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Application of Resurgence Theory to Approximate Inverse Square Potential in Quantum Mechanics

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Abstract

Previous work concluded that the leading term of the energy spectrum of a quantum particle moving in an inverse-square law potential with regulated singularity is non-perturvative in nature. Trans-series and resurgence theory predicts the emergence of perturbative, and potentially logarithmic, corrections to the leading behavior. This talk relates an attempt to systematically calculate such corrections for the first time.

Keywords

Resurgence Theory, Quantum Mechanics, Inverse Square Potential

Cover Page Footnote

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1 Introduction

1.1 Conformal Quantum Mechanics

In Quantum Mechanics, the Schrödinger Equation is the fundamental equation that describes the time-evolution of a physical system in one dimension.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V\Psi(x,t). \tag{1}$$

It can be solved by applying the method of separation of variables. Assume that the wavefunctions can be factored in a time-dependent and space dependent part as

$$\Psi(x,t) = \psi(x)\phi(t), \tag{2}$$

the time-independent Schrödinger Equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{3}$$

When $E > V_{x\to\pm\infty}$, the particle has continuous scattering states. On the other hand, when $E < V_{x\to\pm\infty}$, the particle in general has discrete energy spectrum consisting of bound states. A good example of bound states is harmonic oscillator. For a harmonic oscillator with potential $V = \frac{1}{2}m\omega^2$, the discrete bound state energies are

$$E_n = \hbar\omega(n + \frac{1}{2}). \tag{4}$$

Theoretically, if the expression for potential energy V is known, all energy states of the particle and the corresponding eigenfunctions can be obtained by solving Schrödinger Equation. In real life, however, solving the Schrödinger Equation analytically is impossible for all but the simplest potentials. The energy spectrum can only be obtained analytically for a few types of potentials, such as infinite square well, finite square well, and harmonic oscillator. Therefore, it is often necessary to take advantage of approximation techniques like perturbation method.

The inverse square potential $-\frac{\alpha}{x^2}$ is special because it implies conformal symmetry. It is helpful to first illustrate what conformal symmetry means. Assume the expression for potential energy of the particle is $V(x) = -(\frac{1}{4}+\lambda)\frac{1}{x^2}$ with $\alpha = \frac{1}{4}+\lambda$ and $\lambda > 0$. The reason why coefficient $-\alpha$ has to be less than $-\frac{1}{4}$ will be explained after the example. Then the time-independent Schrödinger Equation becomes

$$-\frac{d^2\psi(x)}{dx^2} - (\frac{1}{4}+\lambda)\frac{1}{x^2}\psi(x) = E\psi(x), \tag{5}$$

where $\frac{\hbar^2}{2m}$ is set to one. To make the conformal symmetry manifest, substitute $x' = \beta x$, the equation turns out to be

$$-\beta^2\frac{d^2}{dx'^2}\psi(\frac{x'}{\beta}) - (\frac{1}{4}+\lambda)\frac{\beta^2}{x'^2}\psi(\frac{x'}{\beta}) = E\psi(\frac{x'}{\beta}) \tag{6}$$

$$-\frac{d^2}{dx'^2}\psi(\frac{x'}{\beta}) - (\frac{1}{4}+\lambda)\frac{1}{x'^2}\psi(\frac{x'}{\beta}) = \frac{E}{\beta^2}\psi(\frac{x'}{\beta}). \tag{7}$$

Let $\psi'(x') \equiv \psi(\frac{x'}{\beta})$, $E' = \frac{E}{\beta^2}$, the Schrödinger Equation becomes

$$-\frac{d^2}{dx'^2}\psi'(x') - (\frac{1}{4}+\lambda)\frac{1}{x'^2}\psi'(x') = E'\psi'(x'). \tag{8}$$

