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Prime Walks in Cyclotomic Fields

Jacob Bond
Macalester College

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Prime walks in cyclotomic fields

By Jacob Bond

For David Larabee (1963-2005),
teacher and mentor

1. Introduction

A prime walk is a sequence of primes which allow one to "walk" from the origin out to infinity "stepping" only on prime numbers and taking "steps" of bounded size. More precisely, a prime walk is a sequence of primes $(\psi_j)_{j=1}^{\infty}$ such that there exists an m satisfying $\psi_1 \leq m$, $|\psi_{j+1} - \psi_j| \leq m$ for all $j \geq 1$, and $\lim|\psi_j| = \infty$. When we consider rational primes, we quickly see that a prime walk does not exist since $(m+1)!+2, (m+1)!+3, \ldots, (m+1)!+m+1$ contains no primes. Because the same can be done for any m, every sequence of primes starting from the origin must have an unbounded difference.

A natural extension to this question asks whether the same is true of the Gaussian integers $\mathbb{Z}[i]$. The Gaussian integers consist of algebraic numbers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. Further, we define a prime to be an integer ψ which is only divisible by its associates, $\psi, -\psi, \psi i, -\psi i$, and the units, $1, -1, i, -i$. It is important to realize that in this case, we are not simply looking for intervals which are prime free, but rather regions, called "moats," which surround the origin. Since T. Motzkin and B. Gordon originally posed this question, various results have been established. The most direct result was the computational construction of a moat of width $\sqrt{26}$. On the other hand, it has been shown that one can not walk to infinity along a straight line, and, more generally, that a walk cannot exist when restricted to a sufficiently small angular sector.

In addition to the Gaussian integers, one may also consider the related problem of the Eisenstein integers $\mathbb{Z}[\omega]$, where $\omega = e^{\pi i/3}$. Just as with the Gaussian integers, the Eisenstein integers are defined $\mathbb{Z}[\omega] = \{a+b\omega \mid a, b \in \mathbb{Z}\}$. J. Haugland [6] presents an argument suggesting that a prime walk should not exist in $\mathbb{Z}[\omega]$. Haugland argues that based on the density of the primes and the symmetry of the gcd function in the Eisenstein integers, such a walk must cross an arbitrarily large region composed only of composite numbers. Thus, a prime walk would not exist in $\mathbb{Z}[\omega]$. However, once we move beyond $\mathbb{Z}[\omega]$, the field loses its lattice structure. Even though the higher cyclotomic fields do not exhibit the nice arrangements of $\mathbb{Z}[\omega]$ and $\mathbb{Z}[i]$, they still represent interesting cases to study.

This paper investigates prime walks to infinity from various perspectives and in three distinct fields, $\mathbb{Z}[\omega]$, $\mathbb{Z}[i]$, and $\mathbb{Z}[\zeta]$ with $\zeta = e^{2\pi i/5}$. P. Loh [7] showed that any prime walk in the Gaussian integers must sweep out some angular sector of positive measure. He further demonstrated that by taking a union of sectors with small arc, one can prohibit a prime walk on a region for which the sum of the angular measures of each sector is arbitrarily close to 2π. In this paper, these results are extended from $\mathbb{Z}[i]$ to $\mathbb{Z}[\omega]$. We also present the argument against a prime walk in $\mathbb{Z}[\omega]$ given by Haugland, filling in the specific details. Additionally, we extend this argument from the Eisenstein integers to the Gaussian integers. Although they give no definitive answer, these results provide additional evidence that no such walk exists on either the Gaussian or Eisenstein integers. Finally, a brief discussion is given on the existence of prime walks in the fifth cyclotomic field $\mathbb{Z}[\zeta] = \{a+b\zeta+c\zeta^2+d\zeta^3 \,|\, a,b,c,d \in \mathbb{Z}\}$ which seem to indicate that a prime walk is possible in this field.

2. Angular sectors in $\mathbb{Z}[\omega]$

We begin our investigation of prime walks with a reformulation of two results by Po-Ru Loh. Loh established that any walk to infinity on the Gaussian primes must sweep out an angle of positive measure. Alternatively, this result shows that by restricting the size of an angular sector, we can prevent the presence of a prime d-walk, a walk with step-size no greater than d. We define $P_i(d, \alpha_0)$ to be the least common multiple of all Gaussian integers within distance d of α_0.

Theorem 2.1. [7] *Given a step-size d and an $\alpha_0 \in \mathbb{Z}[i]$, there exists a ϑ so that no d-walk on the Gaussian primes from α_0 exists which is contained within an angular sector centered at α_0 and having measure ϑ. Let $c = 3\sqrt{2}/8$ and $P = P_i(d, \alpha_0)$. Then*

$$\vartheta = 2\arcsin\left(\frac{c(d+1)^2}{|P|^2}\right) > 2c\frac{d^2}{|P|^2}$$

is such a value for ϑ.

The idea behind this theorem is that while a large region of composite numbers does nothing to prevent a walk in the plane, as one can just walk around it, such a region will prohibit a prime walk confined to an angular sector. A natural next step would be to take the union of sectors on which no prime walk exists. In particular, we will look at unions where each sector within the union has the same angular measure. We will denote the collection of all the unions which are centered at a given point α_0 by $\mathcal{C}$. However, not all unions will have the same common angular measure for its respective sectors.

Theorem 2.2. [7] *Given a step-size d and an $\alpha_0 \in \mathbb{Z}[i]$, there exists a union S among the collection $\mathcal{C}$ of unions of nonoverlapping angular sectors centered at α_0 such that S is a union of n sectors with angular measure φ, $n\varphi = 2\pi - \varepsilon$ for any $\varepsilon > 0$, and no d-walk on the Gaussian primes exists on S.*

As a result of the similarities which exist between the Gaussian and Eisenstein integers, each of these theorems has a natural analogue in the Eisenstein integers. The only difference in the first theorem is the value which is given to c. The second theorem remains exactly the same. Additionally, both proofs run almost exactly the same as in [7] for the Gaussian integers. My extensions of Theorem 2.1 and Theorem 2.2 are given below.

Theorem 2.3. *Given a step-size d and an $\alpha_0 \in \mathbb{Z}[\omega]$, there exists a ϑ so that no d-walk on the Eisenstein primes from α_0 exists which is contained within an angular sector centered at α_0 and having measure ϑ. Let $c = 3\sqrt{3}/8$ and $P = P_\omega(d, \alpha)$. Then*

$$\vartheta = 2 \arcsin\left(\frac{c(d+1)^2}{|P|^2}\right) > 2c \cdot \frac{d^2}{|P|^2}$$

is such a value for ϑ.

Proof. At the outset, let D be a closed disc of radius $R = 2d + 2$ centered at α and let P be a least common multiple of the integers contained in D. Further, we will be working with the lattice given by $L = \{\alpha_0 + xP \mid x \in \mathbb{Z}[\omega]\}$. Now, we will show that for $\gamma \in \mathbb{Z}[\omega]$, if γ is contained in D, $|\gamma| \neq 1$, and $x \neq 0$, then each $\gamma + xP$ will be composite.

