(3)}$, which is satisfied by

$$p' = (0, 1, 2, 3, 2, 1, 0, 1, 0, 1, 0, 1, 0).$$

(c) For $\mathsf{G} = \mathbf{T}$ with $k = 7$, let

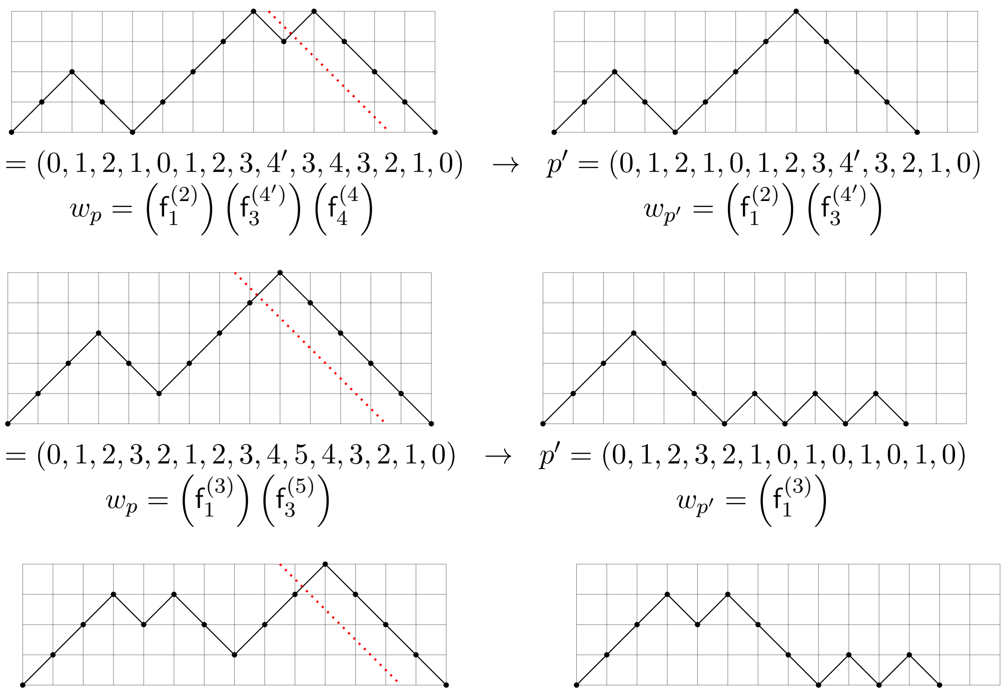
$$p = (0, 1, 2, 3', 2, 3, 2, 1, 2, 3', 4', 3', 2, 1, 0),$$
$$w_p = \left(\mathsf{f}_1^{(3')}\right)\left(\mathsf{f}_2^{3}\right)\left(\mathsf{f}_4^{(4')}\right).$$

The factoring lemma states that we can find a path $p' \in \mathcal{P}_{14}^0$ such that $w_{p'} = \left(\mathsf{f}_1^{(3')}\right)\left(\mathsf{f}_2^{3}\right)$, which is satisfied by

$$p' = (0, 1, 2, 3', 2, 3, 2, 1, 0, 1, 0, 1, 0).$$

Now, we show that the reverse process of Lemma 3.14 can also be done.

Lemma 3.16 (Combination Lemma)**.** *For* $\mathsf{G} = \mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, *and* $\mathbf{I}$, $p \in \mathcal{P}_{2k}^0$, *and* $|\lambda| + i = k + 2$, *elements* $w_p \mathsf{f}_i^{(\lambda)} \in \mathsf{Z}_{k+1}$ *can be written as the sum of basis words* $\{w_p \mid p \in \mathcal{P}_{2(k+1)}^0\}$.



$$p = (0,1,2,1,0,1,2,3,4',3,4,3,2,1,0) \quad\rightarrow\quad p' = (0,1,2,1,0,1,2,3,4',3,2,1,0)$$
$$w_p = \left(\mathsf{f}_1^{(2)}\right)\left(\mathsf{f}_3^{(4')}\right)\left(\mathsf{f}_4^{(4}\right) \qquad w_{p'} = \left(\mathsf{f}_1^{(2)}\right)\left(\mathsf{f}_3^{(4')}\right)$$

$$p = (0,1,2,3,2,1,2,3,4,5,4,3,2,1,0) \quad\rightarrow\quad p' = (0,1,2,3,2,1,0,1,0,1,0,1,0)$$
$$w_p = \left(\mathsf{f}_1^{(3)}\right)\left(\mathsf{f}_3^{(5)}\right) \qquad w_{p'} = \left(\mathsf{f}_1^{(3)}\right)$$

$$p = (0,1,2,3',2,3,2,1,2,3',4',3',2,1,0) \quad\rightarrow\quad p' = (0,1,2,3',2,3,2,1,0,1,0,1,0)$$
$$w_p = \left(\mathsf{f}_1^{(3')}\right)\left(\mathsf{f}_2^{3}\right)\left(\mathsf{f}_4^{(4')}\right) \qquad w_{p'} = \left(\mathsf{f}_1^{(3')}\right)\left(\mathsf{f}_2^{3}\right)$$

Figure 3.1 A visualization of Examples 3.15, (a), (b), and (c), from top to bottom. In each we remove the last peak of our path, and then complete the path so that the only peaks which are added contribute the identity element $\mathsf{f}_i^{(1)}$ to $w_{p'}$. This removal can be seen by the red dashed line in each of the diagrams on the left.

Proof. We will prove by induction on the number of non-identity peaks in p (or factors of w_p). Clearly, if p has no non-identity peaks, $w_p = \mathbf{1}$, and so $w_p\mathsf{f}_i^{(\lambda)} = \mathsf{f}_i^{(\lambda)}$ is a basis word. For some fixed number of non-identity peaks $\ell > 0$, assume that elements $w_p\mathsf{f}_i^{(\lambda)}$ can be expressed in standard form for every path $p \in \mathcal{P}_{2k}^0$ with fewer than ℓ non-identity peaks.

Now, let p be a path with ℓ non-identity peaks, and let $\mathsf{f}_i^{(\lambda)}$ be a generator such that $|\lambda| + i = k+2$. By Lemma 3.14, we can re-express $w_p = w_{p'}\mathsf{f}_j^{(\mu)}$ for $p' \in \mathcal{P}_{2(k-1)}^0$ and $|\mu| + j = k+1$. If $w_p\mathsf{f}_i^{(\lambda)}$ is standard, our proof is complete. If not, then the product must violate conditions (2) or (3) of Equation 3.8.

We proceed by case.

Case 1: Condition (2) of Equation 3.8 is violated. As $|\mu| + j < |\lambda| + i$, it must be the case that $j \geq i$. Then, however, by relation (FS6.1) from Theorem 2.51, $\mathsf{f}_j^{(\mu)}\mathsf{f}_i^{(\lambda)} = \mathsf{f}_i^{(\lambda)}$, and so

$$w_p \mathsf{f}_i^{(\lambda)} = w_{p'} \mathsf{f}_i^{(\lambda)}. \tag{3.17}$$

Recalling that p' must have one fewer non-identity peak than $w_p(p)$, we can now apply our inductive hypothesis to complete this case.