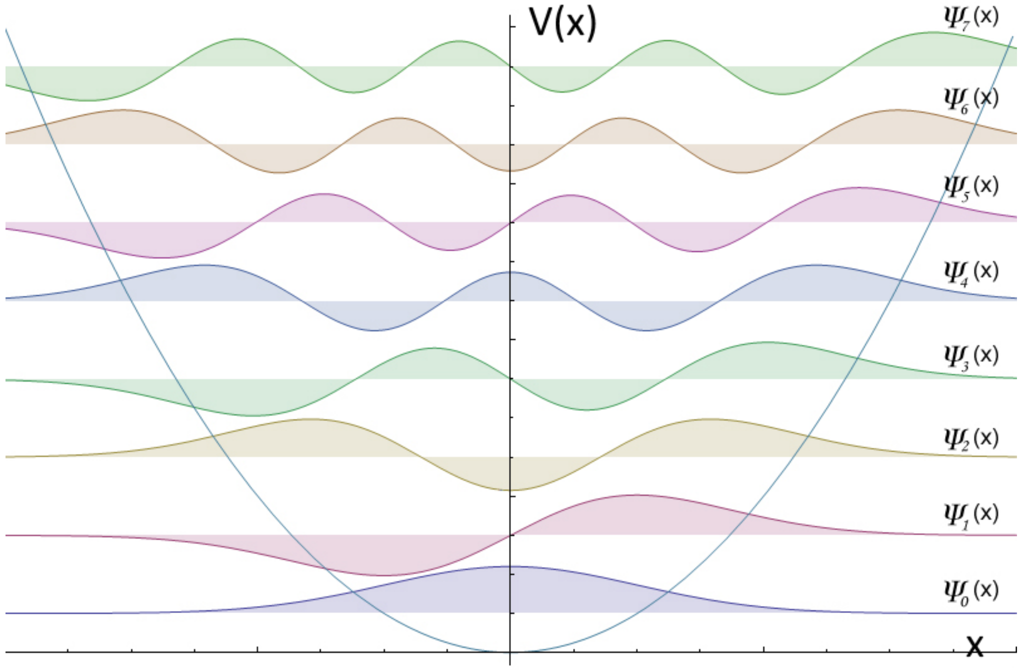

Figure 1: Model of simple harmonic oscillator with the first few wave functions. [1]

Clearly, $\psi'(x')$ is also a solution to the Schrödinger Equation with eigenvalue $E' = \frac{E}{\beta^2}$. Only the potentials with $-\frac{\alpha}{x^2}$ singularity have this property. Note that these potentials have singularity at $x = 0$.

According to K.M. Case [2], the energy spectrum of $-\frac{\alpha}{x^2}$ potentials has either none or infinite numbers of eigenvalues depending on the value of α. When $-\alpha \succeq -\frac{1}{4}$, there is no solution to the Schrödinger Equation. On the other hand, when $-\alpha < -\frac{1}{4}$, there are infinite solutions to the Schrödinger Equation. Therefore, potentials with such conformal symmetry either do not have bound states or have infinite continuous bound states from negative infinity to zero. Case calculated that the energy spectrum of potential $V(x) = -(\frac{1}{4} + \lambda)\frac{1}{x^2}$ has the form

$$E_n \propto e^{-\frac{n\pi}{\sqrt{\lambda}}}. \tag{9}$$

Note that this is not continuous spectrum, since n is discrete. But E_n goes all the way the negative infinity.

1.2 Approximate Conformal Symmetry; Regulated Singularity

M.D. Nguyen [3] gives an example of a potential with approximate conformal symmetry. The potential is in the form

$$V(x; a, \lambda) = -\frac{1}{4}\frac{x^2 + a}{(x^2 + 1)^2} - \lambda\frac{1}{x^2 + 1}. \tag{10}$$

The potential of this form has significant meanings in the field of Quantum Mechanics because it describes the movements of particles in higher-dimensional membranes. In his paper, Nguyen imposes the limitation that $-2 \leq a < \frac{1}{4}$ so that the potential always

