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The classical view of logic dominated the Second International Congress of Philosophy in 1904. Classical logicians, such as Couturat, believed mathematics exist independent of the human mind.[1] For these philosophers, mathematics was an eternal truth awaiting discovery by human reason. Other logicians fiercely opposed this view. At the same Congress, Boutroux contended that philosophers should investigate the human preference for particular forms of mathematics within the "infinity of alternative, equally possible [forms]."[2] Boutroux made this contention because he believed that mathematics lacks another kind of existence besides that conferred on them by humans. Poincaré furthered this objection to classical logic, evidencing the "inescapably 'intuitive' character of logical reasoning."[3] Both argued for a philosophical investigation into the conventions underlying the existent mathematical systems. Thus began the field of intuitionistic mathematics; a philosophy based upon the innately human, intuitive origins of mathematics. Intuitionistic mathematics provides an informative challenge to classical logic, specifically through establishing mathematics as a human construction. Furthermore, Intuitionistic mathematics is able to overcome the consequence of the aforementioned assertion and one of its principle objections; the charge that Intuitionism leads to solipsism.

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[1] G. D. Bowne, The Philosophy of Logic 1880-1908 (Netherlands: Mouton and Company, 1966) p. 127.
[2] Bowne 109.
[3] Bowne 127.

Until 1907 this brief foray exploring the role of intuition in mathematics and logic seemed forgotten. In that year L. E. J. Brouwer published his sometimes poetic and often obscure work, On the Foundations of Mathematics.[4] Alongside his precursors Boutroux and Poincaré, Brouwer objected to the system of classical logic. More importantly, he disagreed with their position as well. He argued that the classical and early intuitionistic logicians mistakenly supplanted the intuition of mathematics with the language of mathematics.[5] For Brouwer, mathematics arises out of intuition. In support of his view, Brouwer drew from Kant's philosophy of a priori synthetic knowledge. According to both philosophers, time is a necessary component of experience.[6] Brouwer further argued that there exists a close connection between time, the Self, and the idea of mathematics. Specifically, the mind experiences a singular moment that gives rise to another, all the while preserving a memory of the first moment. The experience is caused by an awareness of the Self's continuity over time.[7] In Brouwer's terms this simultaneous experience is the perception of a "twoity." Recollecting the twoity at later moments gives humans the notion of unity among twoities. The mind extends these synthetic experiences over again to form the typical conceptions of mathematics, such as infinity and arithmetic. Thus, for Brouwer, the intuitive

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[4] S. C. Kleene and R. E. Vesley, The Foundations of Intuitionistic Mathematics, Studies in Logic and the Foundations of Mathematics Series (Amsterdam: North-Holland, 1965) 1.
[5] Bowne 142.
[6] Michael Detlefsen, "Constructive Existence Claims," The Philosophy of Mathematics Today, ed. Matthias Schirn (Oxford: Clarendon Press, 1998) 316.
[7] Detlefsen 318.

simultaneous experience of moments arising out of each-other is the basis of all mathematics.[8]

Brouwer's claim that mathematics begins in intuition has significant epistemic consequences. First, the assertion privileges the epistemic value of intuitions. [9] This is clear after acknowledging Brouwer's use of yet another preceding philosopher; namely, Descartes. Brouwer adopted the epistemic method of Descartes' Meditations, developing his knowledge of mathematics from the point of intuition. Such a method was required since, for Brouwer, mathematics originates in intuition, just as certain knowledge originates in the intuitive experience of the thinking Self for Descartes. Thus, for both philosophers, certainty is derived from a claim's proximity to intuitions of the Self.[10] Consequently, any mathematical claim distanced from the Self introduces the possibility of error. Such "distance" is created through symbolizing or otherwise representing intuitions; distinctly non-intuitive activities. Any formalization of intuition constitutes "an incomplete communication of information."[11] Intuitions by their nature are language-less experiences so any representation of them in language admits error.