Case 2: Condition (3) of Equation 3.8 is violated, and so it must be the case that $\mathsf{G} = \mathbf{T}$, and that $|\lambda| = |\mu| = 4$, $\lambda \neq \mu$, and $i = \ell + 1$. Thus by the induction relation (FS5) from Theorem 2.51,

$$\mathsf{f}_\ell^{(\mu)}\mathsf{f}_i^{(\lambda)} = \mathsf{f}_\ell^{(\mu)}\mathsf{f}_i^{(\lambda-1)}(\mathbf{1} - \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_3)\mathsf{f}_i^{(\lambda-1)}. \tag{3.18}$$

Then, however, as $i = \ell + 1$ and $|\lambda| = |\mu|$, $|\lambda| + i \leq |\mu| + \ell$, and so relation (FS6.2) from Theorem 2.51 applies, and so the entire product is 0, which completes the case trivially. □

3.3.2 Main Theorem

We can now prove the main theorems of the thesis for $\mathsf{Z}_k(\mathsf{SU}_2)$, $\mathsf{Z}_k(\mathbf{T})$, $\mathsf{Z}_k(\mathbf{O})$, and $\mathsf{Z}_k(\mathbf{I})$.

Theorem 3.19. *As described in Equation* 3.7, *the set* $\{w_p \mid p \in \mathcal{P}_{2k}^0\}$ *is a basis for the algebra* $\mathsf{Z}_k(\mathsf{G})$, *with* $\mathsf{G} = \mathsf{SU}_2$, $\mathbf{T}$, $\mathbf{O}$, *or* $\mathbf{I}$.

Proof. Theorem 2.11 shows that every element of $\mathsf{Z}_k(\mathsf{G})$ can be expressed as the sum of products of elements of $\mathsf{gen}(\mathsf{Z}_k(\mathsf{G}))$. Thus, showing that any product of elements $g \in \mathsf{gen}(\mathsf{Z}_k(\mathsf{G}))$ can be written as a $\mathbb{C}$-linear combination of elements $\{w_p \mid p \in \mathcal{P}_{2k}^0\}$ is sufficient to complete the proof.

This result is the inductive result of Theorem 3.20, as $\mathbf{1} \in \{w_p \mid p \in \mathcal{P}_{2k}^0\}$, and any product is $\mathbf{1}$ multiplied on the right by some sequence of elements $g \in \mathsf{gen}(\mathsf{Z}_k(\mathsf{G}))$. □

We follow this superficially satisfying result with the technical result that undergirds Theorem 3.19.

Theorem 3.20. *For* $\mathsf{G} = \mathsf{SU}_2, \mathbf{T}, \mathbf{O}$, *and* $\mathbf{I}$, *if* $p \in \mathcal{P}^0_{2(k+1)}(\mathsf{G})$ *and* $g \in \mathsf{gen}(\mathsf{Z}_{k+1}(\mathsf{G}))$, *then the product* gw_p *can be written as a sum of words of the form* w_q *for* $q \in \mathcal{P}^0_{2(k+1)}(\mathsf{G})$.

Proof. We prove by induction on k, with the trivial base case $k = 1$. Then, assume for that sake of induction that our result holds for some fixed k, and consider Z_{k+1}.

We claim that if $p \in \mathcal{P}^0_{2(k+1)}(\mathsf{G})$ and $g \in \mathsf{gen}(\mathsf{Z}_{k+1}(\mathsf{G}))$, then gw_p can be written as a sum of words of the form

$$w_q \mathsf{f}_i^{(\lambda)}, \tag{3.21}$$

for $q \in \mathcal{P}^0_{2k}$, and $|\lambda| + i = k + 2$. That $q \in \mathcal{P}^0_{2k}$ implies $w_q \in \mathsf{Z}_k \subset \mathsf{Z}_{k+1}$.

We proceed based on the value of g:

Case 1: $g \in \mathsf{gen}(\mathsf{Z}_k)$. By Lemma 3.14, we can write $w_p = w_{p'}\mathsf{f}_j^{(\mu)}$, for $|\mu| + j = k + 2$, and $p' \in \mathcal{P}^0_{2k}$, $w_{p'} \in \mathsf{Z}_k \subset \mathsf{Z}_{k+1}$. Then, $gw_{p'} \in \mathsf{Z}_k \subset \mathsf{Z}_{k+1}$, and so by induction, we can express $gw_{p'}$ as the sum of elements w_q for $q \in \mathcal{P}^0_{2k}$ implies that $w_q \in \mathsf{Z}_k \subset \mathsf{Z}_{k+1}$, and so $gw_{p'}\mathsf{f}_j^{(\mu)}$ can be written as the sum of elements $w_q\mathsf{f}_i^{(\lambda)}$, meeting the aforementioned conditions. This verifies our claim.

Case 2: $g \in \mathsf{gen}(\mathsf{Z}_{k+1}) \setminus \mathsf{gen}(\mathsf{Z}_k)$, and so $g = \mathsf{f}_k^{(2)}$. Applying Lemma 3.14 twice, we can write $w_p = w_{p'}\mathsf{f}_j^{(\mu)}\mathsf{f}_i^{(\lambda)}$, for $|\mu| + j = k + 1$, $|\lambda| + i = k + 2$, and $p' \in \mathcal{P}^0_{2k}$ so that $w_{p'} \in \mathsf{Z}_{k-1} \subset \mathsf{Z}_{k+1}$. As $w_{p'}$ is entirely the product of generators $\mathsf{f}_m^{(\nu)}$ for which $|\nu| + m \leq k$, relation (FS6.1) from Theorem 2.51 gives $\mathsf{f}_k^{(2)}w_{p'} = w_{p'}\mathsf{f}_k^{(2)}$.

Considering the product $\mathsf{f}_k^{(2)}\mathsf{f}_j^{(\mu)}\mathsf{f}_i^{(\lambda)}$, there are several cases in which we can easily make strong statements.

Case 2 (a) $|\lambda| = k + 1$: By relation (1), our word is $\mathsf{f}_1^{(\lambda)}$.

Case 2 (b) $|\lambda| \neq k + 1$, and $j > 1$: The entire product $\mathsf{f}_k^{(2)}\mathsf{f}_j^{(\mu)}\mathsf{f}_i^{(\lambda)}$ is contained in the subalgebra $\tau\mathsf{Z}_k\tau^{-1}$ as $\tau\mathsf{f}_{k-1}^{(2)}\mathsf{f}_{j-1}^{(\mu)}\mathsf{f}_{i-1}^{(\lambda)}\tau^{-1}$, which we can apply our inductive hypothesis to. Continuing, by induction our product can be written in the form $\tau w_q\tau^{-1}$ for $q \in \mathcal{P}^0_{2k}$, which by the factoring lemma is equal to $\tau w_{q'}\tau^{-1}\mathsf{f}_m^{(\nu)}$ for $q' \in \mathcal{P}^0_{2(k-1)}$ and $|\nu| + m = k + 2$. Then, as $\tau w_{q'}\tau^{-1} \subseteq \mathsf{Z}_k$, $w_{p'}\tau w_{q'}\tau^{-1} \in \mathsf{Z}_k$, and so we may apply our inductive hypothesis to express the entire product as the sum of basis words, w_r for $r \in \mathcal{P}^0_{2k}$. Thus, we can express $w_{p'}\tau w_{q'}\tau^{-1}\mathsf{f}_m^{(\nu)}$ as the sum or elements of the form $w_r\mathsf{f}_m^{(\nu)}$, with the aforementioned constraints.