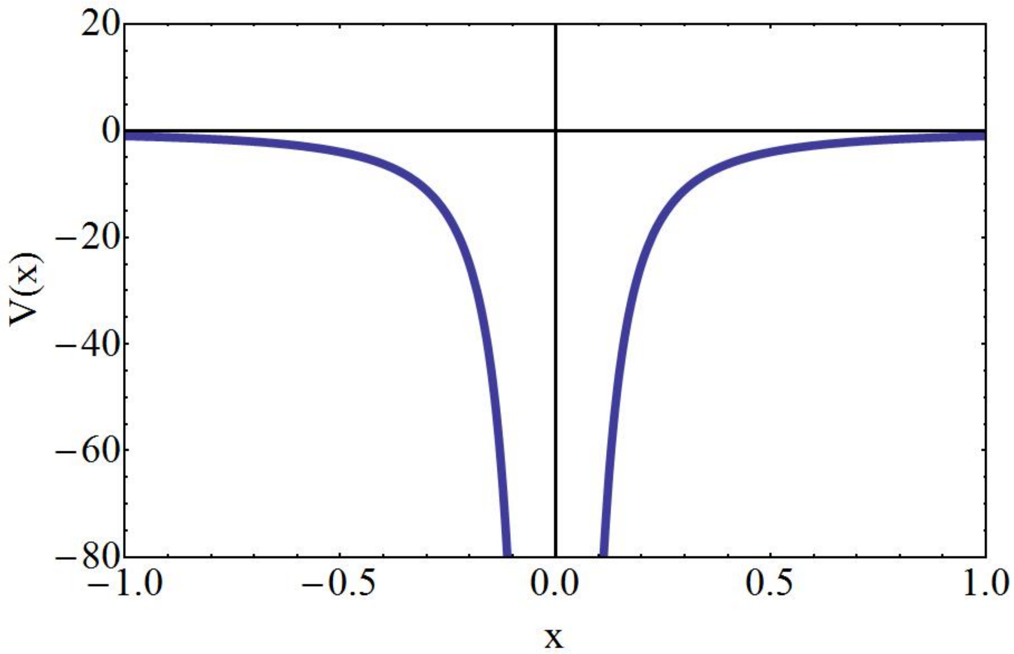

Figure 2: Potential with expression $V = -\frac{1}{x^2}$. There is a singularity at $x = 0$, which implies that the energy levels will go all the way down to negative infinity.

has bound state energies. Therefore, in the special cases of $\lambda \ll 1$ and $-2 \leq a < \frac{1}{4}$, this potential can be viewed as a regulated case for the conformal potential

$$V = -\alpha \frac{1}{x^2}. \tag{11}$$

Furthermore, in the limit of $\lambda \ll 1$, the potential becomes conformal:

$$V(x) = -(\frac{1}{4} + \lambda)\frac{1}{x^2}. \tag{12}$$

However, as Nguyen points out, the time-independent Schrödinger Equation of this potential with conformal symmetry has singularity at $x = 0$. This means that there is no normalizable solution at $E = 0$. By cutting off this singularity and inserting a smooth curve around $x = 0$, Nguyen calculates that the solution to the Schrödinger Equation also turns out to have the form

$$E_n \propto e^{-\frac{n\pi}{\sqrt{\lambda}}}, \tag{13}$$

which is consistent with Case as discussed in the previous section.

Another example of regulated singularity comes from K.S. Gupta [4]. He also looks at the inverse square potential

$$V(x) = -\frac{\alpha}{x^2}. \tag{14}$$

As discussed above, reversed square potential has infinite energy levels at $V(x) < 0$. Gupta uses a process called renormalization to make the energy levels finite. Renormalization process introduces a cutoff b. As shown in Figure 4, the singularity is cut off at $x = \pm b$. The more b approaches 0, the more accurate the renormalization process

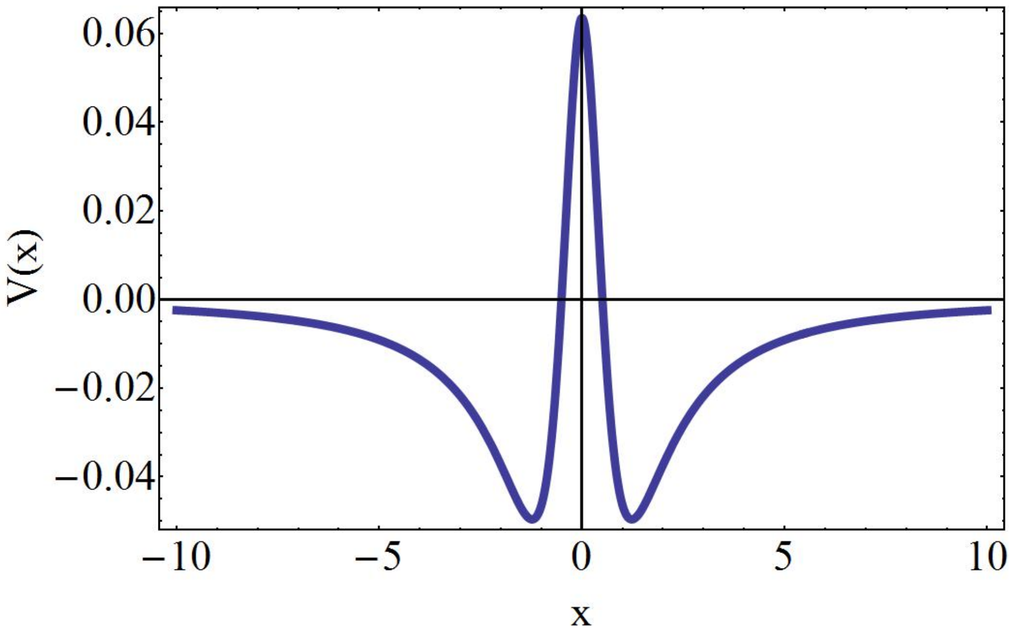

Figure 3: Potential of Equation (10) with $a = -\frac{1}{4}$ and $\lambda = 0.001$.

is. However, a restriction has to be imposed so that singularity will not reappear when $b \to 0$. According to Gupta, α should be a function of b so that the ground energy state always exists. He also imposes the restriction $\alpha = \mu^2 + \frac{1}{4}$ to eliminate the situation with zero energy states. In this expression, μ is defined as a real number ranging form negative infinity to positive infinity.

