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A Model Representation for the Symmetric Group and the Partition Algebra

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A Model Representation of the Symmetric Group and the Partition Algebra

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Abstract

Combinatorics is the art of counting, how many such objects are there. Algebra deals with how objects can interact. Representation theory sits between the two. In particular, it uses combinatorial techniques to prove algebraic questions. Herein I use it to derive information about the symmetric group, S_n, by proving a combinatorial identity. In mathematics though, we always seek the strongest possible theorem, the broadest result. Thus it is natural to consider here not only S_n, but also several other related diagram algebras. The conjecture and part, but not all, of the proof will generalize.

After introducing the necessary definitions, background, and tools, the second chapter, the heart of the paper, contains a proof of the main conjecture. The two main objects of study are the Roichman and Saxl weights, which are functions on permutations. In particular, I prove that the sum of the two weights over all symmetric permutations, those which are their own inverse, is equal and give an explicit calculation for it. Since the weights are always $+1$, -1, or 0, much of the proof will be done by finding and cancelling pairs of elements with opposite sum. This part is aimed at a general audience and requires little background. Once the central result has been proven, I turn to the deeper algebraic background of the problem and explain why it is important. In particular, this result allows us to construct a model which tells us a great deal about S_n.

This leads us naturally to consider the general case, where we state the version of the conjecture. In order to deal with it, we will need several new concepts. These and their properties of which will be introduced next, comparing the general ones to how they specialize for the symmetric case; indeed these will often be identical. We can then give partial results toward its proof. While I am unable to prove the entire result, I solve a special case and point out possible approaches to finish the problem.

This work arose out of a summer research project with my advisor, Tom Halverson, and was continued during the 2005-06 academic year. I am indebted to him for everything, to the NSF for supplying funding, and to the entire math department at Macalester for instilling in me a passion for mathematics and the ability to pursue it.

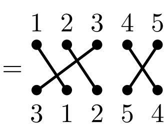
Contents

Chapter 1

Basic Objects

1.1 The Symmetric Group

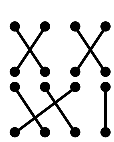
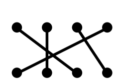
A *permutation* is a rearrangement of the numbers 1 to n. For our purposes we most often view them as diagrams, i.e.

$\sigma =$ 

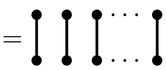
We can simply write σ as $3\,1\,2\,5\,4$, which is called one-line notation because it just reads off the order of the strands. A *cycle* $(p_1, p_2, \ldots p_k)$ is the permutation which sends p_1 to p_2, p_2 to p_3, etc., and finally p_k gets sent to p_1. Two cycles are disjoint if they do not involve any of the same elements. Another way to describe a permutation then is to write it as a product of disjoint cycles. Thus the diagram above would be (123)(45).

For each vertex i we say that $\sigma(i)$ is where i is sent by σ, so above $\sigma(1) = 2$ and $\sigma(3) = 1$. Any vertex which is sent to itself is called a *fixed point* and $\mathrm{fp}(\sigma)$ is the number of fixed points in σ.

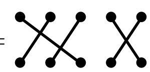
The *symmetric group*, S_n, is the set of all permutations of the elements $\{1, \ldots, n\}$ under the operation of function composition. Composition is done by simply stacking one diagram on top of another and connecting each path from the top to bottom row.

$2134 \cdot 2314 =$  $=$  $= 3241$

Under this operation as the name implies, we do indeed have a group, which means that the operation of composition satisfies associativity, has an identity, and each element has an inverse. Here the identity element, which we call **1**, is simply the permutation

$$1\,2\,3\,\ldots\,n =$$ 

Graphically, the *inverse* of a diagram is obtained by simply flipping it across the horizontal axis. Then σ and σ^{-1} obviously multiply to the identity. For example, using the same σ as before

$$\sigma^{-1} =$$  $$= 2\,3\,1\,5\,4$$

In all but the smallest cases, S_n is not commutative, meaning that $\sigma_1\sigma_2$ is not equal to $\sigma_2\sigma_2$ in general. Results about the symmetric group are too numerous to name, but one example is that any group with a finite number of elements can be found as a subgroup of a suitable sized S_n. Thus we see its centrality in algebra.

A crucial role in this paper will be played by the symmetric permutations, those which are their own inverses. We call such permutations *involutions* and the set of all involutions S_n^s.

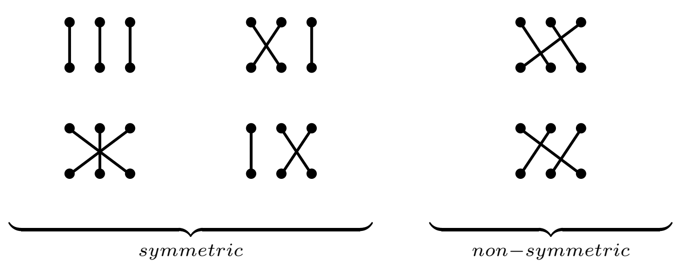


Figure 1.1: S_3, S_3^s is the left four diagrams

As you can see, there are 4 symmetric permutations in S_3. In Table 2.1, we list the sizes S_n and S_n^s for $0 \leq n \leq 10$. There is no closed formula for the number of involutions, but it is clear that they grow much slower than the number of all permutations, which is just $n! = n(n-1)\ldots(2)(1)$

1.2 Partitions and Cycles

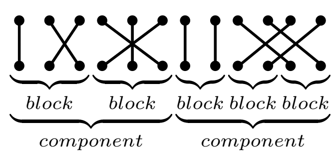
Permutations, and a great number of other things, are related to partitions. A *partition* λ of the positive integer n, denoted $\lambda \vdash n$, is a sequence of

n	0	1	2	3	4	5	6	7	8	9	10
$\|S_n\| = n!$	1	1	2	6	24	120	720	5040	40320	362880	3628800
$\|S_n^s\|$	1	1	2	4	10	26	76	232	764	2620	9496

Table 1.1: The number of permutations and involutions in S_n

nonnegative integers $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ such that $|\lambda| = \lambda_1 + \ldots + \lambda_l = n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$. The length $l(\lambda)$ is the number of nonzero parts of λ. We will sometimes group similar terms and denote the number of copies of each with a superscript, i.e. $\{3, 3, 2, 2, 2\} = \{3^2, 2^3\} \neq \{9, 8\}$ and we would still say that the length is 5. In the notation $\{\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \ldots, \lambda_k^{\alpha_k}\}$, the inequality above is strict, so $\lambda_1 > \lambda_2 > \ldots > \lambda_k$.