To begin, we show that $|P/\gamma| > 2$. Now, if we have that $|P| \geq 6$, then we will have our result for $|\gamma| \leq 3$. To see that $|P| \geq 6$, we first observe that $R = 2d + 2 \geq 4$. As a result, we may fit a square with side ≥ 5 inside of D. Therefore, proper choice of a, b will yield an integer of the form $5a + 5b\omega$ within D. Similarly, we will also have an integer of the form $2a + 2b\omega$ which lies in D. Since these integers are in D, we know that the primes 2 and 5 must divide P and $|P| \geq 10 > 6$. It follows that $|P/\gamma| > 2$ for $|\gamma| \leq 3$.

We now assume that $|\gamma| > 3$ and show that at least one of $\gamma \pm 1, \gamma \pm \omega, \gamma \pm \omega^2$ is contained in D. To show this, we need to have that $|\alpha_0 - \gamma - v| \leq R$, for some unit v, whenever $|\alpha_0 - \gamma| \leq R$. It turns out that we may restrict our attention to $v = \pm\omega, \pm\omega^2$. Continuing, we write $\alpha = a + bi$ and $\gamma = c + di$ and let $m = a - c$ and $n = b - d$. We thus obtain the condition $|m + ni| \leq R$. As a result, we find that

$$|\alpha-\gamma-v| = \sqrt{\left(m \pm \frac{1}{2}\right)^2 + \left(n \pm \frac{\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{m^2+n^2 \pm m \pm n\sqrt{3}+1}.$$

Since $|\alpha-\gamma| \leq R$, we will be guaranteed that $|\alpha-\gamma-v| \leq R$ if $\pm m \pm n\sqrt{3} \leq -1$. For one of $|m|, |n| \geq 1$, we need only choose the proper unit, and thus the proper signs, to achieve the desired inequality. On the other hand, if both $|m|, |n| < 1$, then $\gamma + v$ will obviously be contained in the D. As a result of the fact that γ and v are relatively prime, both γ and $\gamma + v$ must divide P. Consequently, $|P/\gamma| \geq |\gamma + v| \geq |\gamma| - |v| > 2$.

Now that we have established $|P/\gamma| > 2$, we are in a position to prove that $\gamma + xP$ is composite. For $\gamma = 0$, the assertion is trivial. If $\gamma \neq 0$, then because $\gamma \in D$, $\gamma | P$ and

$$|\gamma + xP| = |\gamma|\left|1+\frac{xP}{\gamma}\right| \geq |\gamma|\left(\left|\frac{xP}{\gamma}\right|-1\right) \geq |\gamma|\left(\left|\frac{P}{\gamma}\right|-1\right).$$

But since $|P/\gamma| > 2$, the above inequality tells us that $|\gamma + xP| > |\gamma|$. It follows that $\gamma + xP = \beta\gamma$ for some $|\beta| > 1$ and that $\gamma + xP$ is composite for nonzero x. Thus, we have at most six Eisenstein primes contained in C, all of which are contained in a circle of radius 1 about γ.

Subsequently, we consider an angular sector Γ originating at α_0 and having measure $0 < \vartheta < 2\pi$. Further, define l to be the ray bisecting Γ and also originating at α_0. We claim that it is possible to ensure that l must pass near a point in the lattice L.

In order to achieve this, we transform our coordinate system through the application of a rotation by $-\text{Arg}\, P$, a translation by $-\alpha_0$, and a scaling by $1/|P|$. This transformation places α at the origin and places the unit vectors P and Pi along the x- and y-axes. We denote by Γ' and l' the images of Γ and l under this composition. Additonally, we write m for the slope of l' and $r = R/|P|$. Note that $r < 1$ since $R|P$ but R and P are not associates. As a result of the symmetry exhibited in $\mathbb{Z}[\omega]$, we take $0 \leq m \leq 1/\sqrt{3}$ and use the lemma which follows to approximate m. Taking k to be $4/(r\sqrt{3})$, we are guaranteed integers a, b satisfying both $1 \leq a < 4/(r\sqrt{3})$ and $|am - b| \leq r\sqrt{3}/4$.

Lemma 2.4. [7] *For any real numbers m and k with $k > 1$, there exist integers a and b such that $1 \leq a < k$ and $|am - b| \leq 1/k$.*

Proof. Since proving this result for $\lceil k \rceil$ implies the result for k, we restrict ourselves to $\lceil k \rceil$. That is, we only need consider the cases where $k \in \mathbb{Z}$. Denote $x - \lfloor x \rfloor$ by $\{x\}$ and consider the sequence $(s_i)_{i=1}^{k}$, where $s_i = \{(i-1)m\}$. From this sequence, we form the auxiliary sequence $s_2 - s_1, s_3 - s_2, \ldots, s_k -$

$s_{k-1}, 1 + s_1 - s_k$, the sum of which is 1. Using the pigeonhole principle, we see that there must be at least one term in our auxiliary sequence which is no greater than $1/k$. Consequently, there are j_1 and j_2, $0 \le j_1 < j_2 < k$, $|\{j_2 m\} - \{j_1 m\}| \le 1/k$. Let $a = j_2 - j_1$ and $b = \lfloor j_2 m \rfloor - \lfloor j_1 m \rfloor$. Then $1 \le a < k$ and

$$\begin{aligned} |am - b| &= |am - (\lfloor j_2 m \rfloor - \lfloor j_1 m \rfloor)| \\ &= |j_2 m - j_1 m - \lfloor j_2 m \rfloor + \lfloor j_1 m \rfloor| \\ &= |(j_2 m - \lfloor j_2 \rfloor m) - (j_1 m - \lfloor j_1 m \rfloor)| \le 1/k. \end{aligned}$$ □

Before continuing, we define X to be the point with coordinates (a, b), Y to be the point of intersection between l and the line $\operatorname{Re} z = a$, and l'_1, l'_2 to be the rays bounding Γ'. Owing to the fact that $\operatorname{Im} X = b$ while $\operatorname{Im} Y = am$, we find that the distance $XY \le r\sqrt{3}/4$. We will now show that $d(Y, l'_i) < r\sqrt{3}/4$. As we have noted, $m \le 1/\sqrt{3}$ and therefore the distance $\alpha Y \le 2a/\sqrt{3} < 8/(3r)$ due to the restriction on a. From this, we find that $d(Y, l'_i) < 8\sin(\vartheta/2)/(3r)$. Taking

$$\vartheta = 2 \arcsin\left(\frac{3\sqrt{3}(d+1)^2}{8|P|^2}\right),$$

we obtain

$$\begin{aligned} \sin\left(\frac{\vartheta}{2}\right) &= \frac{3\sqrt{3}(d+1)^2}{8|P|^2} \\ &= \frac{3\sqrt{3} \cdot 2^2 (d+1)^2}{8 \cdot 4|P|^2} \\ &= \frac{3\sqrt{3}}{32}\left(\frac{2d+2}{|P|}\right)^2 = \frac{3r^2\sqrt{3}}{32}, \end{aligned}$$

resulting in $d(Y, l'_i) < r\sqrt{3}/4$. It follows that

$$d(X, l'_i) \le XY + d(Y, l'_i) < \frac{r\sqrt{3}}{2}.$$

In the next place, denote by C the circle with radius r which is centered at X. Because $d(X, l_i) < r\sqrt{3}/2$, we know that each l_i subtends an arc of C of measure $> \arccos\sqrt{3}/2 = \pi/3$. Further, due to the fact that a, the x-coordinate of X, is > 1, while r is less than 1, C does not contain the origin and creates two regions within Γ' which are separated by a distance larger than r.