Gupta's expression of inverse square potential $V(x) = -\frac{\mu^2+\frac{1}{4}}{x^2}$ and Nguyen's expression $V(x) = -\frac{\lambda+\frac{1}{4}}{x^2}$ is very similar. μ and $\sqrt{\lambda}$ are roughly analogous. Then, the Hamiltonian H of Gupta's inverse square potential becomes

$$H = -\frac{d^2}{dx^2} - \frac{\alpha(b)}{x^2}, \tag{15}$$

where $\alpha(b)$ now is a function of b and H is defined in the range $[b, \infty]$. The approximate energy levels of this regulated potential turns out to be

$$E_n = -e^{-\frac{2n\pi}{\mu}} \left[\frac{2}{be^{\gamma}}\right]^2 [1 + O(\mu)]. \tag{16}$$

As $b \to 0$, the whole expression goes to negative infinity, which is consistant with Nguyen's and Gupta's result.

In my paper, I seek to expand this result by using resurgence theory and trans-series method. For the first time, I propose that the energy spectrum calculated by Case and Minh is just the first term of a trans series. Trans-series predicts that the bound states of potential with this form should look like

$$\begin{aligned} E_n \propto\, & e^{-\frac{n\pi}{\sqrt{\lambda}}}(a_{00} + a_{01}\lambda + a_{02}\lambda^2 + a_{03}\lambda^3 + ...) \\ & + e^{-\frac{2n\pi}{\sqrt{\lambda}}}(a_{10} + a_{11}\lambda + a_{12}\lambda^2 + a_{13}\lambda^3 + ...) + ... \end{aligned} \tag{17}$$

with, possibly, a logarithm term. For the purpose of this paper, I will focus on deriving the first few terms.

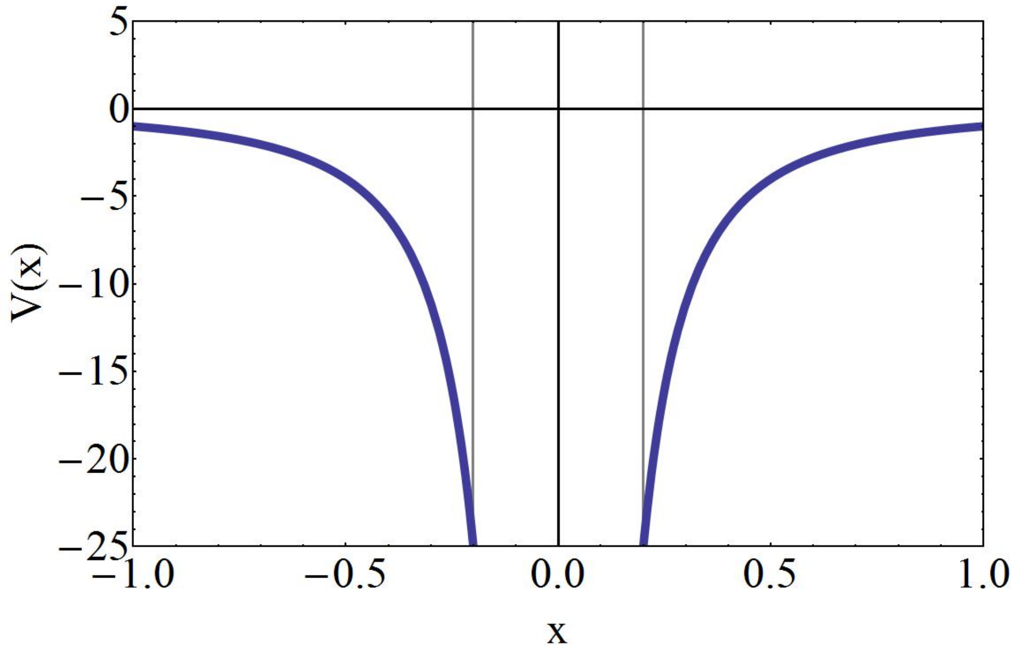

Figure 4: Regulated reversed square potential by chopping off the singularity at $x = \pm b$ where $b = 0.2$.

2 Analysis

2.1 Resurgence Theory

Resurgence theory connects perturbative physics and non-perturbative physics. It argues that all the information of the non-perturbative expansion of a potential is contained in the expression of the perturbative expansion.