Case 2 (c) $|\lambda| \neq k+1$, and $j = 1$: As $|\mu|+j = k+1$, it follows that $|\mu| = k$, and because w_p is a basis word, basis conditions (2) give that $|\lambda| = k$ and $i = 2$. Thus, $gw_p = \mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)}$. Then, Lemmas 3.22 and 3.42 complete our claim in this case, as all basis words are in the form described by the claim.

Applying our claim and Lemma 3.16 successively to any product gw_p for $g \in \mathsf{gen}(\mathsf{Z}_{k+1})$ and $p \in \mathcal{P}^0_{2(k+1)}$ produces a sum of basis words w_q for $q \in \mathcal{P}^0_{2(k+1)}$. □

3.3.3 Additional Lemmas

Here we cover the proofs of several smaller lemmas which are essential to the proof of Theorem 3.20, but whose proofs are too cumbersome to be included in the proof for the theorem.

Lemma 3.22. *Let* $\mathsf{G} = \mathsf{SU}_2$, $\mathbf{T}$, $\mathbf{O}$, *or* $\mathbf{I}$ *such that* $\frac{d_{(0)}}{d_{(1)}} = \frac{1}{2}$. *For* $\lambda \in \Lambda$ *such that* $(\lambda - 1) \notin \mathsf{br}(\mathsf{G})$, *and* $k = |\lambda|$, *the product*

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)}$$

can be expressed as the sum of basis words. Explicitly, for $k \geq 2$,

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} = \frac{d_{(0)}}{d_{(1)}}\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{f}_2^{(\lambda)} + \frac{d_{(0)}}{d_{(1)}}\frac{d_{(\lambda)}}{d_{(\lambda-1)}}\sum_{\nu=\lambda+1}\mathsf{f}^{(\nu)}, \tag{3.23}$$

$$= \frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{f}_2^{(\lambda)} + \frac{d_{(\lambda)}}{2d_{(\lambda-1)}}\sum_{\nu=\lambda+1}\mathsf{f}^{(\nu)}. \tag{3.24}$$

which is a linear combination of basis words.

Proof. For $k = 1$, this statement is trivial because $\mathsf{f}_i^{(1)} = \mathbf{1}$ for all $i \geq 1$ and $\mathsf{f}_1^{(2)}$ is a basis element. For $k \geq 2$ and $(\lambda - 1) \notin \mathsf{br}(\mathsf{G})$, the induction relation from Theorem 2.51 gives that

$$\mathsf{f}_1^{(\lambda)} = \mathsf{f}_1^{(\lambda-1)}(\mathbf{1} - \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1})\mathsf{f}_1^{(\lambda-1)},$$

and the definition of a shifted idempotent in Equation 2.49

$$\mathsf{f}_2^{(\lambda)} = \tau\ \mathsf{f}_1^{(\lambda)}\ \tau^{-1}.$$

Thus, using relations from Theorem 2.51,

$$\begin{aligned}
\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} &= \mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda-1)}\left(\mathbf{1}-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\right)\mathsf{f}_1^{(\lambda-1)}\tau\ \mathsf{f}_1^{(\lambda)}\tau^{-1} \\
&= \mathsf{f}_1^{(\lambda-1)}\mathsf{f}_k^{(2)}\left(\mathbf{1}-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\right)\mathsf{f}_1^{(\lambda-1)}\tau\ \mathsf{f}_1^{(\lambda)}\tau^{-1} && \text{by (FS6.1)} \\
&= \mathsf{f}_1^{(\lambda-1)}\mathsf{f}_k^{(2)}\left(\mathbf{1}-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\right)\mathsf{f}_1^{(\lambda-1)}\sigma_{k-1}\sigma_k\mathsf{f}_1^{(\lambda)}\tau^{-1} && \text{by (FS2)} \\
&= \mathsf{f}_1^{(\lambda-1)}\mathsf{f}_k^{(2)}\left(\mathbf{1}-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\right)\left(\mathsf{f}_1^{(\lambda)}-\mathsf{e}_k\mathsf{f}_1^{(\lambda)}+\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1}
\end{aligned}$$

Recalling that $\mathsf{f}_k^{(2)} = \mathbf{1} - \frac{d_{(0)}}{d_{(1)}}\mathsf{e}_k = \mathbf{1} - \frac{1}{2}\mathsf{e}_k$, we use this to expand the above product, giving

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} = \mathsf{f}_1^{(\lambda-1)}\left(\mathbf{1}-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}-\frac{1}{2}\mathsf{e}_k+\frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{e}_{k-1}\right)\left(\mathsf{f}_1^{(\lambda)}-\mathsf{e}_k\mathsf{f}_1^{(\lambda)}+\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1}. \quad (3.25)$$

We further expand the above product, ignoring terms which are zero by relations (FS2) and (FS6.2):

$$\begin{aligned}
\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} = \mathsf{f}_1^{(\lambda-1)}\Bigg(&\mathsf{f}_1^{(\lambda)}-\mathsf{e}_k\mathsf{f}_1^{(\lambda)}+\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}+\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&-\frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}+\mathsf{e}_k\mathsf{f}_1^{(\lambda)}-\frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}-\frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&+\frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{e}_{k-1}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\Bigg)\tau^{-1}. \quad (3.26)
\end{aligned}$$

While most terms in the above equation are in usable forms for our pursoses, we need to simply three of them. First, using relation (FS4):

$$\begin{aligned}
-\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{e}_{k-1}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} &= -\left(\frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\right)\left(\frac{d_{(\lambda-1)}}{d_{(\lambda-2)}}\right)\mathsf{f}_1^{(\lambda-2)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&= -\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}. \quad (3.27)
\end{aligned}$$

By the same relation, (FS3), and (TL2),

$$\frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{e}_{k-1}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} = \left(\frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\right)\left(\frac{d_{(\lambda-1)}}{d_{(\lambda-2)}}\right)\mathsf{e}_k\mathsf{f}_1^{(\lambda-2)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}$$
$$= \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}. \tag{3.28}$$

Finally, by relations (FS3) and (TL2),

$$-\frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} = -\frac{1}{2}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} = -\frac{1}{2}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}$$
$$= -\frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \tag{3.29}$$

Now, we return to Equation 3.26 with substitutions from Equation 3.27, Equation 3.28, and Equation 3.29, noting that $\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k = \mathsf{e}_k\mathsf{f}_1^{(\lambda-1)}$ by relation (FS3). Combining like terms and simplifying, we find that

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} = \left(\mathsf{f}_1^{(\lambda)} - \frac{1}{2}\left(\frac{d_{(\lambda-1)}+d_{(\lambda-2)}}{d_{(\lambda-1)}}\right)\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{f}^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1}$$
$$= \left(\frac{d_{(\lambda)}}{2d_{(\lambda-1)}}\mathsf{f}_1^{(\lambda)} - \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{f}^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1} \tag{3.30}$$
$$+ \left(\left(1 - \frac{d_{(\lambda)}}{2d_{(\lambda-1)}}\right)\mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\mathsf{f}^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1}$$

Using the fact that $2d_{(\lambda)} = d_{(\lambda+1)} + d_{(\lambda-1)}$, we further simplify to find that

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} = \frac{d_{(\lambda)}}{2d_{(\lambda-1)}}\left(\mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda)}}\mathsf{f}^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1}$$
$$+ \frac{d_{(\lambda-2)}}{2d_{(\lambda-1)}}\left(\mathsf{f}_1^{(\lambda)} - \mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \mathsf{f}^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1}.$$

Finally, Lemmas 3.31 and 3.36 complete our proof. □

The following lemmas assist in the proof of Lemma 3.22.