Furthermore, according to Brouwer, "intuition proceeds independently of… ...formalization"[12] such that "a mathematical entity is not necessarily predeterminate, and may, in its state of free growth, at some time acquire a property which it did not posses

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[8] D. van Dalen, ed., Brouwer's Cambridge Lectures on Intuitionism (Cambridge: Cambridge University, 1981) 4.
[9] Detlefsen 319.
[10] Detlefsen 316.
[11] Kleene and Vesley 93.
[12] A. Heyting, Intuitionism an Introduction, Studies in Logic and the Foundations of Mathematics Series (Amsterdam: North-Holland, 1966) 5.

before."[13] This curious phenomenon occurs because Intuitionists redefine the meaning of mathematical assertions, in opposition to the meaning described by classical logicians. Classical logicians believe that a statement is either true or false. Brouwer countered this by claiming that (i) the assertion of a mathematical statement is that it is provable, (ii) the assertion of the negation of a mathematical statement is that the assumption of the statement in a proof leads to an absurdity and (iii) some particular assertions may be neither proven true nor absurd.[14] Brouwer's demonstration of the necessity for this redefinition is illustrated below in his argument against the Law of the Excluded Middle. Formalizations are both distanced from intuition and have meanings that are subject to change. This gives intuitions an epistemic privilege above non-intuitive mathematical claims.

Such privilege becomes significant in relation to the second epistemic consequence of Brouwer's claim that mathematics originates intuitively; that is, formal mathematics becomes a human construction. If intuition is the only ground for certain knowledge in mathematics, any non-intuitive claim is chosen by the mathematician. Such choice over non-intuitive mathematical claims entails that truths are created, rather than discovered.[15] Thus, formal mathematical claims are constructions because they are non-intuitive. Moreover, since the Self is the source of knowledge, anything epistemic assertion that exists must exist through the Self.[16] This entails that mathematical systems are merely constructions of the mathematician's mind since epistemic entities only obtain existence through intuitions of the Self. Because

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[13] van Dalen 92.
[14] van Dalen 92.
[15] Detlefsen 320.
[16] Detlefsen 321.

mathematical formalizations are chosen and do not obtain epistemic existence through human minds they are necessarily constructed.

The assertion that mathematical systems are constructions may at first appear to be insignificant. However, it results in important consequences for formal mathematical systems such as classical logic. In such systems there is a "distinctness of mathematics from the language in which mathematics is expressed."[17] The "distinctness" comes to pass because mathematics is an intuition whereas the language of mathematics is a human construction.[18] According to Brouwer, formal mathematics developed from mathematicians' construction of symbolic parallels to their intuitive experience. Thus, classical logicians create the formalizations that Brouwer cautions as fallible. The problem is that classical logicians believe in the "existence of immutable properties of time and space, properties independent of experience and of language."[19] This belief leads them to ignore the Intuitionist's warning that formalizations are fallible "[T]he classical mathematician thinks of himself as reasoning about an objective, external domain of entities", whereas the Intuitionist recognizes the constructed nature of mathematical entities.[20] The incautious reasoning of the classical logicians leads to demonstrably meaningless claims. Brouwer's proof against Law of the Excluded Middle demonstrates both

[17] Kleene and Vesley 1.

[18] S. Shapiro, ed., Intensional Mathematics, Studies in Logic and the Foundations of Mathematics Series (Amsterdam: North-Holland, 1985) 23.

[19] van Dalen 1.

[20] A. Kino, J. Myhill and R. E. Vesley, eds., Intuitionism and Proof Theory, Studies in Logic and the Foundations of Mathematics Series (Amsterdam: North-Holland, 1970) 101.

his meaning of "meaningless" and the significant consequence of formalizing mathematics.