According to G.V. Dunne and M. Unsal [5], for quantum mechanical systems with harmonic minima, perturbation method leads to divergent and non-alternating series. Since the sum of $E_{perturbative}$ is divergent, it has conflict with the actual energy levels of the particle that converges to a certain number. Therefore, the non-perturbative expansion must be in such a way that cancels out all the divergent parts of the perturbative expansion. The non-perturbative expansion is given by the tunneling effect. This exact relation between perturbative expansion and non-perturbative expansion is called Resurgence Theory.

First, take double-well potential as an example (Figure 5). Dunne and Unsal set up the potential in the form

$$V_{DW}(x) = x^2(1 + gx)^2, \tag{18}$$

where g is just a coefficient with the restriction $g > 0$. The perturbative expansion $E_{DW}(B, g^2)$ of the energy levels has the form

$$E_{DW}(B, g^2) = 2B - 2g^2\Big(3B^2 + \frac{1}{4}\Big) - 2g^4\Big(17B^3 + \frac{19}{4}B\Big) \\ - 2g^6\Big(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\Big) - 2g^8 ... \tag{19}$$

where $B \equiv n + \frac{1}{2}$. The summation of all the polynomials of B turns out to be negative infinity, which is not physically possible. There must be other corrections to the exact

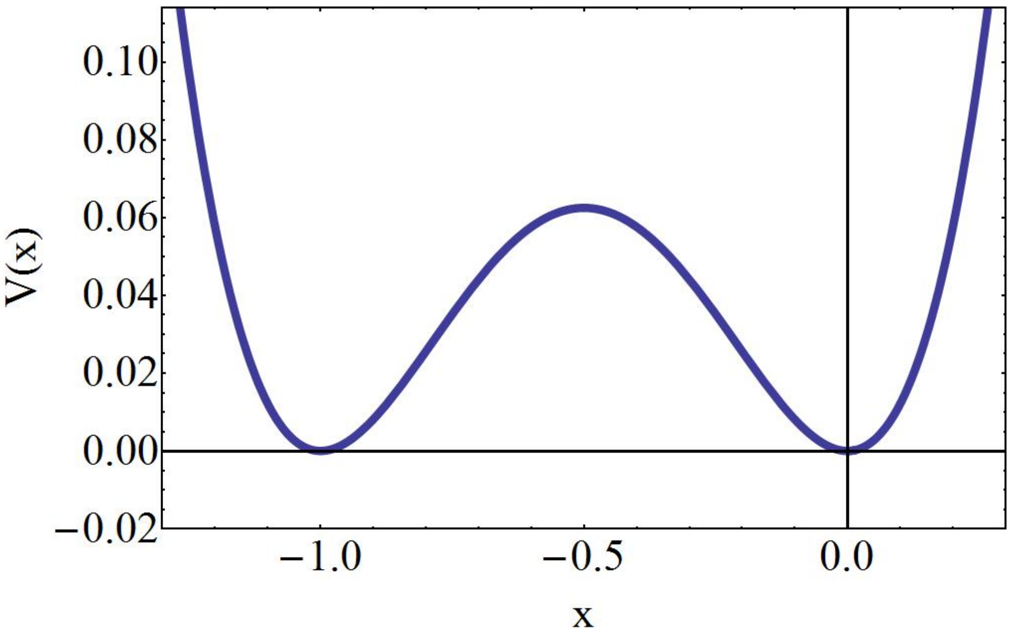

Figure 5: Potential $V(x) = x^2(1 + gx)^2$ with $g = 1$.

energy levels of the double-well potential to cancel out the divergent terms. According to Figure 5, there is the quantum tunneling effect through the middle barrier. The quantum tunneling effect is the quantum mechanical phenomenon where a particle tunnels through a potential barrier which classically not allowed. Dunne and Unsal [5] calculate that the non-perturbative expansion A_{DW} due to the tunneling effect of particles is

$$A_{DW}(B, g^2) = \frac{1}{3g^2} + g^2\Big(17B^2 + \frac{19}{12}\Big) + g^4\Big(125B^3 + \frac{153}{4}B\Big) + g^6\Big(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\Big) + g^8 \ldots \quad (20)$$

Therefore, the exact relation between non-perturbative expansion A_{DW} and perturbative expansion E_{DW} is

$$\frac{\partial E}{\partial B} = -\frac{g^2}{S}\Big(2B + g^2\frac{\partial A}{\partial g^2}\Big), \quad (21)$$

where S is a coefficient derived from the tunneling effect. The exact energy states can be obtained by adding up E_{DW} and A_{DW}. In this way, all the divergent terms will cancel out each other, bringing the exact energy levels to finite values. The significance of Resurgence Theory is that the exact energy levels can be obtained by only calculating the perturbative expansion. Once the perturbative expansion is known, Equation (22) can be applied to solve for non-perturbative expansion. Then, by adding them up, the exact energy states will be obtained.