Lemma 3.31. *If* $\lambda \notin \mathsf{br}(\mathsf{G})$, *and* $k = |\lambda|$,

$$\sum_{\nu=\lambda+1}\mathsf{f}_1^{(\nu)} = \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda)}\mathsf{e}_k + \frac{d_{(\lambda-2)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \tag{3.32}$$

Proof. From the induction relation of Theorem 2.51,

$$\sum_{\nu=\lambda+1} f_1^{(\nu)} = \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}} \mathsf{f}_1^{(\lambda)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \tag{3.33}$$

$$= \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}} \left(\mathsf{f}_1^{(\lambda-1)} - \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_{k-1} \mathsf{f}_1^{(\lambda-1)} \right) \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \tag{3.34}$$

$$= \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda)}} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_{k-1} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \tag{3.35}$$

Then, $\mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k = \mathsf{e}_k \mathsf{f}_1^{(\lambda-1)}$, and $\mathsf{f}_1^{(\lambda-1)} \mathsf{f}_1^{(\lambda)} = \mathsf{f}_1^{(\lambda)}$, so out proof is complete □

Lemma 3.36. *For $\lambda \in \Lambda(\mathsf{G})$ and $k = |\lambda|$,*

$$\mathsf{f}_1^{(\lambda-1)} \mathsf{f}_2^{(\lambda)} = \left(\mathsf{f}_1^{(\lambda)} - \mathsf{e}_k \mathsf{f}_1^{(\lambda)} + \mathsf{f}^{(\lambda-1)} \mathsf{e}_{k-1} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \tau^{-1} \tag{3.37}$$

Proof. Recall that

$$\mathsf{f}_1^{(\lambda-1)} \mathsf{f}_2^{(\lambda)} = \mathsf{f}_1^{(\lambda-1)} \tau \mathsf{f}_1^{(\lambda)} \tau^{-1} \tag{3.38}$$

$$= \mathsf{f}_1^{(\lambda-1)} \sigma_1 \cdots \sigma_{k-1} \sigma_k \mathsf{f}_1^{(\lambda)} \tau^{-1} \tag{3.39}$$

by relation (FS2) from Theorem 2.51,

$$\mathsf{f}_1^{(\lambda-1)} \mathsf{f}_2^{(\lambda)} = \mathsf{f}_1^{(\lambda-1)} \sigma_{k-1} \sigma_k \mathsf{f}_1^{(\lambda)} \tau^{-1} \tag{3.40}$$

$$= \left(\mathsf{f}_1^{(\lambda)} - \mathsf{e}_k \mathsf{f}_1^{(\lambda)} + \mathsf{f}^{(\lambda-1)} \mathsf{e}_{k-1} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \omega^{-1} \tag{3.41}$$

□

We now show a result analogous to Lemma 3.22 for the branch idempotents described in Section 2.2.2. For $\mathsf{G} = \mathbf{T}, \mathbf{O}$, and $\mathbf{I}$, and distinct $\lambda, \mu \in \Lambda(\mathsf{G})$ such that $|\lambda| = |\mu| = |\beta| + 1$,

$$\mathsf{f}_1^{(\lambda)} + \mathsf{f}_1^{(\mu)} = \mathsf{f}_1^{(\beta)} \left(\mathbf{1} - \frac{d_{(\beta-1)}}{d_{(\beta)}} \mathsf{e}_{|\beta|} \right) \mathsf{f}_1^{(\beta)}.$$

Thus the products $\mathsf{f}_{|\lambda|}^{(2)} \mathsf{f}_1^{(\lambda)} \mathsf{f}_2^{(\lambda)}$, $\mathsf{f}_{|\lambda|}^{(2)} \mathsf{f}_1^{(\lambda)} \mathsf{f}_2^{(\mu)}$, $\mathsf{f}_{|\lambda|}^{(2)} \mathsf{f}_1^{(\mu)} \mathsf{f}_2^{(\lambda)}$, and $\mathsf{f}_{|\lambda|}^{(2)} \mathsf{f}_1^{(\mu)} \mathsf{f}_2^{(\mu)}$ cannot be expressed as the sum of elements $\{w_p \mid p \in \mathcal{P}^0_{2(|\lambda|+1)}\}$. Thus, we must take a different approach to find a similar result.

Lemma 3.42. *For $\mathsf{G} = \mathbf{T}$, $\mathbf{O}$, and $\mathbf{I}$, and for distinct $\lambda, \mu \in \Lambda(\mathsf{G})$ such that $|\lambda| = |\mu| = |\beta| + 1$, the products*

1. $\mathsf{f}_{|\lambda|}^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)}$,

2. $\mathsf{f}_{|\lambda|}^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\mu)}$,

3. $\mathsf{f}_{|\lambda|}^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)}$, *and*

4. $\mathsf{f}_{|\lambda|}^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\mu)}$

can be expressed as the sum of words $\{w_p \mid p \in \mathcal{P}^0_{2(|\lambda|+1)}\}$.

Example 3.43. Explicitly, the results of Lemma 3.42 are as follows.

T: In this case $\beta = 2$, and if $\lambda = 3$ and $\mu = 3'$ the expressions are

$$\begin{aligned}
\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)} &= \frac{1}{3}\mathsf{f}_1^{(4)} - \frac{1}{3}\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3)} + \mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)} + \frac{1}{2}\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3)}\\
\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3')} &= -\frac{2}{3}\mathsf{f}_1^{(4')} + \frac{2}{3}\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3')}\\
\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3')} &= -\frac{2}{3}\mathsf{f}_1^{(4)} + \frac{2}{3}\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3)}\\
\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3)} &= \frac{1}{3}\mathsf{f}_1^{(4')} - \frac{1}{3}\mathsf{f}_1^{(2)}\mathsf{f}_2^{(3')} + \mathsf{f}_1^{(3')}\mathsf{f}_2^{(3')} + \frac{1}{2}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3')}
\end{aligned}$$

O: In this case $\beta = 3$, and if $\lambda = 4$ and $\mu = 4'$ the expressions are

$$\begin{aligned}
\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4)} &= \frac{3}{8}\mathsf{f}_1^{(5)} - \frac{3}{8}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(4)} + \mathsf{f}_1^{(4)}\mathsf{f}_2^{(4)} + \frac{1}{2}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4)}\\
\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4')} &= \frac{9}{8}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(4')} - \frac{1}{2}\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4')} - \frac{1}{4}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4')}\\
\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4)} &= -\frac{3}{5}\mathsf{f}_1^{(5)} + \frac{3}{8}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(4)}\\
\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4')} &= -\frac{3}{4}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(4'} + \mathsf{f}_1^{(4)}\mathsf{f}_2^{(4')} + \frac{3}{2}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4')}
\end{aligned}$$