For Brouwer, the LEM is a formal assertion devoid of mathematical meaning. Such meaningless claims have the same epistemic value as those pointed out in the conclusion of Wittgenstein's Tractatus Logico Philosophicus. The claims made by classical logicians are not so much wrong as they are nonsensical. This is because, by necessity, such claims do not describe things in the world, since the only real mathematical entities are human intuitions. The LEM is particularly susceptible to Intuitionist critique since it necessitates the existence of entities that are yet to be constructed. For intuitionists, sensibility is faithfulness to the intuitive origins of mathematics. Non-performable and therefore nonsensical constructions are too far removed from intuition. Since formal mathematics is a human construction prone to error, it is patently erroneous to formalize over non-constructed, non-intuited entities. The constructions of classical logicians such as the LEM commit such error.

Brouwer used the following argument to demonstrate the fallibility of this Law and the fallibility of constructing mathematical formalizations in general.[21] Write down the expansion of π with the expansion of .333333… below it. Call the lower expansion (that of .333333...) R. When, and if, the expansion of π creates a sequence …0123456789… halt the expansion of R. Call the digit where the 9 in the π sequence of ...0123456789... occurs μ. This digit also marks when the R expansion halts. Now, suppose R cannot be a rational number. This means the digit μ cannot occur. Therefore, no sequence of …0123456789… occurs in π. But if no such sequence occurs in π the expansion of R continues forever.

[21] van Dalen 6.

However, this would make R a rational number because it is equal to one-third. Thus, a contradiction results from the supposition that R is a rational number. At this point classical logicians would assert that R must be a rational number. Intuitionists such as Brouwer deny this. If R is a rational number there must be two known integers, call them p and q, such that p divided by q is equal to R. However, this requires either knowledge of the sequence …0123456789… in π, or the ability to show that the sequence never occurs. Since the sequence is not yet known to occur in the construction of π it is impossible to assert the number that is R.

This argument is more accessible in symbolic terms. First, make the supposition that R is not rational. Symbolize this as "¬p". This resulted in a contradiction that is then inferred to result in ¬¬p. From this negation of a falsity, classical logicians assert a positive truth, symbolized as "p". Intuitionists deny this assertion because it is based on the LEM. This law asserts that all statements, including mathematical ones, are either true or false (symbolized as pv¬p). This is fine for a classical logician because they believe in the objective mathematical truth or falsity of p. Intuitionists, however, point out that the value of the R expansion is as yet unknown, and therefore no assertion can be made regarding it. Roughly, this argument shows that for intuitionists, ¬¬p does not imply p.[22] For intuitionists to accept the LEM some method must exist to determine which of p or ¬p is true for any proposition p. Unfortunately, no such method exists.

Such a formal presentation of intuitionistic logic seems to contradict one of Brouwer's main beliefs. He argued that any attempt to bring mathematics outside the internal intuitive experience results in fallibility and

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[22] Heyting 100.

loss of meaning. He also criticized others for developing formal mathematical systems. This seems hypocritical, given that Brouwer expressed mathematical arguments for intuitionism in both writing and speech, such as his proof against the LEM. Though he rejected a formal construction of mathematics, Brouwer did admit to the usefulness of formalizing intuitively correct constructions.[23] Heyting, a student of Brouwer, later claimed that Brouwer's objection to formalization only applied to formalization without consideration for the meaning of what is formalized.[24] Brouwer's own words provide evidence for Heyting's claim. According to Brouwer, classical logic studies the constructions of mathematicians, rather than the intuitions that give rise to mathematics.[25] Essentially, classical logic formalizes mathematical constructions, whereas Intuitionists formalize intuitions only. Classical logic is problematic because their formalizations are in ignorance of the source of mathematical meaning, specifically intuitions.