2.2 New Potential

My goal is to apply Resurgence Theory to Nguyen's potential

$$V(x; a, \lambda) = -\frac{1}{4}\frac{x^2 + a}{(x^2 + 1)^2} - \lambda\frac{1}{x^2 + 1}. \quad (22)$$

I will consider the special cases when $-2 \leq a < \frac{1}{4}$ and $\lambda \ll 1$. According to Figure 3 and Figure 4, this potential has some similarities comparing to double-well potential. Both of them have two minima and a potential barrier in the middle, which indicates the quantum tunneling effect. Therefore, it is reasonable to assume that the perturbative expansion and non-perturbative expansion of this potential also satisfy Resurgence Theory. The perturbative expansion can be calculated by doing Taylor Expansion at the minimum points. The non-perturbative effect due to the quantum tunneling effect can be calculated from perturbative expansion using Resurgence Theory.

My first step is to calculate the perturbative expansion at the minimum points. The minima of the potetential are at

$$x_{min} = \pm\sqrt{\frac{1-a}{2\lambda+\frac{1}{2}} - 1}. \tag{23}$$

For consistency with Dunne and Unsal's paper and for simplicity, I assume that

$$\sqrt{\frac{1-a}{2\lambda+\frac{1}{2}} - 1} = \frac{1}{g}. \tag{24}$$

To calculate the perturbative expansion, I need to do Taylor expansion at minimum points $x = \pm\frac{1}{g}$. Because the potential is symmetric along y-axis, I pick $x = \frac{1}{g}$ for my calculation:

$$\begin{aligned} V(x) = V(\frac{1}{g}) + V'(\frac{1}{g})(x-\frac{1}{g}) + \frac{1}{2}V''(\frac{1}{g})(x-\frac{1}{g})^2 \\ + \frac{1}{6}V^{(3)}(\frac{1}{g})(x-\frac{1}{g})^3 + \frac{1}{24}V^{(4)}(\frac{1}{g})(x-\frac{1}{g})^4 + ... \end{aligned} \tag{25}$$

Since the function is at its minima,

$$V'(\frac{1}{g}) = 0. \tag{26}$$

Also, the first term $V(\frac{1}{g})$ is just a constant:

$$V(\frac{1}{g}) = \frac{(2\lambda+\frac{1}{2})^2}{4(a-1)}, \tag{27}$$

which is the depth of the well counting from $x = 0$. Note that, since $-2 \leq a < \frac{1}{4}$, the minimum is below the x-axis. Therefore, the dominant term is $V''(\frac{1}{g})(x-\frac{1}{g})^2$. I calculate this term by doing the general derivatives first:

$$\frac{dV}{dx} = -\frac{1}{4}\Big(\frac{2x}{(x^2+1)^2} - \frac{2(x^2+a)}{(x^2+1)^3}\cdot 2x\Big) + \lambda\cdot\frac{2x}{(x^2+1)^2}, \tag{28}$$

and

$$\begin{aligned} \frac{d^2V}{dx^2} = -\frac{1}{4}\Big[\frac{2}{(x^2+1)^2} - 2\cdot\frac{2x}{(x^2+1)^3}\cdot 2x - 4\cdot\frac{3x^2+a}{(x^2+1)^3} + 12\cdot\frac{x(x^2+a)\cdot 2x}{(x^2+1)^4}\Big] \\ + \lambda\Big[\frac{2}{(x^2+1)^2} - 2\cdot\frac{2x}{(x^2+1)^3}\cdot 2x\Big]. \end{aligned} \tag{29}$$

Then, I plug in $x = \frac{1}{g}$:

$$\frac{d^2V}{dx^2} = -\frac{1}{4}\Big[\frac{2}{((\frac{1}{g})^2+1)^2} - 2(\frac{1}{g})^2 \cdot \frac{2}{((\frac{1}{g})^2+1)^3} \cdot 2 - 4 \cdot \frac{3(\frac{1}{g})^2+a}{((\frac{1}{g})^2+1)^3} + 24(\frac{1}{g})^2 \cdot \frac{(\frac{1}{g})^2+a}{((\frac{1}{g})^2+1)^4}\Big] + \lambda\Big[\frac{2}{(\frac{1}{g}^2+1)^2} - 8(\frac{1}{g})^2 \cdot \frac{1}{((\frac{1}{g})^2+1)^3}\Big]. \tag{30}$$