I: In this case $\beta = 5$, and if $\lambda = 6$ and $\mu = 6'$the expressions are

$$\begin{aligned}
\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6)} &= \frac{2}{6}\mathsf{f}_1^{(7)} - \frac{5}{6}\mathsf{f}_1^{(5)}\mathsf{f}_2^{(6)} + \frac{3}{2}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6)} + \mathsf{f}_1^{(6')}\mathsf{f}_2^{(6)}\\
\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6')} &= \frac{5}{6}\mathsf{f}_1^{(5)}\mathsf{f}_2^{(6')} - \mathsf{f}_1^{(6)}\mathsf{f}_2^{(6')} + \frac{3}{4}\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6')}\\
\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6)} &= -\frac{3}{6}\mathsf{f}_1^{(7)} + \frac{5}{4}\mathsf{f}_1^{(5)}\mathsf{f}_2^{(6)} - \frac{3}{4}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6)} - \frac{1}{2}\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6)}\\
\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6')} &= -\frac{5}{4}\mathsf{f}_1^{(5)}\mathsf{f}_2^{(6')} + \frac{3}{2}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6')} + 2\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6')}
\end{aligned}$$

Expressions which capture each of these formulas in terms of generics terms such as $d_{(\lambda)}$, $d_{(\mu)}$, and $d_{(\beta)}$ were not immediately apparent, and the proof must be completed on a case-by-case basis.

We now prove Lemma 3.43.

Proof. Recall that $\lambda, \mu = (\beta + 1)$ with $\lambda \neq \mu$, and define $k = |\lambda| = |\mu|$. We prove the lemma in two cases.

Case 1: We show the result for expressions 2 and 3 in the statement of Lemma 3.43,

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\mu)} \quad \text{and} \quad \mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)},$$

for fixed λ and μ. These expressions are symmetric with regard to the interchange of λ and μ, and so we proceed without specifying values (such as "λ labels the module with higher dimension") for as long as possible to keep our proof general among groups and possible values of λ. Thus, without loss of generality, we examine the expression $\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)}$.

First, recall from the induction relation of Theorem 2.51 that for any of the groups included in our statement,

$$\mathsf{f}_k^{(2)} = \mathbf{1} - \frac{1}{2}\mathsf{e}_k. \tag{3.44}$$

Proceeding with this expansion,

$$\begin{aligned} \mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)} &= \left(\mathbf{1} - \frac{1}{2}\mathsf{e}_k\right)\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)} \\ &= \mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)} - \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)}. \end{aligned}$$

Noting that $\mathsf{f}_2^{(\lambda)} = \tau\ \mathsf{f}_1^{(\lambda)}\tau{-}1$, we can further expand the above expression using relations from Theorem 2.51 as follows:

$$\begin{aligned} \mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)} &= \left(\mathsf{f}_1^{(\mu)}\tau\mathsf{f}_1^{(\lambda)} - \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\mu)}\tau\mathsf{f}_1^{(\lambda)}\right)\tau^{-1} \\ &= \left(\mathsf{f}_1^{(\mu)}\sigma_k\mathsf{f}_1^{(\lambda)} - \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\mu)}\sigma_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1} && \text{by (FS2)} \\ &= \left(-\mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1} && \text{by (FS6.2)} \\ &= \left(-\mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\mu)}}{2d_{(\lambda-1)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1} && \text{by (FS4)} \\ &= \left(-\mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\mu)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau^{-1} && \text{by (FS2) and (FS6.1)} \end{aligned}$$

Recall that the induction relation gives that

$$\mathsf{f}_1^{(\lambda)} + \mathsf{f}_1^{(\mu)} = \mathsf{f}_1^{(\lambda-1)} \left(\mathbf{1} - \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}} \mathsf{e}_{k-1} \right) \mathsf{f}_1^{(\lambda-1)},$$

and so as $\mathsf{f}_1^{(\lambda-1)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_1^{(\lambda-1)} = \mathsf{f}_1^{(\lambda)}$ by relation (FS6.1),

$$\mathsf{f}_1^{(\mu)} = \mathsf{f}_1^{(\lambda-1)} \left(\mathbf{1} - \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}} \mathsf{e}_{k-1} - \mathsf{f}_1^{(\lambda)} \right) \mathsf{f}_1^{(\lambda-1)}. \tag{3.45}$$

Thus for some scalar a

$$\begin{aligned}
\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)} &= \Bigg(\frac{d_{(\mu)}}{2d_{(\lambda-1)}} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} - (1-a) \left(\mathsf{f}_1^{(\mu)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \\
&\qquad -(a) \left(\mathsf{f}_1^{(\lambda-1)} \left(\mathbf{1} - \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}} \mathsf{e}_{k-1} - \mathsf{f}_1^{(\lambda)} \right) \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \Bigg) \tau^{-1} \\
&= \Bigg(\frac{d_{(\mu)}}{2d_{(\lambda-1)}} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} - (1-a) \left(\mathsf{f}_1^{(\mu)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \\
&\qquad + (a) \Bigg(-\mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k \mathsf{f}^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_{k-1} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \\
&\qquad\qquad + \mathsf{f}_1^{(\lambda-1)} \mathsf{f}_1^{(\lambda)} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \Bigg) \Bigg) \tau^{-1}.
\end{aligned}$$

Applying relations (FS3) and (FS6.1),

$$\begin{aligned}
\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\lambda)} = \Bigg(&\frac{d_{(\mu)}}{2d_{(\lambda-1)}} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} + (1-a) \left(-\mathsf{f}_1^{(\mu)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \\
&+(a) \left(-\mathsf{e}_k \mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}} \mathsf{f}_1^{(\lambda-1)} \mathsf{e}_{k-1} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} + \mathsf{f}_1^{(\lambda)} \mathsf{e}_k \mathsf{f}_1^{(\lambda)} \right) \Bigg) \tau^{-1}.
\end{aligned} \tag{3.46}$$

From here, we must proceed based on the dimensions for each group.

For $\mathsf{G} = \mathbf{T}$, setting $a = 0$ and $\lambda = 3$ in Equation 3.46 becomes

$$\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3)} = \left(\frac{2}{6} \mathsf{e}_3 \mathsf{f}_1^{(3)} - \mathsf{f}_1^{(3')} \mathsf{e}_3 \mathsf{f}_1^{(3)} \right) \tau^{-1} \tag{3.47}$$

and the same value of a and $\lambda = 3'$ gives

$$\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3')} = \left(\frac{2}{6} \mathsf{e}_3 \mathsf{f}_1^{(3')} - \mathsf{f}_1^{(3)} \mathsf{e}_3 \mathsf{f}_1^{(3')} \right) \tau^{-1}. \tag{3.48}$$

From the above equations, it is straightforward to confirm the results given in Example 3.43, using Lemma 3.60 and Lemma 3.36.