If we now return to our original coordinate system, we are guaranteed a moat of width $R = 2d + 2$ which contains at most six primes and separates Γ into two pieces. Because these six primes are all located on a circle of radius 1, we are guaranteed to have a region of width greater than d which contains no primes. Thus, no d-walk can exist on our angular sector giving us the desired result. □

Just as in the Gaussian integers, we proceed by taking unions of angular sectors in $\mathbb{Z}[\omega]$. The next theorem is our extension to the Eisenstein integers of Loh's result about angular sectors in the Gaussian integers.

Theorem 2.5. *Given a step-size d and an $\alpha_0 \in \mathbb{Z}[\omega]$, there exists a union S among the collection C of unions of nonoverlapping angular sectors centered at α_0 such that S is a union of n sectors with angular measure φ, $n\varphi = 2\pi - \varepsilon$ for any $\varepsilon > 0$, and no d-walk on the Eisenstein primes exists on S.*

Proof. Our goal is to demonstrate that for each ε, $0 < \varepsilon < 1$, we can find an S, consisting of n angular sectors of measure φ and on which no d-walk exists, so that $n\varphi/2\pi > 1 - \varepsilon$. Due to the symmetry of the lattice in $\mathbb{Z}[\omega]$, we may restrict ourselves to the interval $[0, \pi/6]$. In the first place, we will choose only among angular sectors for which we have a moat that is sufficiently far from α. By doing so, we may ensure that the separation between angular sectors at the point that a moat does arise is sufficiently large to prevent a step across angular sectors.

We will being applying Theorem 2.3 to walk with step-size d', and for this reason, we will use a prime notation for the conclusions of Theorem 2.3. We know that we will have a moat of width d' whenever we have a disc of radius $r' = 2d' + 2$ centered at the point $a' + b'\omega$. As a result of the symmetry which is exhibited by $\mathbb{Z}[\omega]$, we have the restrictions $0 \le b' \le a'/\sqrt{3} < 2a'/3$ and $a' < k' = 4/(r'\sqrt{3})$. In order to ensure that the desired moat is sufficiently far from α_0, we place a further restriction on a', the x-coordinate of the disc. In particular, we will not include angular sectors for which $a' < \varepsilon_1 k'$, where ε_1 is yet to be defined. Because the point a' is determined by the slope of the angular bisector, m', we use to place a restriction on a'. That is, we exclude slopes m' for which there exists an a', b' satisfying Lemma 2 with $a' < \varepsilon_1 k'$. To accomplish this, we rewrite our restriction as $|m' - b'/a'| \le 1/(a'k')$. Using this restriction, we can determine which values for m' within the interval $[0, 2/3]$ are admissible.

It turns out that we will be excluding subintervals from $[0, 2/3]$ for each value of a'. Because a' must be an integer, we will determine the lengths of the intervals which are excluded for each a'. For $a' = 1$, we must have that $b' = 0$ and thus $|m'| \le 1/(a'k')$. This results in the excluded interval $[0, 1/k']$, having length $1/k'$. Because any other (a', b') pair with $b' = 0$ will result in a subinterval of $[0, 1/k']$, we need not consider $b' = 0$ for $a' \ge 2$. Continuing, for each $a' \ge 2$, we may choose any b' from among $\{1, 2, \ldots, \lfloor 2a'/3 \rfloor\}$. Since each (a', b') pair will yield the excluded interval $[b'/a' - 1/(a'k'), b'/a' + 1/(a'k')]$, we obtain an excluded interval of length $2/(a'k')$. Thus, the combined length of excluded intervals for each a' is

$$< \frac{2a'}{3}\frac{2}{a'k'} = \frac{4}{3k'}.$$

Although we will have some repetition, for instance the interval corresponding to $(4,2)$ will be a subinterval of that corresponding to $(2,1)$, we will still be excluding a combined length of $< 4/(3k'$ for each $a' \geq 2$. Since the length of the interval for $a' = 1$ is $1/k'$, the total combined length is $< 4/(3k') \cdot \varepsilon_1 k' = 4\varepsilon_1/3$, as we are excluding intervals with $a' < \varepsilon_1 k'$.

In spite of the fact that we have been working with intervals of slopes while we are actually interested in angular intervals for $\arctan m'$, we are still guaranteed a total length of $< 4\varepsilon_1/3$. To see this, we note that

$$\frac{d \arctan m'}{dm'} = \frac{1}{(1+m'^2)} \leq 1$$

and thus the interval lengths will be decreased as we take the arctan. Setting $\varepsilon_1 = \varepsilon\pi/16$ yields at most $\varepsilon/2$ for the ratio of excluded intervals to the total interval $\pi/6$. Thus, we have at least $1 - \varepsilon/2$ of the original interval remaining from which to choose $\arctan m'$ and gurantee that $|a'| \geq \epsilon_1 k'$. This achieves our first objective.

We now wish to ensure that our angular sectors are separated by gaps of width at least $d/|P(d')|$ when we are at a distance of $\varepsilon_1 k'$ from α_0. This will be accomplished by using a large value for d' relative to d. To proceed, we wish to find an upper bound for the ratio of the size of a gap to the size of a sector. From Theorem 2.3, we know that a lower bound on the sector size is $\vartheta_s = c_1 d'^2/|P(d')|^2$. On the other hand, the gaps must have a width that is $d/|P(d')|$ at a distance of $\varepsilon_1 k'$. This yields the bound ϑ_g

$$2 \arcsin\left(\frac{d/(2|P(d')|)}{\varepsilon_1 k'}\right) < \frac{d}{|P(d')|\varepsilon_1 k'} = \frac{c_2 d r'}{|P(d')|\varepsilon_1} = \frac{c_2 d}{|P(d')|\varepsilon_1} \cdot \frac{2(d'+1)}{|P(d')|}$$
$$< \frac{c_3 d d'}{|P(d')|^2 \varepsilon_1} = \vartheta_g$$

on the size of the gap, and the upper bound

$$\frac{\vartheta_g}{\vartheta_s} = \frac{c_3 d d'}{|P(d')|^2 \varepsilon_1} \left(\frac{c_1 d'^2}{|P(d')|^2}\right)^{-1} = \frac{d}{d'} \cdot \frac{c_4}{\varepsilon_1}$$

on the ratio of gap to sector size. This ratio can be made less than $\varepsilon/2$ for each value of d by choosing d' sufficiently large.