Given a partition $\mu = \{\mu_1, \mu_2, \ldots, \mu_l\} \vdash n$ we define the *μ-blocks* as the consecutive sets of length $\mu_1, \mu_2, \ldots, \mu_l$. These sets will then contain all the elements from 1 to n. Thus the blocks of $\mu = \{4, 3, 3\}$ are the sets $\{1, 2, 3, 4\}, \{5, 6, 7\}$ and $\{8, 9, 10\}$. The *μ-components* of a partition are obtained by lumping together all blocks of the same size, so for the previous we would have $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8, 9, 10\}$. For the partition $\{3^2, 2^3\}$, the following permutation would break down as



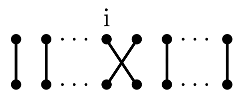
Notice that in the first component we do not have any stands which cross between blocks while in the second, we do. Also we do not have any strands crossing between components. This distinction will prove crucial.

Define the *break set* of a partition as the set of the last elements in each block, that is

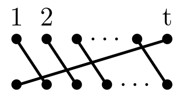
$$B(\mu) = \{\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \ldots\}$$

For example: $B(\{3, 3, 2, 2, 2\}) = \{3, 6, 8, 10, 12\}$

Next define a *transposition* s_i as the permutation which switches the elements i and i+1, $\gamma_1 = 1$ and let $\gamma_t = s_{t-1}s_{t-2}\ldots s_1$ for $t > 1$. Thus

$s_i =$ 

(1.2.1)

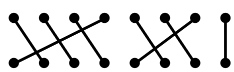
$\gamma_t =$ 

(1.2.2)

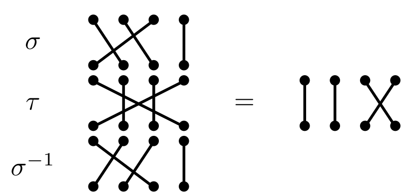
For any two permutations σ_1 and σ_2 of size n_1 and n_2 define $\sigma_1 \otimes \sigma_2$ as the permutation in $S_{n_1+n_2}$ formed by placing σ_2 to the right of σ_1. For a partition $\mu = \{\mu_1, \mu_2, \ldots, \mu_l\} \vdash n$, define

$$\gamma_\mu = \gamma_{\mu_1} \otimes \gamma_{\mu_2} \otimes \cdots \otimes \gamma_{\mu_l} \in S_n \tag{1.2.3}$$

Then for instance if $\mu = \{4, 3, 1\}$ we have that

$\gamma_\mu =$ 
(1.2.4)

We define another natural action on S_n called *conjugation*. Given two permutations σ and τ we conjugate by multiplying $\sigma\tau\sigma^{-1}$. We say that two permutations σ_1 and σ_2 are *conjugate* if there exists a third permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.



The *cycle type* of a permutation $\sigma \in S_n$ is the partition $\rho(\sigma) \vdash n$ given by the lengths of the disjoint cycles in σ. A basic result is that conjugation does not affect cycle type, so each $\sigma \in S_n$ is then conjugate to $\gamma_{\rho(\sigma)}$. For ease of notation we will sometimes write $\sigma = \{\rho_1, \rho_2, \ldots, \rho_l\}$ to refer to the cycle type of a permutation and the reader should not confuse these curly brackets with the round brackets of cycle notation.

Lemma 1.2.1. *A permutation is symmetric iff $\rho(\sigma) = \{2^\alpha, 1^\beta\}$ and σ is symmetric iff $\tau\sigma\tau^{-1}$ is also symmetric for any τ, i.e conjugation does not change symmetry.*

Proof. Suppose we have a cycle of size k on the elements $p_1, p_2, \ldots p_k$. If $\sigma(\sigma(p_1)) = p_1$ then $\sigma(p_1) = \sigma^{-1}(p_1)$ so recalling the definition of a cycle, we have that $p_2 = p_k$. But then k must be either 1 or 2. But conjugation doesn't change cycle type, so $\rho(\sigma) = \{2^\alpha, 1^\beta\}$ iff $\rho(\tau\sigma\tau^{-1}) = \{2^\alpha, 1^\beta\}$ iff $\tau\sigma\tau^{-1}$ is symmetric.

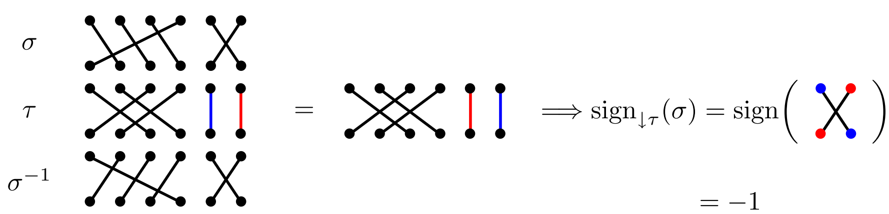
□

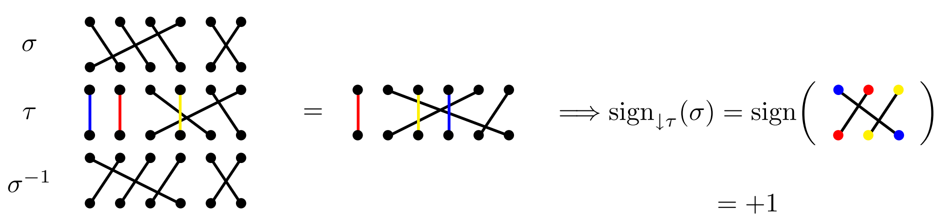
The *length* of a permutation $\ell(\sigma)$ is the minimal number of transpositions necessary to build it, which turns out to be the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$. We say that the *sign* of a permutation is $(-1)^{\ell(\sigma)}$. Since the length of a cycle of size k is $k-1$ and length adds for disjoint cycles, we can use the cycle type of a permutation to compute its sign. If $\rho(\sigma) = \{\rho_1, \ldots, \rho_l\}$ then

$$\ell(\sigma) = (\rho_1 - 1) + \ldots + (\rho_l - 1) = (\rho_1 + \ldots + \rho_l) - (1 + \ldots 1) = n - l$$
$$\text{sign}(\sigma) = (-1)^{\ell(\sigma)} = (-1)^{n-l}$$