Nevertheless, the critique of hypocrisy is minor compared to the problem that rests in the roots of intuitionism itself. This problem is a logical consequence of the Intuitionist's claim that mathematics is a human construction. Explicitly, how is any communication of mathematics a sensible and worthwhile activity? What allows Brouwer and the Intuitionists to decide and formalize their so-called "intuitively correct constructions"? Mathematics arises out of an intuitive experience and thereafter is subject to fallibility and loss of meaning due to the limitations

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[23] Joan Moschovakis, "Intuitionistic Logic," The Stanford Encyclopedia of Philosophy, ed. Edward N. Zalta, Winter 2002 <http://plato.stanford.edu/archives/win2002/entries/logic-intuitionistic/>.
[24] Kleene and Vesley 3.
[25] Bowne 143.

of language.[26] Such problems arise since formal systems are never adequate as they are linguistically removed from mathematical intuition.[27] It seems that intuitionism is caught in the trap of solipsism. Furthermore, an intuitionist cannot rely on past mathematical constructions. Unless such constructions are remembered they must be recorded in some manner, thereby removing them from the certainty of intuition. It appears that Intuitionistic mathematicians must admit to an extreme form of solipsism; a lack of certainty in the knowledge of one's very Self over time.

This is a serious charge against Intuitionism. Mathematics is a valuable method and a science that most would wish to retain. The proliferation and development of mathematics itself are evidence enough. Even Brouwer himself stands in agreement. He felt mathematics was central to human experience, finding its source in the intuitions of the Self.[28] Brouwer also believed mathematics is central to "human epistemic life" since the synthetic experience is applied to human experience in the world, per Kant.[29] If mathematics is so vital to both empirical investigation and human experience, why does Brouwer make its communicability, when not meaningless, at the very least prone to error?

Intuitionist Mathematicians can make at least three responses to the charge of solipsism. First, since mathematical systems are the product of human construction mathematicians must change their objectives. The underlying accusation in the indictment of solipsism is that it limits the certainty of mathematical investigation; an admittedly valuable activity. Intuitionists respond by reframing

[26] Kleene and Vesley 93.
[27] Heyting 5.
[28] Detlefsen 331.
[29] Detlefsen 331.

mathematics into an enterprise more cautious than that of the classical logicians. The purpose of classical logic is an attempt to penetrate the "objective, external domain of [mathematical] entities."[30] By sweeping aside this platonistic account and establishing mathematics as a human construction, the intention of mathematics must be directed by the exclusion of error.[31] This is because there are no formal mathematical truths to discover. The accusation of solipsism loses its teeth since mathematicians must abandon their hope for non-intuitive certainty. This response assuages the concern that Intuitionistic mathematical beliefs are too limited by giving mathematics a new purpose. Accordingly, the charge of solipsism loses some of its persuasiveness.

Nevertheless, merely redefining mathematics seems somewhat dissatisfactory as a response to the accusation that Intuitionistic Mathematics leads to solipsism. A second response attempts to justify Intuitionistic formalizations. According to Brouwer, "inner experience reveals how, by unlimited unfolding of the basic intuition [of the Self over time], much of 'separable' mathematics can be rebuilt in a suitably modified form."[32] For Brouwer, some mathematical constructions achieve a transmission of truth from the certainty of intuition. However, these particular constructions must be constructed according to a general form. Brouwer called this form of construction a "law-like choice sequence."[33] That is, Brouwer allows formal construction so long as mathematicians utilize some constant method of determining the epistemic value of constructions. This method enforces

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[30] Kino, Myhill and Vesley 101.
[31] Detlefsen 320.
[32] van Dalen 5.
[33] Richard Tieszen, "Intuitionism, Meaning Theory and Cognition," History and Philosophy of Logic 21 (2000): 179-194.

the equal epistemic value of the constructions; the constructions would be true relative to each other, if not objectively true. The constant method is the form of construction that matches human intuition. Humans experience a constancy of form in their intuition of the "unfolding" Self over time. Moreover, though always the same, intuitions are constantly recreated, and can be separated into distinct experiences. This allows mathematicians to use constructions over time since they share the form of constant, yet endlessly repeated intuitions.[34] The constancy and repetition of intuition give mathematicians the general form to follow when constructing mathematics. So long as constructions share the form of intuitions, Intuitionists can accept formalization.