We now create our union of angular sectors. We select angle bisectors from $[0, \pi/6]$ to place in our set S_b. Each bisector that we add removes at most $\vartheta_s + \vartheta_g$ from what remains of $[0, \pi/6]$. At the same time, each bisector adds a measure of at least ϑ_s to our set S. Due to the fact that we begin with $1 - \varepsilon/2$ of $[0, \pi/6]$ to choose from, we end up with sectors comprising at least

$$\left(1 - \frac{\varepsilon}{2}\right) \frac{\vartheta_s}{\vartheta_s + \vartheta_g} > \left(1 - \frac{\varepsilon}{2}\right) \frac{1}{1 + \varepsilon/2} > 1 - \varepsilon$$

of the interval $[0, \pi/6]$. □

3. *A walk on the complex primes*

J. Haugland [6] formulated a heuristic argument that a prime walk in $\mathbb{Z}[\omega]$, $\omega = e^{\pi i/3}$, does not exist. Although not a formal proof, it seems likely that this argument would hold and thus show that no prime walk exists. In this section, we present the argument and work out the specific details. The argument is intended to show the likelihood of the hypothesis in Theorem 3.1, that is, for any M there is a T satisfying statement (1) below. We denote the norm of a $\gamma \in \mathbb{Z}[\omega]$ by $N(\gamma)$. For $\gamma = a + b\omega$, $N(\gamma) = \gamma\overline{\gamma} = a^2 + ab + b^2$.

Theorem 3.1. *If for any M, there exists a positive integer T such that*

$$\begin{gathered} \textit{If } (\beta_j)_{j=1}^{\infty} \textit{ is a sequence of Eisenstein integers} \\ \textit{for which } \lim|\beta_j| = \infty \textit{ and } \gcd(N(\beta_j), T) = 1 \textit{ for all } j, \\ \textit{then } |\beta_{j+1} - \beta_j| \geq M \textit{ for infinitely many } j. \end{gathered} \tag{1}$$

then for any infinite sequence of primes $(\psi_i)_{i=1}^{\infty}$, $\psi_i \in \mathbb{Z}[\omega]$, the difference sequence $\Delta\psi_i = |\psi_{i+1} - \psi_i|$ is unbounded.

If such a T exists, this theorem would then be applied to the concept of a prime walk by letting M be a bound on step-size. Then for any step-size, any sequence which satisfies the hypothesis of statement (1) above must contain infinitely many steps of size greater than M. This implies that every prime sequence, and thus every prime walk, has an unbounded difference sequence.

Proof. Assume the above hypothesis and let our sequence (β_j) be composed of primes $\psi \in \mathbb{Z}[\omega]$ such that $\lim|\beta_j| = \infty$ and $N(\beta_j) > T^2$ for all j. Because our sequence consists only of primes ψ, $N(\beta_j) = p$ or p^2, for some rational prime $p > T$ ([5] Proposition 9.1.2, pg. 110). In either case, $\gcd(N(\beta_j), T) = 1$ and (β_j) satisfies the given two conditions. Thus $|\beta_{j+1} - \beta_j| \geq M$ for infinitely many j.

Any sequence of primes (ψ_i) for which $\lim|\psi_i| = \infty$ must have a member from which point on $N(\psi_i) > T$. From this member on, (ψ_i) is a sequence satisfying the hypothesis of statement (1). There must then be a ψ_i for which $|\psi_{i+1} - \psi_i| \geq M$. This can be accomplished for any M and the difference sequence $\Delta\psi_i = |\psi_{i+1} - \psi_i|$ must be unbounded. □

The important piece of (1) which we will be working with is the requirement that $\gcd(N(\gamma), T) = 1$. What is important about these γ is that they exhibit a large amount of symmetry in $\mathbb{Z}[\omega]$ which we will be exploiting. In particular, we will consider the triangle, which we will denote by $\mathcal{T}$, formed from the line segments stretching from the origin to T, T to $T\omega$, and $T\omega$ back to the origin. We will now show exactly how this symmetry will help us.

Lemma 3.2. *The distribution of γ with $\gcd(N(\gamma), T) = 1$ is symmetric with respect to the transformations $\gamma \to \gamma\omega$, $\gamma \to \gamma + T$, and $\gamma \to \overline{\gamma}$.*

Proof. What is important is that for $\gamma = a + b\omega$, $N(\gamma) = a^2 + ab + b^2$, and we have

$$N(\gamma\omega) = (-b)^2 + (-b)(a + b) + (a + b)^2 = a^2 + ab + b^2 = N(\gamma)$$
$$N(a + b\omega + T) = a^2 + 2aT + T^2 + ab + abT + b^2 \equiv N(a + b\omega) \pmod T$$
$$N(\overline{\gamma}) = (a + b)^2 + (a + b)(-b) + (b)^2 = a^2 + ab + b^2 = N(\gamma)$$

Further, because $N(\gamma) - (N(\gamma) - T) = T$, $\gcd(N(\gamma), T) = \gcd(N(\gamma) - T, T)$. ([3] pg. 7, Proposition 1.3) Thus, the gcd function is unchanged under the transformations $\gamma \to \gamma\omega$, $\gamma \to \gamma + T$, and $\gamma \to \overline{\gamma}$. □

On account of this lemma, we see that we can rotate $\mathcal{T}$ by $\pi/6$, reflect $\mathcal{T}$ over the real axis, and translate $\mathcal{T}$ to the left or right by T without changing the arrangement within $\mathcal{T}$ of γ for which $\gcd(N(\gamma), T) = 1$. Thus, the plane consists entirely of copies of $\mathcal{T}$ and thus any result established for $\mathcal{T}$ is applicable to the entire plane. In particular, we will use the fact the density of the γ for which $\gcd(N(\gamma), T) = 1$, that is, the ratio of the number lattice points with norm relatively prime to T to the total number of lattice points, may be made arbitrarily small.

Proposition 3.3. *Given any $\epsilon > 0$, there exists a positive integer T such that the density of integers $\gamma \in \mathbb{Z}[\omega]$ for which $\gcd(N(\gamma), T) = 1$ is less than ε.*

Proof. We now wish to determine the density of $\gamma \in \mathcal{T}$ which have a norm relatively prime to T. We will achieve this result by establishing the density of $\gamma \in \mathcal{T}$ for which $\gcd(N(\gamma), T) \neq 1$. Because the $\gcd(N(\gamma), T) \neq 1$ if and only if there is some rational prime p which divides T satisfying $\gcd(N(\gamma), p) \neq 1$, we restrict our attention to $\gcd(N(\gamma), p)$

Let $p \equiv 1 \pmod 6$ be a rational prime and note that for some prime $\psi \in \mathbb{Z}[\omega]$, $p = \psi\overline{\psi}$ ([5] Proposition 9.1.4, pg. 110). For $\gcd(N(\gamma), p) \neq 1$, we have that $N(\gamma) = px$ for some positive integer x. Then $\psi\overline{\psi}x = \gamma\overline{\gamma}$ and either $\psi|\gamma$ or $\overline{\psi}|\gamma$.