1.3 The Saxl Weight

Given two permutations, define the sign of σ restricted to τ, $\text{sign}_{\downarrow\tau}(\sigma)$, as the normal sign evaluated on the action of σ on the fixed points of τ. Thus



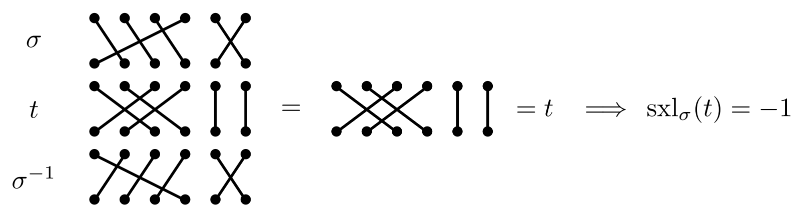


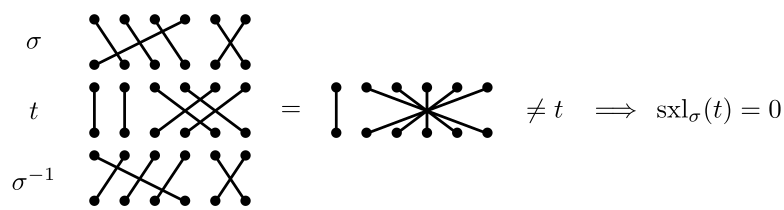
Lemma 1.3.1. $\mathrm{sign}_{\downarrow \mathbf{1}}(\sigma) = \mathrm{sign}(\sigma)$ *and* $\mathrm{sign}_{\downarrow \tau}(\mathbf{1}) = +1$ *where* $\mathbf{1}$ *is the identity permutation.*

We now define the *Saxl weight* of a symmetric permutation t with respect to a permutation σ as

$$\mathrm{sxl}_\sigma(t) = \begin{cases} \mathrm{sign}_{\downarrow t}(\sigma) & \text{if } \sigma t \sigma^{-1} = t \\ 0 & \text{if } \sigma t \sigma^{-1} \neq t \end{cases} \tag{1.3.1}$$

To illustrate, the Saxl weight with respect to the permutation $\sigma = (1234)(56)$ of the following permutations are





More examples can be found in Figure 3.1. The name here comes from J. Saxl, who used this weight implicitly in [Sxl] to derive important algebraic properties of S_n. Similarly, we will use the Saxl weight later in the construction of an algebraic object, but for now we are only concerned with its combinatorial properties. In particular, we will be comparing it to another weight.

1.4 The Roichman Weight

The Roichman weight, like the Saxl weight, will have a deeper meaning later, but for now we can view it simply as a function on two permutations. Recalling that $\rho(\sigma)$ is the cycle type of σ and $B(\mu)$ is the set of break points of a partition, then for a permutation in S_n, we compute the σ-*Roichman weight* as

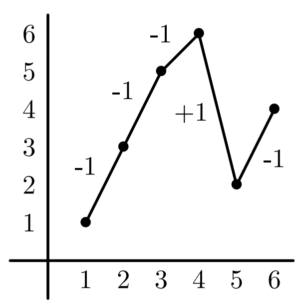
$$\mathrm{rwt}_\sigma(\tau) = \prod_{i=1}^{n-1} r_i \quad \text{where } r_i = \begin{cases} -1 & \text{if } \tau^{-1}(i) < \tau^{-1}(i+1) \text{ and } i \notin B(\rho(\sigma)) \\ 0 & \text{if } \tau^{-1}(i-1) > \tau^{-1}(i) < \tau^{-1}(i+1) \text{ and } i-1, i \notin B(\rho(\sigma)) \\ +1 & \text{otherwise} \end{cases}$$

In almost all cases we will be calculating the Roichman weight of a symmetric permutation, so we can ignore the inverses. For a easier conception of this, consider the sequence of numbers within each block of $\rho(\sigma)$. Call a sequence of numbers $a_1, a_2, \ldots, a_n$ *down-up* if there exists an i such that $a_{i-1} > a_i < a_{i+1}$. Then the Roichman weight will be zero over any block where the inverse of the permutation is down-up. If a block is not down-up then the inverses must first increase, then decrease, and so the Roichman weight will be (-1) raised to the number of times the sequence of the inverses increases within a block since all other terms contribute a factor of +1.

Graphically, we can draw τ^{-1} as a function on the points 1 to n and multiply by -1 whenever the permutation increases, by +1 whenever it decreases, and by 0 if it ever goes down and then back up, where we only care about what happens within each block. Thus

$$\tau = 1\,5\,2\,6\,3\,4$$

$$\tau^{-1} = 1\,3\,5\,6\,2\,4 =$$



$$B(\{6\}) = \{6\} \Longrightarrow \mathrm{rwt}_{\{6\}}(\tau) = (-1)(-1)(-1)(+1)(0) = 0$$

$$B(\{4,2\}) = \{4,6\} \Longrightarrow \mathrm{rwt}_{\{4,2\}}(\tau) = (-1)(-1)(-1)(+1)(-1) = +1$$

$$B(\{2,2,2\}) = \{2,4,6\} \Longrightarrow \mathrm{rwt}_{\{2,2,2\}}(\tau) = (-1)(+1)(-1)(+1)(-1) = -1$$