Since this is only the perturbative expansion of the potential, it is important to rule out the quantum tunneling effect. By assuming that the two minima are far apart, the middle barrier will be infinitely wide so that no particle can tunnel through. Therefore $x_{min} \to \infty$. Note that in Equation (32), all the g's are multiple of $\frac{1}{g^2}$. Therefore, it is safe to assume that

$$g^2 \to 0, \frac{1}{g^2} \to \infty. \tag{31}$$

Then, Equation (30) can be simplified as

$$V''(\frac{1}{g}) \simeq \frac{3}{2}g^4 - 2ag^6 - 6\lambda g^4 \tag{32}$$

This equation can be further simplified by using Equation (24) which gives the relationship between a, g, and λ:

$$\frac{1}{g^2} + 1 = \frac{1-a}{2\lambda + \frac{1}{2}}. \tag{33}$$

The fact that $\frac{1}{g^2} \to \infty$ means that equation (33) becomes

$$\frac{1}{2} - 2\lambda \simeq 1 + ag^2. \tag{34}$$

Therefore, I plug in Equation (34) into Equation (30) and get

$$V''(\frac{1}{g}) = 3g^4 + ag^6. \tag{35}$$

Equation (34) requires ag^2 to be a finite number on the order of 1 to 10. Therefore, $V''(\frac{1}{g})$ is on the order of g^4. According to equation (25), Taylor expansion of potential $V(x)$ at minina turns out to be

$$V(x) \simeq C + \frac{1}{2}k(x - \frac{1}{g})^2 \tag{36}$$

where

$$V''(\pm\frac{1}{g}) = k = 3g^4 + ag^6, C = V(\frac{1}{g}). \tag{37}$$

This clearly resembles double well potential. According to the prediction of trans-series, the perturbative expansion of this potential should be in the form

$$V(x) = c_0 g^4 (x - \frac{1}{g})^2 + c_1 g^6 (x - \frac{1}{g})^4 + c_2 g^8 ... \tag{38}$$

where c_n's are constants to be determined. By shifting minimum on the right side to the origin and assuming that the minima are far apart, the expression can be re-written as

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{39}$$

where $\omega = \sqrt{\frac{k}{m}}$ and $x = x - \frac{1}{g}$. Then, this problem is turned into a single well potential problem.

The perturbative expansion of this simplified single well potential can be applied to Equation (39). Perturbative theory states that the first order correction is in the form

$$H = H^0 + \beta H^1 \tag{40}$$

where H^0 is the original function and H^1 is the first order perturbation. Also, the first order correction of wave function and energy levels are

$$\psi_n = \psi_0 + \beta\psi_n^1 + ... \tag{41}$$

$$E_n = E_n^0 + \beta E_n^1 + ... \tag{42}$$

I plug them back into time-independent Schrödinger Equation and get

$$H\psi_n = E_n\psi_n \tag{43}$$

$$(H^0 + \beta H^1)(\psi_n^0 + \beta\psi_n^1) = (E_n^0 + \beta E_n^1)(\psi_n^0 + \beta\psi_n^1). \tag{44}$$

This equation can be further simplified as

$$H^0\psi_n^1 + H^1\psi_n^0 = E_n^1\psi_n^0 + E_n^0\psi_n^1. \tag{45}$$

Then, I multiply both sides by ψ_n^{0*} and simplify the expression again to get

$$E_n^1 = \langle\psi_n^0|\, H' \,|\psi_n^0\rangle \tag{46}$$

where H' can be calculated from the first order correction of potential $V(x) = \frac{1}{2}k(x \pm \frac{1}{g})^2$.

Rising and lowering operator method requires that the contributions of all the terms with odd number of operators is zero. Therefore, there is no contribution from the term $V^{(3)}$. The first order correction of the potential $V(x)$ is then the term $\frac{1}{24}V^{(4)}(\frac{1}{g})(x - \frac{1}{g})^4$. After a series of calculation and simplification, $V^{(4)}(\frac{1}{g})$ turns out to be

$$\frac{1}{24}V^{(4)}(\frac{1}{g}) = \frac{1}{4}(80g^6 + 55ag^8), \tag{47}$$

and $V^{(4)}(\frac{1}{g})$ is on the order of g^6 as expected.

Then, I can calculate the first-order perturbative correction using the results. According to Equation (40),

$$H = H^0 + H^1 \tag{48}$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{4}(80g^6 + 55ag^8) \tag{49}$$

$$H^0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \tag{50}$$

$$H^1 = \frac{1}{4}(80g^6 + 55ag^8). \tag{51}$$

Therefore, E_n^1 can be calculated using Equation (46) and Equation (51). Using the method of rising and lowering operators, I get the first term of the perturbative expansion:

$$E_n^1 = \frac{\hbar^2}{16m}(n^2 + n + \frac{3}{4})\frac{80g^2 + 55ag^4}{3 + ag^2}. \tag{52}$$