Subsequently, denote by S the set of all $\gamma \in \mathbb{Z}[\omega]$ with $N(\gamma) \leq T$ and denote by S' the set consisting of $\gamma \in \mathbb{Z}[\omega]$ satisfying $N(\gamma) \leq T/p$. As we have seen, if $\gcd(N(\gamma), p) \neq 1$, then either $\psi|\gamma$ or $\overline{\psi}|\gamma$. Each member of S not relatively prime to ψ will be of the form $\psi\alpha$, $\alpha \in S'$.

What is important is that S and S' are the ellipses given by

$$\frac{1}{T}(x^2 + xy + y^2) = 1,$$
$$\frac{p}{T}(x^2 + xy + y^2) = 1,$$

respectively and the sets cover areas of $\pi T\sqrt{4/3}$ and $\pi(T/p)\sqrt{4/3}$ ([8] Equation 92). We now show that the number of lattice points contained in S and S' is asymptotic with the area of the sets. We cover each lattice point with a square of side 1. Then the total area covered by the squares is equal to the number of lattice points covered. Now, we know that the whole area of the squares is not quite contained in the ellipse as some of the squares will both inside of and outside of the ellipse. However, if we decrease the length of our major axis by $\sqrt{2}$, the we will have none of these partial squares in the ellipse. On the other hand, increasing the major axis by $\sqrt{2}$ will include all of the partial squares. Thus, the area of the squares, and thus the number of lattice points, lies between the area of the contracted ellipse, $\pi(T-\sqrt{2})b$, and the extended ellipse, $\pi(T+\sqrt{2})b$, where $2b$ is the length of the minor axis. Taking the ratio of these values with the original area yields

$$\frac{\pi(T-\sqrt{2})b}{\pi Tb} \leq \text{Area of squares} \leq \frac{\pi(T+\sqrt{2})b}{\pi Tb}.$$

Further, letting T approach infinity and substituting the number of lattice points for the area of the squares, we arrive at the inequalities

$$\lim_{T\to\infty}\left(1-\frac{\sqrt{2}}{T}\right) < \lim_{T\to\infty}\frac{\#\text{ of lattice points}}{\text{Area of ellipse}} < \lim_{T\to\infty}\left(1+\frac{\sqrt{2}}{T}\right).$$

Both the left and right hand limits are equal to 1 and thus the number of lattice points is asymptotic to the area of the ellipse.

We now show that for each prime p which divides T, the density of γ which are relatively primes is $(1-1/p)^2$. First, the ratio of the areas of the ellipses, 1/p, will be the density of γ which are not relatively prime to ψ. Thus, the density of γ which are relatively prime to ψ is $1-1/p$.The same is true for $\overline{\psi}$ and as a result the γ which are relatively prime to both ψ and $\overline{\psi}$ will have density $(1-1/p)^2$. That is, the γ for which $\gcd(N(\gamma),p)=1$ have density $(1-1/p)^2$. Because we are interested in the general property $\gcd(N(\gamma),T)=1$ rather than $\gcd(N(\gamma),p)=1$, we must determine the density of γ which are relatively prime to a composite number T. This density will be equal to the product of the densities of γ such that $\gcd(N(\gamma),p)=1$ for each $p|T$. For this reason, we will examine the product of densities taken over primes $p\equiv 1 \pmod 6$, noting that the primes $p\equiv 5 \pmod 6$ all lie on the real line.

Further taking the log of this product, we find that

$$\begin{aligned}\log\Big(\prod_{\substack{p\equiv 1 \,(\mathrm{mod}\ 6)\\ p \text{ prime}}}(1-\frac{1}{p})^2\Big) &= \sum_{\substack{p\equiv 1 \,(\mathrm{mod}\ 6)\\ p \text{ prime}}} 2\log(1-\frac{1}{p}) \\ &= \sum_{\substack{p\equiv 1 \,(\mathrm{mod}\ 6)\\ p \text{ prime}}} -2(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}\cdots) \qquad [2] \\ &< \sum_{\substack{p\equiv 1 \,(\mathrm{mod}\ 6)\\ p \text{ prime}}} -\frac{2}{p}\end{aligned}$$

Combining this result with a theorem of Dirichlet, which says that that $\sum 1/p$, taken over the primes $p \equiv 1 \pmod 6$, diverges ([4], pg. 34), we find that $\prod(1-1/p)^2 \to e^{-\infty} = 0$. The density of γ which satisfy $\gcd(N(\gamma), T)$ can be made arbitrarily small by taking the product over sufficiently many primes. For this reason, we let $T = n!$ for a sufficiently large n. □

What we are trying to do in this argument is show that there are large regions which are free of any primes. We have established that in a large enough triangle $\mathcal{T}$, we can ensure that our γ are sufficiently spread out. The next step is to look at the integers which are close to each γ.

Proposition 3.4. *Asymptotically, there are $\pi r^2\sqrt{4/3}$ lattice points within a distance r of a given lattice point.*

Proof. Consider $\overline{D}(\gamma;\sqrt{s})$, the closed disc of radius $\sqrt{s}$ centered at γ. We will determine the longterm average for the number of lattice points which lie on a circle of given radius. Let the number of lattice points contained within the disc $\overline{D}(\gamma;\sqrt{1})$ be a_1. Further, if we denote by a_{n+1} the number of lattice points which are in $\overline{D}(\gamma;\sqrt{n+1}) - \overline{D}(\gamma;\sqrt{n})$, the number of lattice points in $\overline{D}(\gamma;\sqrt{s})$ is $\sum_{i=1}^{s} a_i$.

In order to proceed, center a hexagon with inradius 1/2 at each lattice point in the disc. These hexagons will have area $\sqrt{3}/2$ ([9] Equation 4), and because each hexagon corresponds to a unique lattice point, the area covered by the hexagons will be equal to the number of lattice points in the disc multiplied by $\sqrt{3}/2$. Near the edges of $\overline{D}(\gamma;\sqrt{s})$, there will be hexagons which extend outside of the disc. Because the inradius of the hexagons is 1/2, we find that the circumradius is $1/\sqrt{3}$ and two points in a hexagon are within a distance of $2/\sqrt{3}$. Thus, increasing the radius of the disc by $2/\sqrt{3}$ will ensure that all partial hexagons within the original disc will be completely contained within the enlarged disc, $\overline{D}(\gamma;\sqrt{s}+2/\sqrt{3})$. Similarly, decreasing the radius by $2/\sqrt{3}$ will result in a disc which completely excludes any partial hexagons within $\overline{D}(\gamma;\sqrt{s})$. The total area of hexagons within $\overline{D}(\gamma;r)$ must lie between the areas of the enlarged and contracted discs, $\pi(s - 4\sqrt{s}/\sqrt{3} + 4/3)$ and $\pi(s + 4\sqrt{s}/\sqrt{3} + 4/3)$.