Thinking about these in the second way, we can calculate the Roichman weight within each block

$$\mathrm{rwt}_{\{6\}}(\tau) = \underbrace{1 \quad 3 \quad 5 \quad 6 \quad 2 \quad 4}_{(-1)(-1)(-1)(0)(+1)} = 0$$

$$\mathrm{rwt}_{\{4,2\}}(\tau) = \underbrace{1 \quad 3 \quad 5 \quad 6}_{(-1)(-1)(-1)} * \underbrace{2 \quad 4}_{(-1)} = +1$$

$$\mathrm{rwt}_{\{2,2,2\}}(\tau) = \underbrace{1 \quad 3}_{(-1)} * \underbrace{5 \quad 6}_{(-1)} * \underbrace{2 \quad 4}_{(-1)} = -1$$

More examples of this can also be found in Figure 3.1. The Roichman weight is named for Y. Roichman, who defined it in [Roi] in order to compute the characters of S_n. It was converted to a weight on permutations by A. Ram in [Ram].

Chapter 2

Combinatorics

2.1 Conjecture

When we compare the Roichman and Saxl weight in Figure 3.1, there appears to be some relation between them. Computing, we see that each is always either $+1$, -1, or 0, and that the occurrences of these seem to correlate in some fashion. In fact, it appears that whenever the Saxl weight is non-zero, the Roichman weight is equal to it and all other non-zero instances of the Roichman weight cancel. This observation proves to be false, though the first case does not appear until S_6. Nevertheless, if we fix a σ and sum over all symmetric permutations, we have equality. This is the main conjecture, that for all σ

$$\sum_{t \in S_n^s} \mathrm{sxl}_\sigma(t) = \sum_{t \in S_n^s} \mathrm{rwt}_\sigma(t) \tag{2.1.1}$$

Here we fix a σ in each row and calculate the Saxl and Roichman weights relative to it for all involutions in S_4. We have picked one element with each cycle type, which is given above it. Why we have picked these representatives will become clear shortly. Notice the thick lines divide the columns according to how many fixed points they have and that the sums are in fact equal within each set. This leads to a slightly stronger form of the conjecture, namely that if T^f is the set of all symmetric permutations in S_n with exactly f fixed points, then for all σ

$$\sum_{t \in T^f} \mathrm{sxl}_\sigma(t) = \sum_{t \in T^f} \mathrm{rwt}_\sigma(t) \tag{2.1.2}$$

Notice that the original form is an immediate corollary of this as $S_n^s = \bigcup T^f$.

S_4	1 2 3 4	2 1 3 4	3 2 1 4	4 2 3 1	1 3 2 4	1 4 3 2	1 2 4 3	2 1 4 3	3 4 1 2	4 3 2 1	Total
$\{1^4\}$ rwt	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+10
sxl	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+10
$\{2,1^2\}$ rwt	−1	+1	+1	+1	−1	−1	−1	+1	−1	+1	0
sxl	−1	+1	0	0	0	0	−1	+1	0	0	0
$\{2^2\}$ rwt	+1	−1	−1	+1	+1	−1	−1	+1	+1	+1	+2
sxl	+1	−1	0	0	0	0	−1	+1	+1	+1	+2
$\{3,1\}$ rwt	+1	0	+1	0	−1	−1	+1	0	−1	+1	+1
sxl	+1	0	0	0	0	0	0	0	0	0	+1
$\{4\}$ rwt	−1	0	0	0	0	−1	+1	0	0	+1	0
sxl	−1	0	0	0	0	0	0	0	0	+1	0

Figure 3.1

2.2 Proof

We now begin the real work.

Conjecture 2.2.1. *If σ and τ are conjugate, then $\sum_{t \in T^f} \mathrm{sxl}_\sigma(t) = \sum_{t \in T^f} \mathrm{sxl}_\tau(t)$.*

Proof. This is at least believable just looking at computations. The conjugation action simply changes which t are fixed by σ and τ but does not change the sign. A proof, however, requires machinery developed in the next chapter where we show that $\sum_{t \in T^f} \mathrm{sxl}_\sigma(t)$ is the trace of a matrix and a basic fact in linear algebra tells us that the trace is invariant under conjugation.

□

We will assume this for now. So let us simply pick a nice representative for each cycle type and begin to compute.

Proposition 2.2.2. *If $\sigma \in S_{2a+1}$ has cycle type $\{2a+1\}$ and $\sigma t \sigma^{-1} = t$ for $t \in S^s_{2a+1}$ then $t(i) = i$ for all i and hence $\mathrm{sxl}_\sigma(t) = +1$.*

Proof. By the lemma, we may assume $\sigma = (1, 2, \ldots, 2a+1)$. If any of the points of t are not fixed, then since t is symmetric, they must come in pairs. But there still must be at least one vertex left over which is fixed, assume that it is 1. Then $t(2) = \sigma t \sigma^{-1}(2) = 2$ and repeating we have that every i is fixed so t must be the identity. But then we are just computing the regular sign of σ which is positive because it is a cycle of odd length. □

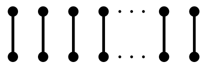
Proposition 2.2.3. *If σ has cycle type $\{2a\}$ and $\sigma t \sigma^{-1} = t$ for $t \in S_{2a}$ then either $t(i) = i$ for all i or $t(i) = i + a$ for all i. In the first case $sxl_\sigma(t) = -1$, while in the second $\mathrm{sxl}_\sigma(t) = +1$.*

Proof. Again, let $\sigma = (1, 2, \ldots, 2a)$. By the same argument as the previous case, we must have that if any i is fixed by t then they all must be. Suppose then that no i is fixed. Let $t(1) = b$. Then $\sigma t \sigma^{-1}(1) = b+1 \Rightarrow t(2) = b+2 \Rightarrow t(n) = b+n$. But since t is symmetric we have that $1 = t(t(1)) = t(b+1) = 2b+1 \Rightarrow b = 0$ or $b = a$. In the first case we have the identity, so we are again computing the regular sign of the permutation, while in the second we have no fixed points, so the Saxl weight is trivially positive. □

Theorem 2.2.4. *Let $\sigma = (1, 2, \ldots, n) \in S_n$ if n is odd and $\sigma = (1, \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n, \frac{n}{2}, \frac{n}{2}-1, \ldots, 2)$ if n is even and hence σ has cycle type $\{n\}$. Then $\sum_{t \in T^f} \mathrm{sxl}_\sigma(t) = \sum_{t \in T^f} \mathrm{rwt}_\sigma(t)$.*

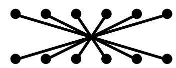
Proof. If the sequence $t(1), t(2), \ldots, t(n)$ ever decreases then increases, then $\mathrm{rwt}_\sigma(t) = 0$. Hence there are three cases where $\mathrm{rwt}_\sigma(t) \neq 0$ which therefore contribute to the sum.