Brouwer's principle objection to classical logic, and formalized mathematics in general, is that they utilize unproven constructions.[35] The Law of the Excluded Middle is an example of such use. Because the formalizations used in the LEM are as yet not constructed, such formalizations are completely removed from intuition. This entails that they fail to meet Brouwer's requirement that constructions follow the general form of intuitions. Moreover, such mathematical formalizations lack existence because they do not begin from intuitions of the Self. Again, this is because the Self is the measure of existence.[36] Lacking a method to construct such formalizations, mathematicians must exclude their use. However, since other constructions proceed from intuition, Brouwer allows their use. This both relieves Intuitionists from the requirement of constructing mathematical

[34] Tieszen 180.
[35] van Dalen 5.
[36] Detlefsen 321.

formalizations each time they are used and undermines the accusation that Intuitionism leads to solipsism.

The third response to the allegation that Intuitionistic mathematics results in solipsism utilizes the shared nature of mathematical intuitions. Brouwer certainly believed that mathematical formalizations were chosen; such was the source of his assertion that mathematical systems are constructs. Nevertheless, in drawing from the philosophy of Kant, Brouwer made very similar claims to Kant's argument that humans use an a priori category of time for organizing experience. According to Brouwer, humans use the intuition of mathematics in their perception of existence. Unlike Kant, Brouwer believed that the utilization of mathematics is freely chosen by humans. The twoity that develops out of a perception of the Self over time is applied to the world *by choice*, rather than by necessity.[37] Nevertheless, Brouwer did assert that mathematical intuitions are the result of "certain organizational features of the human intellect."[38] For Brouwer, the common organizational feature is the intuition of unity-in-plurality over time because "the self of the creator's self-awareness (which is ultimately the object [and source] of her knowledge) is not... ...a product of her legislative decision."[39] Thus, humans cannot choose their Self; rather they can choose to share the common intuition that arises out of an awareness of the Self. Because humans can choose to follow the natural organization of their intuitions, there is a "stock of mathematical entities [that] is a real thing, for each person, and for humanity."[40]

Tieszen holds a similar view, arguing from the standpoint of cognitive psychology. He believes it is

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[37] Detlefsen 333.
[38] Detlefsen 333.
[39] Bowne 144.
[40] van Dalen 90.

perfectly sensible that human beings possess some "isomorphic cognitive structure" that directs communication.[41] If Tieszen is right, it appears that humans are free to "act in accordance with" intuitions that arise out of a structure shared by all of humanity.[42] Furthermore, this entails that humans can choose to construct similar formalizations based upon those shared intuitions. Thus, it is apparent from both Brouwer and Tieszen's conception of intuition that Intuitionism allows for the communication of mathematical formalizations. Intuitionistic mathematics thereby avoids the charge that it results in solipsism.

Even if the Intuitionist's responses are deemed inadequate, Intuitionistic Mathematics retains an epistemic appeal. At the very beginnings of Intuitionistic mathematics in the International Congress of 1904, Russell, an eminent defender of Couturat's position, admitted classical logic's inadequacy as demonstrated by the new "intuitionist" position.[43] Russell's admission was vindicated in 1931 by Gödel's Incompleteness Theorem. In contrast, the constructions of Intuitionistic Mathematics have been shown to be both sound and complete.[44] Nevertheless, Brouwer had another motivation to argue that mathematical formalizations are human constructions. Detlefsen argues that Brouwer's constructivist position was "no doubt in part due to his mistrust of all things corporate."[45] Brouwer believed that communication is often a hidden form of coercion. For Brouwer, "language, formalization, [and] communication are... ...all tools of the imperialists of the will in their

[41] Tieszen 185.
[42] Detlefsen 320.
[43] Bowne 9.
[44] Shapiro 26.
[45] Detlefsen 330.

attempts to conform the will of the autonomous individual to their own."[46] Classical logicians engage in such imperialism and deny human agents creative freedom by purporting a platonistic account of mathematics. These logicians are convinced of their account because they ignore the intuitive and consequently constructed nature of mathematics. Thus, far from trying to channel mathematics into solipsism, Brouwer and the Intuitionistic mathematicians are defenders of creative freedom.

[46] Detlefsen 330.