For $\mathsf{G} = \mathbf{O}$, setting $a = 0$ and $\lambda = 4$ if Equation 3.46 becomes

$$\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4)} = \left(\frac{2}{8}\mathsf{e}_4\mathsf{f}_1^{(4)} - \mathsf{f}_1^{(4')}\mathsf{e}_4\mathsf{f}_1^{(4)}\right)\tau^{-1} \tag{3.49}$$

while setting $a = \frac{1}{4}$ and $\lambda = 4'$ gives

$$\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4')} = \left(-\frac{1}{8}\mathsf{e}_4\mathsf{f}_1^{(4')} - \frac{3}{4}\mathsf{f}_1^{(4)}\mathsf{e}_4\mathsf{f}_1^{(4')} + \frac{3}{16}\mathsf{f}_1^{(3)}\mathsf{e}_3\mathsf{e}_4\mathsf{f}_1^{(4')} + \frac{1}{4}\mathsf{f}_1^{(4')}\mathsf{e}_4\mathsf{f}_1^{(4')}\right)\tau^{-1}. \tag{3.50}$$

From the above equations, it is straightforward to confirm the results given in Example 3.43, using Lemma 3.60 for the first equation and Lemma 3.36 for both.

For $\mathsf{G} = \mathbf{I}$, setting $a = \frac{3}{4}$ and $\lambda = 6$ Equation 3.46 becomes

$$\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6)} = \left(-\frac{6}{12}\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{1}{4}\mathsf{f}_1^{(6')}\mathsf{e}_6\mathsf{f}_1^{(6)} + \frac{5}{8}\mathsf{f}_1^{(5)}\mathsf{e}_5\mathsf{e}_6\mathsf{f}_1^{(6)} + \frac{3}{4}\mathsf{f}_1^{(6)}\mathsf{e}_6\mathsf{f}_1^{(6)}\right)\tau^{-1} \tag{3.51}$$

while setting $a = -\frac{3}{4}$ and $\lambda = 6'$ gives

$$\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6')} = \left(\frac{13}{12}\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{7}{4}\mathsf{f}_1^{(6')}\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{5}{8}\mathsf{f}_1^{(5)}\mathsf{e}_5\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{3}{4}\mathsf{f}_1^{(6)}\mathsf{e}_6\mathsf{f}_1^{(6)}\right)\tau^{-1} \tag{3.52}$$

From the above equations, it is straightforward to confirm the results given in Example 3.43, using Lemma 3.60 for the first equation and Lemma 3.36 for both.

Case 2: We show the result for expressions 1 and 4 in the statement of Lemma 3.42,

$$\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} \quad \text{and} \quad \mathsf{f}_k^{(2)}\mathsf{f}_1^{(\mu)}\mathsf{f}_2^{(\mu)},$$

for fixed λ and μ. Again, due to the symmetry of λ and μ, we proceed without loss of generality to examine only $\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)}$.

We proceed in parallel to the first case. Recalling Equation 3.44,

$$\begin{aligned}\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} &= \mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} - \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} \\ &= \left(\mathsf{f}_1^{(\lambda)} - \mathsf{f}_1^{(\lambda)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} - \frac{1}{2}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\tau{-}1.\end{aligned}$$

Then, we can apply Equation 3.45, so that for some scalar a,

$$\begin{aligned}\mathsf{f}_k^{(2)}\mathsf{f}_1^{(\lambda)}\mathsf{f}_2^{(\lambda)} = \Bigg(&\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda)} - d_{(\lambda-1)}}{2d_{(\lambda-1)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} - (a)\mathsf{f}_1^{(\lambda)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\ &+(1-a)\left(-\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda-1)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}\right)\Bigg)\tau^{-1} \quad (3.53)\end{aligned}$$

From here, we must proceed based on the dimensions for each group.

For $\mathsf{G} = \mathbf{T}$, setting $a = 1$ and $\lambda = 3$ in Equation 3.53 becomes

$$\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3)}\mathsf{f}_2^{(3)} = \left(\mathsf{f}_1^{(3)} - \frac{1}{6}\mathsf{e}_3\mathsf{f}_1^{(3)} - \mathsf{f}_1^{(3)}\mathsf{e}_3\mathsf{f}_1^{(3)}\right)\tau^{-1} \quad (3.54)$$

and the same value of a and $\lambda = 3'$ gives

$$\mathsf{f}_3^{(2)}\mathsf{f}_1^{(3')}\mathsf{f}_2^{(3')} = \left(\mathsf{f}_1^{(3')} - \frac{1}{6}\mathsf{e}_3\mathsf{f}_1^{(3')} - \mathsf{f}_1^{(3')}\mathsf{e}_3\mathsf{f}_1^{(3')}\right)\tau^{-1}. \quad (3.55)$$

From the above equations, it is straightforward to confirm the results given in Example 3.43, using Lemma 3.60 and Lemma 3.36.

For $\mathsf{G} = \mathbf{O}$, setting $a = 1$ and $\lambda = 4$ if Equation 3.53 becomes

$$\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4)}\mathsf{f}_2^{(4)} = \left(\mathsf{f}_1^{(4)} + \frac{1}{8}\mathsf{e}_4\mathsf{f}_1^{(4)} - \mathsf{f}_1^{(4)}\mathsf{e}_4\mathsf{f}_1^{(4)}\right)\tau^{-1} \quad (3.56)$$

while setting $a = \frac{3}{2}$ and $\lambda = 4'$ gives

$$\begin{aligned}\mathsf{f}_4^{(2)}\mathsf{f}_1^{(4')}\mathsf{f}_2^{(4')} = \Bigg(&\mathsf{f}_1^{(4')} + \frac{1}{4}\mathsf{e}_4\mathsf{f}_1^{(4')} - \frac{3}{8}\mathsf{f}_1^{(3)}\mathsf{e}_3\mathsf{e}_4\mathsf{f}_1^{(4')} - \frac{3}{2}\mathsf{f}_1^{(4')} \\ &-\frac{1}{2}\mathsf{f}_1^{(4)}\mathsf{e}_4\mathsf{f}_1^{(4')}\Bigg)\tau^{-1}. \quad (3.57)\end{aligned}$$

From the above equations, it is straightforward to confirm the results given in Example 3.43, using Lemma 3.60 and Lemma 3.36.

For $\mathsf{G} = \mathbf{I}$, setting $a = \frac{3}{4}$ and $\lambda = 6$ Equation 3.53 becomes

$$\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6)}\mathsf{f}_2^{(6)} = \Big(\mathsf{f}_1^{(6)} - \frac{4}{12}\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{5}{12}\mathsf{f}_1^{(5)}\mathsf{e}_5\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{3}{2}\mathsf{f}_1^{(6)}\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{1}{2}\mathsf{f}_1^{(6')}\mathsf{e}_6\mathsf{f}_1^{(6)}\Big)\tau^{-1} \quad (3.58)$$

while setting $a = -\frac{3}{4}$ and $\lambda = 6'$ gives

$$\mathsf{f}_6^{(2)}\mathsf{f}_1^{(6')}\mathsf{f}_2^{(6')} = \Big(\mathsf{f}_1^{(6)} - \frac{9}{12}\mathsf{e}_6\mathsf{f}_1^{(6)} - \frac{10}{12}\mathsf{f}_1^{(5)}\mathsf{e}_5\mathsf{e}_6\mathsf{f}_1^{(6)} - 2\mathsf{f}_1^{(6)}\mathsf{e}_6\mathsf{f}_1^{(6)} - \mathsf{f}_1^{(6')}\mathsf{e}_6\mathsf{f}_1^{(6)}\Big)\tau^{-1} \quad (3.59)$$

From the above equations, it is straightforward to confirm the results given in Example 3.43, using Lemma 3.60 and Lemma 3.36.