Case I: $t(1) < t(2) < \ldots < t(n)$

$t =$  , t fixes all of $1 - n$

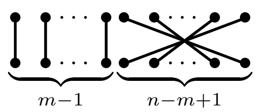
We then have that t is the identity, in which case we have that $\mathrm{rwt}_\sigma(t) = \mathrm{sxl}_\sigma(t) = (-1)^{n-1}$ since each r_i in the Roichman weight computation will be (-1) and there will be $n-1$ of them.

Case II: $t(1) > t(2) > \ldots > t(n)$, n even

We then have that $t =$ 

In this case we know that n must be even since t must be the product of disjoint two cycles. This is the second case in Proposition 3.2.3, but also each r_i will be positive in the Roichman weight, so $\mathrm{rwt}_\sigma(t) = \mathrm{sxl}_\sigma(t) = +1$.

Case III: $t(1) < t(2) < \ldots < t(m) > t(m+1) > \ldots > t(n)$ where k can be 1 iff n is odd.

Here $t =$ 

There will be exactly two such diagrams with a given number of fixed points, one with all of them out front and one with all but one out front and one in the middle of the inverted section. Both of these will have Saxl weight zero as by the previous propositions neither can fix t, but they will have opposite Roichman weights because the number of (-1)'s in the product will be different by one and hence the sums will remain equal.

□

Corollary 2.2.5. *For any σ with cycle type $\{n\}$*

$$\sum_{t \in S_n^s} \mathrm{sxl}_\sigma(t) = \sum_{t \in S_n^s} \mathrm{rwt}_\sigma(t) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{2.2.1}$$

We can then combine these results with the following lemmas to obtain the general statement.

Lemma 2.2.6. *If σ is a permutation which fixes the symmetric diagram t, then x and $t(x)$ are in cycles of the same length in σ.*

Proof. If $\sigma\ t\ \sigma^{-1} = t$ then $\sigma\ t = t\ \sigma$, so t and σ commute. Suppose then that x is contained in the cycle of length l and $t(x)$ is contained in a cycle of length m within σ. This means that $\sigma^l(x) = x$ and $\sigma^m(t(x)) = t(x)$. But then $t(\sigma^m(x)) = t(x)$ so $\sigma^m(x) = x$ and m = l since they are assumed to be the smallest powers which take x and $\sigma(x)$ back to themselves. □

Lemma 2.2.7. *If a symmetric permutation t sends two vertices from one block of the partition $\rho(\sigma)$ into two different blocks, then $\sigma\, t\, \sigma^{-1} \neq t$ and hence* $\mathrm{sxl}_\sigma(t) = 0$

Proof. Suppose vertices i and j are in the same block relative to $\rho(\sigma)$ but $t(i)$ and $t(j)$ are not. We know that since σ acts as a full cycle on each block, that there exists an m such that $\sigma^{-m}(i) = j$. But then $\sigma^m\, t\, \sigma^{-m}(i) = \sigma^m\, t(j)$ is in the same block as $t(j)$, hence it cannot equal i, hence $\sigma\, t\, \sigma^{-1} \neq t$. □

Theorem 2.2.8. *If $\sigma \in S_n$ has cycle type $\{\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \ldots, \lambda_m^{\alpha_m}\}$ and $\mathrm{fp}(\tau)$ is the number of fixed points of τ, then*

$$\sum_{t\,\in\, S_n^s} \mathrm{sxl}_\sigma(t) = \prod_i \sum_{t \in S^s_{\lambda_i^{\alpha_i}}} \mathrm{sxl}_{(\lambda_i^{\alpha_i})}(t) \tag{2.2.2}$$

and

$$\sum_{t\,\in\, S^s_{\lambda^\alpha}} \mathrm{sxl}_{\lambda^\alpha}(t) = \begin{cases} \sum_{\tau \in S^s_\alpha} \lambda^{\frac{\alpha - \mathrm{fp}(\tau)}{2}} & \text{if } \lambda \text{ is odd} \\ 0 & \text{if } \lambda \text{ is even but } \alpha \text{ is odd} \\ \sum_{\tau = \{2^\beta\}} \lambda^\beta = \left[\frac{(2\beta-1)!}{(\beta-1)!2^{\beta-1}}\right] \lambda^\beta & \text{if } \lambda \text{ is even and } \alpha = 2\beta \end{cases} \tag{2.2.3}$$

Proof. Applying the lemmas, we see that the only symmetric permutations which contribute to the sum are those which do not have strands crossing between components. But in that case we can notice that the components act independently and so the sum of the Saxl weights of all combinations is merely the product of the sum on each. To compute the sum of the Saxl weights on each component, we consider three cases:

Case I: λ is odd
Since any permutations which splits blocks have Saxl weight zero, we know that the sum must be equal to the sum of those that don't. But then we can view this as a symmetric permutation among the α blocks and sum independently over each cycle then multiply the sums. For such a permutation, any block which is fixed must fix all of its members, while a block which is switched with another can have any permutation which conjugates with the full cycle, of which there are exactly λ. Finally, notice that there are exactly $\frac{\alpha - \mathrm{fp}(\tau)}{2}$ such blocks, so we get the quantity above.