We observe that the area of the hexagons is equal to $(\sqrt{3}/2)\sum_{i=1}^{s} a_i$, while the area of the original disc is πs. Using this, we find that the ratio of hexagon area to the area of the original disc must satisfy

$$\frac{\pi}{\pi s}\left(s - \frac{4\sqrt{s}}{\sqrt{3}} + \frac{4}{3}\right) < \frac{\sqrt{3}}{2\pi s}\sum_{i=1}^{s} a_i < \frac{\pi}{\pi s}\left(s + \frac{4}{\sqrt{3s}} + \frac{4}{3}\right).$$

In particular, as the radius $\sqrt{s} \to \infty$, and consequently s as well, the above inequalities become

$$\lim_{\sqrt{s}\to\infty}\left(1 - \frac{4}{\sqrt{3s}} + \frac{4}{3s}\right) < \lim_{\sqrt{s}\to\infty}\left(\pi s\sqrt{\frac{4}{3}}\right)^{-1}\sum_{i=1}^{s} a_i < \lim_{\sqrt{s}\to\infty}\left(1 - \frac{4}{\sqrt{3s}} + \frac{4}{3s}\right).$$

However, both the left and right hand limits are equal to 1 and the middle limit must be as well. In other words, asymptotically, there are $\sum_{i=1}^{r^2} a_i = \pi r^2\sqrt{4/3}$ lattice points in a circle of radius r. (See [3] Exercise 9.11, pg. 300) □

All of these pieces are then put together into a coloring argument. In the first place, the γ for which $\gcd(N(\gamma), T) = 1$ are colored red, while the rest on $\mathbb{Z}[\omega]$ are colored white. From the white numbers, we pick out all numbers which are at least $r = (1/2)\sqrt{m^2+1}$ from any red numbers and color them blue. The remaining white numbers, as well as the red numbers, are then colored yellow.

What we obtain is a yellow disc of radius r centered at each γ. As we have seen, there are $A \approx \pi r^2\sqrt{4/3}$ lattice points in each disc. Then the density of blue numbers will be $1 - Ad$, where d is the density of red numbers. As noted, we can make d arbitrarily small and thus bring the density of blue numbers as close to 1 as we would like.

In our triangle $\mathcal{T}$, we create an equivalence relation. Let $\gamma_1 \sim \gamma 2$ if we have a monochromatic chain of β_j with endpoints γ_1 and γ_2, such that $|\beta_{n+1} - \beta_n| = 1$.

Lemma 3.5. *There is at least one equivalence class which touches all three edges of the triangle with vertices at* $(0, T, T\omega)$.

Proof. Assume, to the contrary, that there are no equivalence classes which touch all three edges of the triangle $\mathcal{T}$. There must be an equivalence class touching at least two edges, as any of the three corners satisfies this property. As a result of the symmetry exhibited by $\mathcal{T}$, given a class touching two given edges, there will be a copy of the class touching any other two edges. Due to this and the fact that the classes cannot cross, it makes sense to talk about the equivalence class closest to a third edge. Let C, which we will arbitrarily color yellow, be the equivalence class touching two edges and closest to a third. Denote by L_1, L_2 the edges which C touches and by L_3 the edge which C does not touch.

Let x_1, x_2 be the points of C contained in L_1, L_2, respectively, which are closest to L_3. Similarly, let y_1, y_2 be the blue points which are one unit further along L_1, L_2 than x_1, x_2. We now partition the vertices into two sets: $V_1 = C$ and $V_1^C = \mathcal{T} - C$. Then the edge set $E = E(V_1, V_1^C)$, which contains all edges between a vertex of V_1 and a vertex of V_1^C, will separate V_1 from V_1^C. Observe that all edges must connect a yellow vertex to a blue vertex. Define the set V_2 to be the set of blue vertices to which an edge $e \in E(V_1, V_1^C$ is incident. Note that since $G[\mathcal{T} - V_2]$ contains no edges $e \in E(V_1, V_1^C)$, V_2 is a separating set. From V_2 we select a set V_2' such that $y_1, y_2 \in V_2'$ and V_2' is minimal separating set of V_1, V_1^C.

We now form the subgraph $H = G[V_2]$ which is either connected or not. If H is connected, then we have found a blue class extending from y_1 to y_2

which separates C from L_3. For this reason, we assume that H is not connected. Because H is minimal, each vertex in H may have degree at most two. Because H is disconnected, as we traverse H beginning at y_1 there must be a first vertex u which has degree one. What is important is that in G each vertex surrounding u must either be in V_1, V_2', or $\mathcal{T} - V_1 - V_2'$. We know that one vertex, v is a member of H. If all of the remaining vertices were contained in V_1, $V_2' - \{u\}$ would be a separating set since these vertices are connected to the same vertices in $G[\mathcal{T} - V_2']$ as $G[\{u\} \cup \mathcal{T} - V_2']$. This fact is in contradiction to the assumption that V_2' was minimal.

Thus, beginning with v and moving counterclockwise around u, we must come across a first vertex, v_1 which is not a member of V_1 and a last vertex v_n which is not a member of V_1. Neither the point preceding v_1 nor the point succeeding v_n may be contained in $V_1^C - V_2'$. For this reason, they must both be in V_2', and, since the only adjacent vertex contained in V_2' is v, the two points must coincide. But then all of the vertices around u which are not v must be members of $V_1^C - V_2'$. Just as with V_1, this cannot happen and we have reached a contradiction. H must be connected. □

Lemma 3.6. *There is at most one equivalence class which touches all three edges of the triangle with vertices at* $(0, T, T\omega)$.

Proof. We begin by assuming that there is more than one equivalence class which touches all three edges. Let C_0 be a class which touches all three edges and let $C_0' \subseteq C_0$ be a minimal subset. That is, let C_0' be such that C_0' touches each edge, but no proper subset of C_0' touches all three edges. Because C_0' touches all three sides and is a minimal subset, it has at least two endpoints, x_1 and x_2 which are on distinct edges, E_1 and E_2. Now, we continue C_0' outside of the triangle $\mathcal{T}$ so that we form a simple closed curve Γ_0 containing the path from x_1 to x_2 contained in C_0'. Now, because we have at least two equivalence classes which touch all three edges, we consider a second equivalence class C_1, as well as a minimal subset C_1'. Further, we choose points y_1 and y_2 which lie on E_1 and E_2 and continue our class into a simple closed curve Γ_1, which does not intersect Γ_0, as above.

One of the curves Γ_0 and Γ_1 must contain a point which is closer to E_3 than any point on the other curve. Without loss of generality, we will assume that Γ_1 is closer. Because the curves cannot intersect, Γ_0 cannot cross Γ_1 at any point. Any chain from E_3 to Γ_0 must cross Γ_1 an odd number of times. Thus, by the Jordan Curve Theorem, these two points must be in unique components of the complement of Γ_1. (See [1], pg. 334) Thus, Γ_0 does not touch all three edges and we have a contradiction. □

Proposition 3.7. *There is exactly one equivalence class which touches all three edges of the triangle with vertices at* $(0, T, T\omega)$.