□

Finally, we prove a lemma which is integral to the above proof.

Lemma 3.60. *For* $\mathsf{G} = \mathbf{T}$, $\mathbf{O}$, *and* $\mathbf{I}$, *and for distinct* $\lambda, \mu \in \Lambda(\mathsf{G})$ *such that* $|\lambda| = |\mu| = |\beta| + 1$. *Then,*

$$\sum_{\mu=\beta+1} \mathsf{f}_1^{(\mu)} = \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_1^{(\mu)}\mathsf{e}_{|\lambda|}\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda)}}\mathsf{f}^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}. \quad (3.61)$$

Proof. From the induction relation of Theorem 2.51,

$$\begin{aligned}
\sum_{\nu=\lambda+1} f_1^{(\nu)} &= \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&= \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\left(\mathsf{f}_1^{(\lambda-1)} - \frac{d_{(\lambda-2)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{f}_1^{(\lambda-1)} - \mathsf{f}_1^{(\mu)}\right)\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&= \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} \\
&\quad + \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)}
\end{aligned}$$

Then, $\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_k = \mathsf{e}_k\mathsf{f}_1^{(\lambda-1)}$, $\mathsf{f}_1^{(\lambda-1)}\mathsf{f}_1^{(\lambda)} = \mathsf{f}_1^{(\lambda)}$, and further, as $\mathsf{f}_1^{(\mu)}\mathsf{f}_1^{(\lambda)} = 0$, it follows that $\mathsf{f}_1^{(\mu)}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} = -\mathsf{f}_1^{(\mu)}\sigma_k\mathsf{f}_1^{(\lambda)}$. Then, by relation (FS2) from Theorem 2.51, $\mathsf{f}_1^{(\mu)}\sigma_k\mathsf{f}_1^{(\lambda)} = \mathsf{f}_1^{(\mu)}\tau\mathsf{f}_1^{(\lambda)}$, and so

$$\sum_{\nu=\lambda+1} f_1^{(\nu)} = \mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} + \frac{d_{(\lambda-2)}}{d_{(\lambda)}}\mathsf{f}_1^{(\lambda-1)}\mathsf{e}_{k-1}\mathsf{e}_k\mathsf{f}_1^{(\lambda)} - \frac{d_{(\lambda-1)}}{d_{(\lambda)}}\mathsf{f}_1^{(\mu)}\tau\mathsf{f}_1^{(\lambda)}. \quad (3.62)$$

The above equation is imply another form of Equation 3.61, so our proof is complete. □

Chapter 4

Alternate Bases for $\mathsf{Z}_k(\mathsf{G})$.

In the course of investigation, we encountered another basis for $\mathsf{Z}_k(\mathsf{G})$, which we conjecture would work for every finite subgroup, rather that just SU_2 and the exceptional subgroups. Further, when we compared this basis with the one presented in the previous chapter, the change of basis matrix was triangular, hinting at a deeper connection. In this chapter, we will describe this basis, and briefly explain the (lengthy) proof which we found for all subgroups but $\mathbf{C}_n$ and $\mathbf{D}_n$ for large values of k.

4.1 Infinite Cyclic and Binary Dihedral Subgroups

In this section we introduce two subgroups of SU_2 that are infinite analogues of the cyclic and binary dihedral subgroups, which play an important roles in elucidating certain aspects of the basis given in this chapter. Let $\mathbf{C}_\infty$ be the free group with presentation $\langle g, g^{-1} \mid gg^{-1} = 1\rangle$, and let $\mathbf{D}_\infty$ be the group presented by $\langle g, h \mid h^{-1}gh^{-1} = g^{-1}\rangle$. All content in this section is taken from [BBH].

We can use a restriction of the defining representation for SU_2 to construct $\mathbf{C}_\infty$- and $\mathbf{D}_\infty$- module structures on $\mathsf{V} = \mathbb{C}^2$, which we describe here. Let ω be a complex unit such that there is no integral solution to the equation $\omega^i = 1$ for i. The generator of $\mathbf{C}_\infty$ has a matrix with respect to our standard basis $\{\mathsf{v}_{-1}, \mathsf{v}_{+1}\}$ given by

$$[g] = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}.$$

The generators of $\mathbf{D}_\infty$ have matrices with respect to the same standard basis given by

$$[g] = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \text{and} \quad [h] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

4.1.1 Irreducible Modules

The infinite cyclic subgroup $\mathbf{C}_\infty$ has been shown to have only 1-dimensional irreducible modules indexed by $\mathbb{Z}$, where $\mathbf{C}_\infty^{(i)}$ is defined by the action $g \cdot \mathsf{v}_\emptyset = \omega^i \mathsf{v}_\emptyset$. Thus, $\mathsf{V} = \mathbf{C}_\infty^{(1)} \oplus \mathbf{C}_\infty^{(-1)}$. Further, tensoring by V is encoded in a manner similar to other cyclic groups:

$$\mathsf{V} \otimes \mathbf{C}_\infty^{(i)} = \mathbf{C}_\infty^{(i-1)} \oplus \mathbf{C}_\infty^{(i+1)}.$$

The infinite binary dihedral group $\mathbf{D}_\infty$ has irreducible submodules indexed by $0, 0', 1, 2, \ldots$, where 0 and $0'$ are 1-dimensional, and the remaining are two dimensional. Further, $\mathsf{V} \cong \mathbf{D}_\infty^{(1)}$, and tensoring by V is encoded as

$$\mathsf{V} \otimes \mathbf{D}_\infty^{(i)} = \begin{cases} \mathbf{D}_\infty^{(i-1)} \oplus \mathbf{D}_\infty^{(i+1)} & \text{if } i = 2, 3, \ldots, \\ \mathbf{D}_\infty^{(0)} \oplus \mathbf{D}_\infty^{(0')} \oplus \mathbf{D}_\infty^{(2)} & \text{if } i = 1, \\ \mathbf{D}_\infty^{(1)} & \text{if } i = 0, 0'. \end{cases}$$

4.1.2 Representation Graphs and Bratteli Diagrams

We can define representation graphs and Bratteli diagrams for $\mathbf{C}_\infty$ and $\mathbf{D}_\infty$ as we did for other groups. Both graphs share the property of connectedness and simplicity, like the finite counterparts, and an unsurprising similarity to the corresponding graphs for $\mathbf{C}_n$ and $\mathbf{D}_n$

4.1.3 Centralizer Algebra

Define the centralizer algebras $\mathsf{Z}_k(\mathbf{C}_\infty)$ and $\mathsf{Z}_k(\mathbf{D}_\infty)$ as would be expected, that is

$$\mathsf{Z}_k(\mathbf{C}_\infty) = \mathsf{End}_{\mathbf{C}_\infty}(\mathsf{V}^{\otimes k}),$$
$$\mathsf{Z}_k(\mathbf{D}_\infty) = \mathsf{End}_{\mathbf{C}_\infty}(\mathsf{V}^{\otimes k}).$$

Then, [BBH] shows that

$$\begin{aligned} \mathsf{Z}_k(\mathbf{C}_\infty) &\cong \mathsf{Z}_k(\mathbf{C}_n), && \text{For } \tilde{n} > k \\ \mathsf{Z}_k(\mathbf{D}_\infty) &\cong \mathsf{Z}_k(\mathbf{D}_n), && \text{For } n > k. \end{aligned}$$

This isomorphism gives cause to study these infinite subgroups. As has been stated earlier, this chapter provides a (provable) basis for $\mathsf{Z}_k(\mathbf{C}_n)$ with $k < n$ and for $\mathsf{Z}_k(\mathbf{C}_n)$ with $k < \tilde{n}$, and so provides a basis for all centralizers $\mathsf{Z}_k(\mathbf{C}_\infty)$ and $\mathsf{Z}_k(\mathbf{D}_\infty)$.