Case II: λ is even α is odd
We then must have a block of size λ which is sent to itself, and so by changing whether our permutation is positive or negative on this smaller set, we can cancel all terms.

Case III: λ is even and $\alpha = 2\beta$, i.e. α is even, too.
As we saw in the previous case, if any block gets sent to itself, it yields a factor of 0 to the product of the sums. Hence all blocks must be switched. Also, since any such permutations will have no fixed points, we know that the sum of the Saxl weight is simply the number of such permutations times the number of choices within each block. But there are $\prod_i (2\beta - 2i - 1) = \frac{(2\beta-1)!}{(\beta-1)!2^{\beta-1}}$ ways to pair 2β elements into β pairs, while within each block we still have λ choices, yielding the above result.

□

Inspired by this result, we seek a similar one for the Roichman weight. First, however, we need a better understanding of when the Roichman weight is non-zero with respect to the full partition. Remember that we already showed that the symmetric permutations with non-zero $\{n\}$-Roichman weight are exactly those with the form $1, 2, \dots, m, n, n-1, \dots, n-m+1$. We now give a general characterization for non-symmetric permutations.

Lemma 2.2.9. *If $\sigma \in S_n$ has non-zero $\{n\}$-Roichman weight, then $\sigma^{-1}(1) = 1$ or n.*

Proof. Then the sequence $\sigma^{-1}(1), \sigma^{-1}(2), \dots \sigma^{-1}(n)$ must not be down-up and since 1 must appear somewhere in it, it must either be first or last, meaning it is $\sigma^{-1}(1)$ or $\sigma^{-1}(n)$.

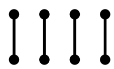
□

Corollary 2.2.10. *There are exactly 2^{n-1} permutations with non-zero $\{n\}$-Roichman weight in S_n and they can each be labeled by a string of $n-1$ letters each either f or ℓ, denoting whether that strand is the first or last. Call this the name of the permutation. The Roichman weight of a permutation thus named is simply $(-1)^f$ where f is the number of strands which are firsts, not counting the last strand.*

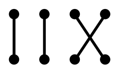
Proof. We simply apply the lemma repeatedly to get the characterization. This then gives us the total number and the calculation follows from the fact that each first strand gives a negative one in the product of the Roichman weight but each last gives positive one.

□

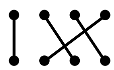
For instance, the following are all permutations in S_4 with non-zero $\{4\}$-Roichman weight.



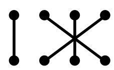
$$= f, f, f \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^3 = -1 \tag{2.2.4}$$



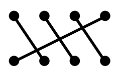
$$= f, f, \ell \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^2 = +1 \tag{2.2.5}$$



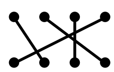
$$= f, \ell, f \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^2 = +1 \tag{2.2.6}$$



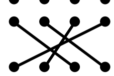
$$= f, \ell, \ell \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^1 = -1 \tag{2.2.7}$$



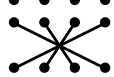
$$= \ell, f, f \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^2 = +1 \tag{2.2.8}$$



$$= \ell, f, \ell \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^1 = -1 \tag{2.2.9}$$



$$= \ell, \ell, f \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^1 = -1 \tag{2.2.10}$$



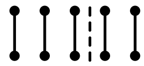
$$= \ell, \ell, \ell \Longrightarrow \mathrm{rwt}(\sigma) = (-1)^0 = +1 \tag{2.2.11}$$

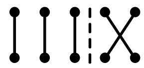
Notice that the first, second, fourth, and eighth are symmetric. Generally, such a diagram will be symmetric iff its name is $f, \ldots, f, \ell, \ldots, \ell$, so there are exactly n of them corresponding to when we switch from f to ℓ.

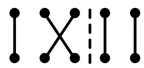
Now observe that in examining $\sum_{t \in S_n^s} \mathrm{rwt}_\sigma(t)$ if we only look at the contribution of the symmetric permutations which do not cross strands between components of σ, then we can sum over the components independently and so these give us a contribution of

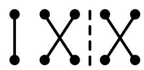
$$\prod_i \sum_{t \in S^s_{\lambda_i^{\alpha_i}}} \mathrm{rwt}_{(\lambda_i^{\alpha_i})}(t)$$

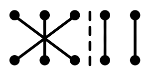
which is what we want. Hence the sum over all t's which do cross must to be zero. These are all permutations with non-zero $\{3,2\}$ - Roichman weight which do not cross between groups.

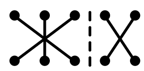
 $$\Longrightarrow \mathrm{rwt}_{\{3,2\}}(\sigma) = +1 * -1 = -1 \tag{2.2.12}$$

 $$\Longrightarrow \mathrm{rwt}_{\{3,2\}}(\sigma) = +1 * +1 = +1 \tag{2.2.13}$$

 $$\Longrightarrow \mathrm{rwt}_{\{3,2\}}(\sigma) = -1 * -1 = +1 \tag{2.2.14}$$

 $$\Longrightarrow \mathrm{rwt}_{\{3,2\}}(\sigma) = -1 * +1 = -1 \tag{2.2.15}$$

 $$\Longrightarrow \mathrm{rwt}_{\{3,2\}}(\sigma) = (-1)^2 * -1 = -1 \tag{2.2.16}$$

 $$\Longrightarrow \mathrm{rwt}_{\{3,2\}}(\sigma) = (-1)^2 * +1 = +1 \tag{2.2.17}$$

These can be grouped to see that the sum of their Roichman weights will simply be the product of the sums of the Roichman weights with respect to $\{3\}$ and $\{2\}$. Hence we must have that all of those which do cross, can be paired off and cancelled. The next theorem will show how to do this generally.