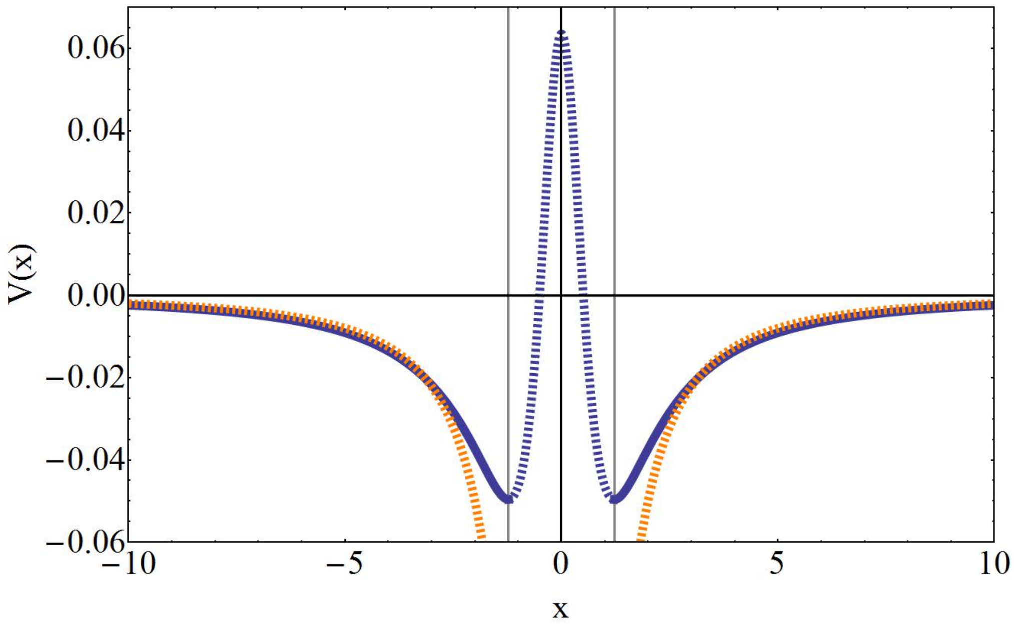

Figure 6: Regulated potential with the middle barrier cut off at $x_{min} = \pm 1.22$. The approximate inverse square potential is $V(x) = -\frac{0.26}{x^2}$

Change n to B by doing the substitution $B = n + \frac{1}{2}$ like what Dunne and Unsal did in their paper on Resurgence Theory, the first order perturbative expansion of the energy levels becomes

$$E_n^1 = \frac{80g^2 + 55ag^4}{3 + ag^2}(B^2 + \frac{1}{2}). \tag{53}$$

This term has the correct order of magnitude comparing to the first order correction term $-2g^2(3B^2 + \frac{1}{4})$ in Equation (19) for double-well potential. The next term would be in the order of g^4 and B^4.

However, there is a fundamental flaw in assuming that this potential is analogous to double-well potential. My assumption that the two minima are far apart requires that $a \to -\infty$ because

$$\sqrt{\frac{1-a}{2\lambda + \frac{1}{2}} - 1} = \frac{1}{g} \tag{54}$$

and $\frac{1}{g} \to \infty$. On the other hand, as mentioned at the beginning of my analysis, the condition $-2 \leq a < \frac{1}{4}$ is imposed to make sure that there is always solutions to the Schrödinger Equation. Therefore, I conclude that the analogy between Nguyen's potential and the double-well potential is not feasible.

2.3 Regulated Potential

Since using Perturbative Theory is not feasible, I then turn to Gupta's method of regulated singularity. Figure 3 indicates that, when $x \to \pm\infty$, the behavior of Nguyen's potential is asymptotically approaching $x = 0$. In other words, $x \to 0$ is the dominant behavior of this potential. Therefore, by cutting off the barrier between two minima (as demonstrated in Figure 6), the potential becomes an approximate renormalized inverse square potential model with regulated singularity. The approximate energy levels can thus be calculated using Gupta's method.

3 Conclusion

In this paper, I proposed that trans series predicts that the energy levels of potential $V(x;a,\lambda) = -\frac{1}{4}\frac{x^2+a}{(x^2+1)^2} - \lambda\frac{1}{x^2+1}$ should be in the form

$$E_n \propto e^{-\frac{n\pi}{\sqrt{\lambda}}}(a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + ...). \tag{55}$$

While the first term $e^{-\frac{n\pi}{\sqrt{\lambda}}}$ is verified by Case [2] and Nguyen [3], the following higher order terms have not been proven. I attempted to derive these terms using Resurgence Theory and double-well approximation. However, my approach using double-well approximation turned out to be incorrect. A better way of solving this problem is to use Gupta's regulated potential method. Future research on this problem should look into this direction.

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