4.2 Basis

Here, we describe an conjecture which arose during research but we were unable to prove. This conjecture describes a basis for the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$ which we found to hold for all finite subgroups G of SU_2.

Before we begin, however, we must digress. It is known form [BBH] that the Bratteli diagrams $\mathcal{B}_\mathsf{V}(\mathbf{C}_n)$ and $\mathcal{B}_\mathsf{V}(\mathbf{C}_m)$ are isomorphic if $\tilde{n} = \tilde{m}$, which is to say that either $n = m$, or if (without loss of generality) $n > m$, $\frac{n}{2} = m$. In fact, [BBH] shows that $\mathsf{Z}_k(\mathbf{C}_n) \cong \mathsf{Z}_k(\mathbf{C}_m)$ if $\tilde{n} = \tilde{m}$, and so in this section we will describe a basis algorithm for $\mathsf{Z}_k(\mathbf{C}_n)$ with $n = 2\tilde{n}$, which naturally extends to values of n.

For $\beta = \min(\mathsf{br}(\mathsf{G}))$, i.e., the first branch vertex, define

$$\mathsf{b}_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\text{ factors}} \otimes \mathsf{f}_{(\beta+1)'} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-\ell)\text{ factors}}, \qquad 1 \le i \le k-\ell, \qquad \text{for each } \mathsf{G},$$

noting that $(\beta+1)'$ refers to the primed vertex following β in the canonical listing of vertices in $\mathcal{R}_\mathsf{V}(\mathsf{G})$. Further, define

$$\mathsf{c}_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\text{ factors}} \otimes \mathsf{f}_{(n)'} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-n+1)\text{ factors}}, \qquad 1 \le i \le k-n+1, \quad \text{for } \mathsf{G} = \mathbf{D}_n,$$

noting the same priming as abov, and finally, define

$$\mathsf{c}_i^+ = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\text{ factors}} \otimes \mathsf{E}_-^+ \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-\tilde{n}+1)\text{ factors}}, \qquad 1 \le i \le k-n+1, \quad \text{for } \mathsf{G} = \mathbf{C}_n,$$

and

$$\mathsf{c}_i^- = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\text{ factors}} \otimes \mathsf{E}_+^- \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-\tilde{n}+1)\text{ factors}}, \qquad 1 \le i \le k-n+1, \quad \text{for } \mathsf{G} = \mathbf{C}_n,$$

with E_-^+ and E_+^- defined as they were in Theorem 2.11.

We again use the definition of paths and peaks given in Section 3.1, with several additions.

In a path

$$p = (0 = p_0, p_1, \ldots, p_{2k-1}, p_{2k} = 0)$$

on $\mathcal{B}_{\mathsf{V}}(\mathbf{C}_n)$, with $n = 2\tilde{n}$, let $|p_i|$ be the minimal distance from p_i to 0, and define a peak as any element p_i for which $|p_{i-1}| < |p_i|$ and $|p_i| > |p_{i+1}|$.

A peak is *nonstandard* if it is the first peak in our path after a step from β to $(\beta+1)'$. A peak is *standard* if it is not nonstandard.

In our alternate algorithm, we associate to each peak p_i a product of generators $\mathcal{B}(p_i)$ called *block* as follows. Define

$$x = \frac{i + |p_i| - 2}{2} \text{ and } y = \frac{i - |p_i| + 2}{2}. \tag{4.1}$$

For $p_i \neq n$ if $\mathsf{G} = \mathbf{D}_n$, and $p_i \neq \tilde{n}$ if $\mathsf{G} = \mathbf{C}_n$, define

$$\mathcal{B}(p_i) = \begin{cases} \mathbf{1} & \text{if } |p_i| = 1, \\ \mathsf{e}_x \mathsf{e}_{x-1} \cdots \mathsf{e}_y & \text{if } |p_i| > 1 \end{cases} \tag{4.2}$$

if p_i is standard, and

$$\mathcal{B}(p_i) = \begin{cases} \mathsf{b}_y & \text{if } |p_i| = \beta + 1, \\ \mathsf{b}_y \mathsf{e}_x \mathsf{e}_{x-1} \cdots \mathsf{e}_y & \text{if } |p_i| > \beta + 1. \end{cases} \tag{4.3}$$

if p_i nonstandard. Observe that in the above cases, when a block has factors e_i, the number of such factors is $x - y + 1 = |p_i| - 1$, which is independent of the level i in which our peak appears. We now define the two edge cases of our peaks. If $\mathsf{G} = \mathbf{D}_n$ and $p_i = n'$,

$$\mathcal{B}(p_i) = \mathsf{c}_y, \tag{4.4}$$

and if $\mathsf{G} = \mathbf{C}_n$ and $p_i = \tilde{n}$,

$$\mathcal{B}(p_i) = \begin{cases} \mathsf{c}_i^+ & \text{if } p_{i-1} = (n-1) \text{ and } p_{i+1} = (n-1)', \\ \mathsf{c}_i^- & \text{if } p_{i-1} = (n-1)' \text{ and } p_{i+1} = (n-1), \\ \mathsf{e}_x \mathsf{e}_{x-1} \cdots \mathsf{e}_y & \text{otherwise, if } p_i \text{ is standard} \\ \mathsf{b}_y \mathsf{e}_x \mathsf{e}_{x-1} \cdots \mathsf{e}_y & \text{otherwise, if } p_i \text{ is nonstandard} \end{cases} \tag{4.5}$$

For a path in the Bratteli diagram $p \in \mathcal{P}_{2k}^0(\mathsf{G})$, define the *word* w_p of p as the product blocks for each peak in p:

$$w_p = \mathcal{B}(p_{i_1})\mathcal{B}(p_{i_2}) \cdots \mathcal{B}(p_{i_\ell}). \tag{4.6}$$

Conjecture 4.7. *For* $\mathsf{G} = \mathsf{SU}_2, \mathbf{C}_n, \mathbf{D}_n, \mathbf{C}_\infty, \mathbf{D}_\infty, \mathbf{T}, \mathbf{O}$*, or* $\mathbf{I}$*, and* $k \geq 0$*, the set* $\{ w_p \mid p \in \mathcal{P}^0_{2k}(\mathsf{G}) \}$ *as define in Equation* 4.6 *is a basis for* $\mathsf{Z}_k(\mathsf{G})$*.*

While we are not able to prove the above conjecture, computational results show that it holds up to $k = 10$ for each group, and we found a very lengthy proof for all but $\mathsf{Z}_k(\mathbf{C}_n)$ for $k \geq \tilde{n}$ and $\mathsf{Z}_k(\mathbf{D}_n)$ for $k \geq n$. Further, we discovered that a triangular relation seems to exist between the basis described in this chapter an the basis which forms the main result of this thesis.

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