Theorem 2.2.11. *If σ has cycle type $\{\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \ldots, \lambda_m^{\alpha_m}\}$, then*

$$\sum_{t\,\in\, S_n^s} \mathrm{rwt}_\sigma(t) = \prod_i \sum_{t\in S^s_{\lambda_i^{\alpha_i}}} \mathrm{rwt}_{(\lambda_i^{\alpha_i})}(t) \tag{2.2.18}$$

and

$$\sum_{t\,\in\, S^s_{\lambda^\alpha}} \mathrm{rwt}_{\lambda^\alpha}(t) = \begin{cases} \sum_{\tau\in S^s_\alpha} \lambda^{\frac{\alpha-\mathrm{fp}(\tau)}{2}} & \text{if } \lambda \text{ is odd} \\ 0 & \text{if } \lambda \text{ is even but } \alpha \text{ is odd} \\ \sum_{\tau=\{2^\beta\}} \lambda^\beta = \left[\frac{(2\beta-1)!}{(\beta-1)!2^{\beta-1}}\right]\lambda^\beta & \text{if } \lambda \text{ is even and } \alpha = 2\beta \end{cases} \tag{2.2.19}$$

Proof. As we already noted, those t's which do not cross between blocks of different sizes already give us this sum, so we must cancel all the rest. So suppose that t is a symmetric permutation with non-zero Roichman weight which does cross. Then t must take some block into two different blocks. Let block i be the leftmost block which is thus split and let a be the strand which is rightmost in the leftmost block which block i is sent into. By the previous corollary, a must either be the first or last strand among those to the right of it. If it is the first then let b be the rightmost strand which is sent to the right of a and conjugate σ with $(a,\ a+1, \ldots,\ b)$. On the other hand, if it is last, then pick b to be the leftmost strand which is sent to the right of a and conjugate by

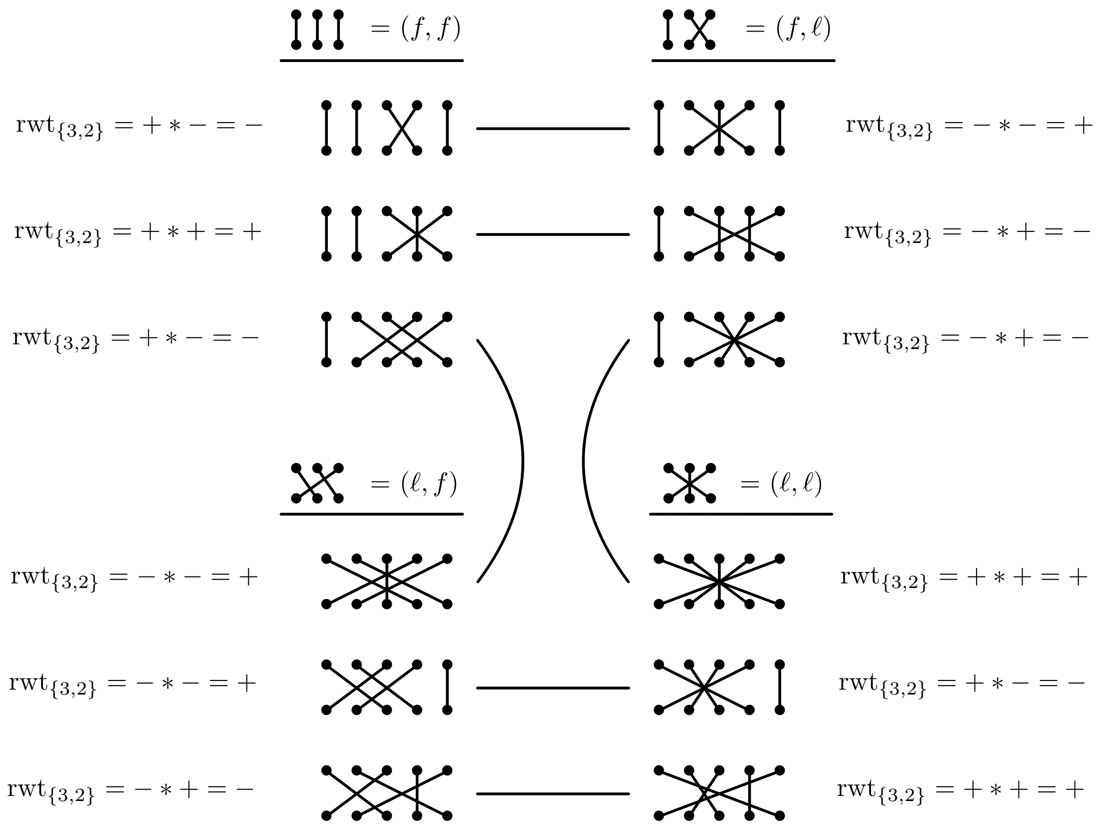


Figure 2.1: Symmetric permutations in S_5 with non-zero Roichman weight on $\{3, 2\}$ which cross between blocks. The header for each group gives the permutation among the first three strands, while the connecting lines indicate which pairs are conjugate with opposite weights. Notice that the first term in the computation, which represents the Roichman weight on the first three strands is opposite between pairs, while the second term, which represents the Roichman weight on the last two strands, is the same.

$(ba,\ a-1, \ldots,\ b)$. I make the following three claims:

1) This changes the Roichman weight on block i

This will change the strand picked from a first to a last and vice-versa but will leave the rest of the strands in block i in the same relative order, so their contribution to the Roichman weight will be the same as before.

2) This does not change the Roichman weight of any other blocks

Since we are conjugating, the inverse cycle we stick on the bottom will affect other blocks. But since a is the rightmost strand which goes to block i and t is symmetric, this must just be crossing strands from separate blocks, which doesn't change the Roichman weight.

3) This is an involution on the set and preserves symmetry
Since it is a conjugation action, not only must it preserve symmetry, but also the number of fixed points. And after applying it b will be the rightmost strand in block i and it will now be the opposite type as a was so applying the process a second time will simple conjugate with the inverse cycle, giving us back the original permutation.

Thus this exactly pairs off positive and negative symmetric permutations which cross, so the sum over them must be zero.