Because we are able to ensure that the density of blue numbers is arbitrarily close to 1, it would make sense that the equivalence class which touches all three edges is blue. If we assume that this is true, then we are able to show that there does not exist a prime walk in $\mathbb{Z}[\omega]$. As we have seen, reflections, rotations, and translations do not alter the distribution of yellow and blue numbers.

Theorem 3.8. *If a blue equivalence class touches all three edges of $\mathcal{T}$, then one cannot walk to infinity on the yellow vertices.*

Proof. We begin by coloring one side of $\mathcal{T}$ purple, one side green, and one side orange as in Figure 1. Now, let $\mathcal{T}'$ be the result of rotating $\mathcal{T}$ through an angle of $2\pi/3$. If we further translate $\mathcal{T}$ by T, the purple line will coincide with the green line, the green line with the orange line, and the orange line with the purple line. However, we know that these translations do change the distribution of γ with norm relatively prime to T. Thus, these lines must be colored identically in our blue and yellow coloring. It follows that the blue classes touching all three sides in adjacent triangles will line up and be connected to one another. In this way, we can tile the plane with copies of $\mathcal{T}$ and end up with a very large blue equivalence class which surrounds all of the yellow classes. Because each prime, other than the finitely many less than n, is surrounded by yellow numbers to a distance r, the distance between two yellow classes is at least $\sqrt{4r^2 - 1} = M$. Thus, one can not step across the large blue equivalence class from one yellow class to another. □

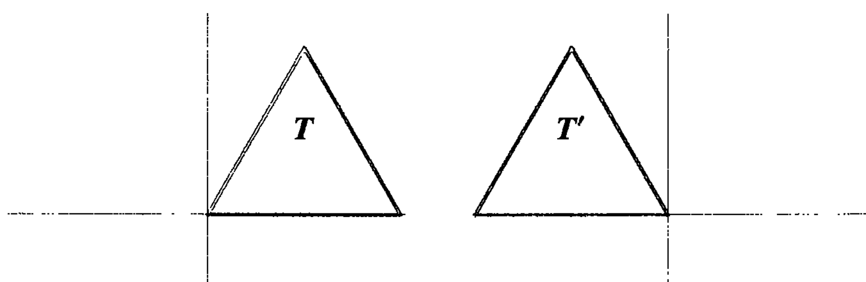

Figure 1: Triangle $\mathcal{T}$ and its image $\mathcal{T}'$ before a translation by T

4. Another walk on the complex primes

We will once again exploit the similarities which exist between $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, this time in order to extend Haugland's argument to $\mathbb{Z}[i]$. Just as in our transition from Theorem 2.1 and Theorem 2.2 to Theorem 2.3 and Theorem 2.5, we can apply the same basic ideas to $\mathbb{Z}[i]$ as we applied to $\mathbb{Z}[\omega]$. The only things that change are the numerical details and for this reason, we present only the relevant portions of the argument.

Lemma 4.1. *The distribution of γ with* $\gcd(N(\gamma), T) = 1$ *is symmetric with respect to the transformations $\gamma \to \gamma i$, $\gamma \to \gamma + T$, and $\gamma \to \overline{\gamma}$.*

Proof. We wish to consider the distribution of $x \in \mathbb{Z}[i]$ for which $\gcd(T, N(x)) = 1$. In the first place,

$$N(xi) = (-b)^2 + (a)^2 = b^2 + a^2 = N(x)$$
$$N(a + bi + T) = (a+T)^2 + (b)^2 = a^2 + 2aT + T^2 + b^2 \equiv N(a+bi) \pmod T$$
$$N(\overline{x}) = (a)^2 + (-b)^2 = a^2 + b^2 = N(x).$$

Further, these transformations, $\{x \to xi, x \to x + T, x \to \overline{x}\}$, generate the group of rigid transformations stabilizing $T\mathbb{Z}[i]$. Consequently, we may restrict our attention to the triangle with vertices $0, T, T + Ti$. □

Proposition 4.2. *Given any $\epsilon > 0$, there exists a positive integer T such that the density of integers $\gamma \in \mathbb{Z}[\omega]$ for which* $\gcd(N(\gamma), T) = 1$ *is less than ε.*

Proof. We now wish to show that the density of $x \in \mathbb{Z}[i]$ for which $\gcd(T, N(x))$ may be made arbitrarily small. In order to begin, we first note that for $p \equiv 1 \pmod 4$, p is representable as a sum of two squares by Theorem 9.1.[3] $\therefore$, $p = a^2 + b^2 = N(\alpha)$ for some $\alpha \in \mathbb{Z}[i]$. Moreover, $N(\alpha) = \alpha\overline{\alpha}$ and as a result $p = \alpha\overline{\alpha}$ is not prime in $\mathbb{Z}[i]$. However, by Thm 9.9, α, as well as $\overline{\alpha}$, is a prime.[3] In conclusion, we find that if $p \equiv 1 \pmod 4$, then in $\mathbb{Z}[i]$, p is the product of a prime and its conjugate, both of whose norms $= p$.

Subsequently, let us consider the set of elts whose norm is $\leq T$ and write $p = \psi\overline{\psi}$. If $x = \gamma\psi$, then $N(x) = N(\gamma)N(\psi) = N(\gamma)p$ and $\gcd(p, N(x)) \neq 1$. The same result holds for $\overline{\psi}$. On the other hand, if $\gcd(p, N(x)) \neq 1$, $N(x) = ap$ and x is divisible by either ψ or $\overline{\psi}$. Thus, $\gcd(p, N(x)) \neq 1$ iff $x = \gamma\psi$ or $x = \gamma\overline{\psi}$ for some $\gamma \in \mathbb{Z}[\omega]$. With this in mind, we observe that each element with norm $\leq T$ which is a multiple of ψ arises as the product of ψ with an element of norm $\leq T/p$. Because the area of the circle containing the elements with norm $\leq T/p$ is $\pi(\sqrt{T/p})^2$, while the circle with radius T has area $\pi(\sqrt{T})^2$, we find that the ratio of multiples of ψ to total elts is $1/p$. We find the same result for the multiples of $\overline{\psi}$. Thus, the density of x with $\gcd(p, N(x)) = 1$ is $(1 - 1/p)^2$. The result follows in the same way as for Proposition 3.3. □

Theorem 4.3. *If a blue equivalence class touches all three edges, then one cannot walk to infinity on the yellow vertices.*

Proof. For $\mathbb{Z}[i]$, we create the triangle $\mathcal{T}_0$ formed by the line segments from 0 to T, from T to $T/2 + Ti/2$, and from $T/2 + Ti/2$ back to 0. In order to continue, we assume that we have a blue class C touching the three edges of our triangle $\mathcal{T}_0$. Before proceeding, we denote the edge connecting 0 to T by E_1, the edge from T to $T/2+Ti/2$ by E_2, and the remaining edge E_3. We now create the following images:

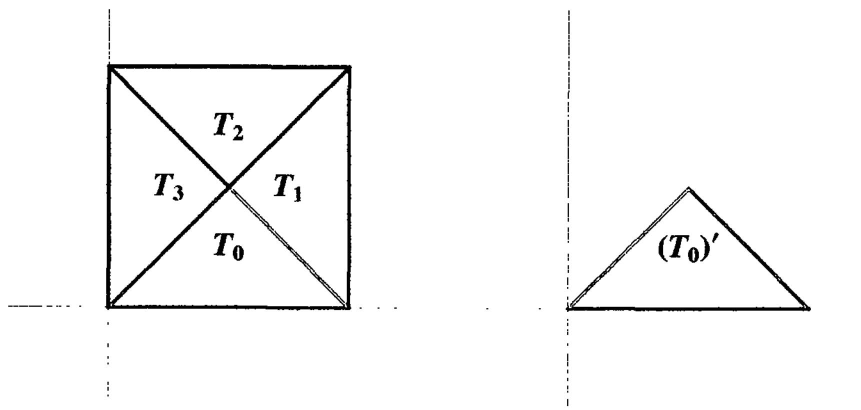

Figure 1: (left) The images $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ formed from $\mathcal{T}_0$

Figure 2: (right) A transformation showing that E_2 and E_3 must coincide

(1) $\mathcal{T}_1$ is the image of $\mathcal{T}_0$ under multiplication by i and addition by T,

(2) $\mathcal{T}_2$ is the image of $\mathcal{T}_1$ under multiplication by i and addition by T,

(3) $\mathcal{T}_3$ is the image of $\mathcal{T}_0$ under conjugation and multiplication by i.

Taking $\mathcal{T}' = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, we have a square composed of four triangles. Observe that the image of C contained in each triangle will match up along the internal edges of the square and will also touch the external edges in the same relative positions. To see this, we begin with $\mathcal{T}_0$, multiplying twice by i, conjugating, and translating by T. This serves to swap edges E_2 and E_3. But these transformations do not effect the arrangement of γ such that $\gcd(N(\gamma), T) = 1$. Thus, the blue equivalence class must touch E_2 in the same place as it touches E_3 so that the images will line up properly. For this reason, we may continue the blue equivalence class from $\mathcal{T}_0$ into the other three triangles. In the same way, we have a blue equivalence class C' which touches all four edges of the square in the same places. We may now apply the appropriate transformations to obtain a tiling of the plane with such squares. As noted, the class C' will touch the edges of each square in the same relative position and thus we will obtain a blue equivalence class which surrounds all yellow numbers. □

5. Higher cyclotomic fields

We conclude with an investigation of higher cyclotomic fields. In particular, we will focus on $\mathbb{Z}[\zeta]$, where $\zeta = e^{2\pi i/5}$, although the same general idea extends to the other cyclotomic fields. Once we get beyond the Eisenstein and Gaussian integers, the higher cyclotomic fields take on a much different structure. They no longer have the nice lattice shape of $\mathbb{Z}[\omega]$ and $\mathbb{Z}[i]$, due to the greater flexibility allowed in taking linear combinations of the roots of unity. In fact, I show that integers in $\mathbb{Z}[\zeta]$ are dense in $\mathbb{C}$. As we will see, the lack of lattice structure seems to admit prime walks into its structure.

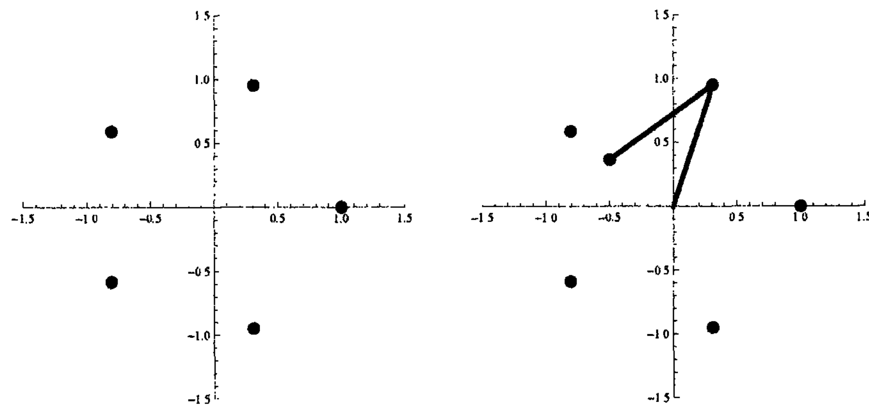

Figure 1: $\zeta + \zeta^3$ breaks from the lattice established by the fifth roots of unity.

Proposition 5.1. *$\mathbb{Z}[e^{2\pi i/5}]$ is dense in $\mathbb{C}$.*

Proof. We are able to exploit an identity involving the Fibonacci numbers F_n. First, we find

$$\begin{aligned}\phi^{-1} &= \frac{\sqrt{5}-1}{2} = 2\cos\left(\frac{8\pi}{5}\right)\\ &= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) - i\sin\left(\frac{8\pi}{5}\right)\\ &= e^{8\pi i/5} + e^{-8\pi i/5} = e^{8\pi i/5} + e^{2\pi i/5} = \frac{1+e^{6\pi i/5}}{e^{-2\pi i/5}}.\end{aligned}$$

Thus, we have the equality $\phi^{-1} = (1+e^{6\pi i/5})/e^{-2\pi i/5}$. But then we know that assymptotically, ϕ^{-1} is F_{n-1}/F_n. That is,

$$\lim_{n\to\infty} F_n(1+e^{6\pi i/5}) = \lim_{n\to\infty} F_{n-1}e^{-2\pi i/5}.$$

And from this, we find

$$\lim_{n\to\infty}\left(F_n - F_n e^{\pi i/5} + F_{n-1}e^{3\pi i/5}\right) = 0.$$

It follows that for any $\gamma \in \mathbb{Z}[\zeta]$ and any $\varepsilon > 0$, there exists an n such that $|\gamma - (\gamma + F_n - F_n\zeta + F_{n-1}\zeta^3)| < \epsilon$. □

Although we do not possess a proof, it seems plausible that the other cyclotomic fields would be dense as well. If this is the case, then it seems a further possibility that the primes may be dense in these fields, and, consequently, that a prime walk of any step size would be admissible. I have found some computational evidence that this is the case in $\mathbb{Z}[\zeta]$. Using the above proposition, we see the the integers

$$(a + F_n) + (b - F_n)\zeta + c\zeta^2 + (d + F_{n-1})\zeta^3 \tag{1}$$

are dense around $a+b\zeta+c\zeta^2+d\zeta^3$. Thus, if (1) is prime for infinitely many n, then we will have a dense set of primes around $a+b\zeta+c\zeta^2+d\zeta^3$. In particular, I have found at least three such n for all integers with coordinates less than 10. Most of the integers have many more n for which (1) is prime. Although this is by no means solid evidence of a prime walk in $\mathbb{Z}[\zeta]$, it does seem to indicate that one may be possible.

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