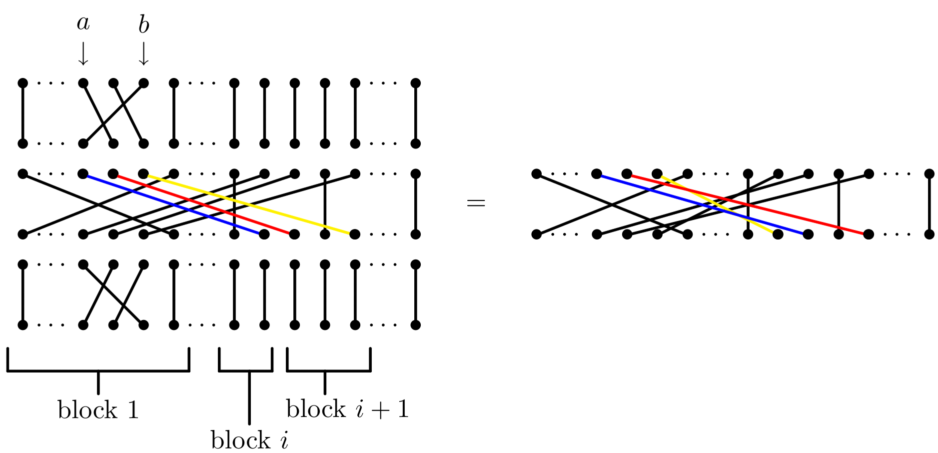


Figure 3.3: An example of the involution. In the original diagram the order of the strands in the first block is $\ell, \ldots, f, f, f, f$ while after the conjugation, it is $\ell, \ldots, f, \ell, f, f$ and no other blocks have their order changed, so the overall sign will have been reversed.

For the calculation, we use the same technique to note that we can cancel all permutations which do not take entire blocks to entire blocks. Hence we must be looking at a, symmetric, permutation on the blocks. The permutations within the blocks must be symmetric as the block it is crossed to will be the inverse and the only permutations which both themselves and their inverses have non-zero Roichman weight are the symmetric ones. If λ is odd, then each crossed block multiplies by λ as there are that many symmetric permutations with non-zero Roichman weight and since they are crossed whether it is positive or negative it multiplies to positive, while each which goes to itself cancels all but one, so does not change the product. Similarly, if λ is even then if any block goes to itself, it multiplies the product by zero, so the product is zero. Thus when α is odd, the product is always zero as we can't match everything up, while if α is even we have λ choices for each transposition of blocks, of which there must be $\alpha/2$, resulting in the formula above.

□

Theorem 2.2.12.

$$\sum_{t\,\in\,S_n^s} \mathrm{sxl}_\sigma(t) = \prod_i \sum_{t\in S^s_{\lambda_i^{\alpha_i}}} \mathrm{sxl}_{\lambda_i^{\alpha_i}}(t) = \prod_i \sum_{t\in S^s_{\lambda_i^{\alpha_i}}} \mathrm{rwt}_{\lambda_i^{\alpha_i}} = \sum_{t\,\in\,S_n^s} \mathrm{rwt}_\sigma(t) \quad (2.2.20)$$

Proof. This is just a combination of the previous two theorems. □

Chapter 3

Representation Theory

We now move into what appears to be a completely unrelated field, representation theory. This theory deals with the groups acting as linear transformations on vector spaces. The number of essentially different ways in which this can happen for a given finite group turns out to finite and so we can list them. This list is inherently connected to the original group and conveys a large deal of information about the it.

3.1 Concepts

We briefly remind the reader of basic group theoretic concepts. Recall that a *group* is simply a set of elements, G, together with an operation, such that the operation is associative, has an identity, and each element has an inverse. An abelian group is one for which all elements commute, i.e. $gh = hg$. A *field* is a set together with two operations which are each group actions and which can be combined in the right way.

A *vector space* over a field, F, which will always be the complex numbers, $\mathbb{C}$, for us, is a set V and an addition rule for two elements in V and a multiplication rule of an element in V by an element in F. These operations must satisfy for all $u, v \in V$ and all $\alpha, \beta \in F$

(1) V is an abelian group under addition
(2) $\alpha(u + v) = \alpha u + \alpha v$
(3) $(\alpha + \beta)v = \alpha v + \beta v$
(4) $(\alpha\beta)v = \alpha(\beta v)$
(5) $1\, v = v$

We will assume that all vector spaces herein are finite dimensional. This means that we can always choose a basis, $\{v_1, \ldots, v_n\}$ where the v_i's are linearly independent and span all of V.

Now let V be a vector space over $\mathbb{C}$ and let G be a group. Then V is an *FG-module* if an operation $g \cdot v$ is defined and satisfies the following conditions for all $u, v \in V, \lambda \in F,$ and $g, h \in G$:

(1) $g \cdot v \in V$
(2) $(g\,h) \cdot v = g \cdot (h \cdot v)$
(3) $e \cdot v = v$ where e is the identity of G
(4) $g \cdot (\lambda\, v) = \lambda(g \cdot v)$
(5) $g \cdot (u + v) = g \cdot u + g \cdot u$

If we define our action on a basis for the vector space, then we will be able to extend it uniquely to all elements, using the above rules. In fact, since the action on the basis contains all the information necessary to construct the entire module, we will often only specify the results of the action on the basis. If we have $\{v_1, \ldots, v_n\}$ is a basis for our vector space, then one concise way of doing this is for each element g in G to label an n by n matrix with the basis elements along the top and side and in the $[i, j]$ position put the coefficient of v_i in $g \cdot v_j$. Doing so for all g in G, the second rule above means that we end up with a homomorphism from G to $GL(n, F)$, the set of all n by n invertible matrices with entries from F. We call this homomorphism a *representation* and, as it contains exactly the same information as the module, we use the two terms interchangeably.

The simplest example is the regular representation of a group. Here any group, G acts on a vector space $V = \mathbb{C}\text{-span}\{v_g | g \in G\}$ by the action $g{\cdot}v_h = v_{gh}$.

For the quaternion group $G = Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle$ we have a representation:

$$\rho : Q_8 \to GL(2, \mathbb{C}) : a \rightarrowtail \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } b \rightarrowtail \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that in the above examples, the given definition determines $g \cdot v$ for all g and v. For instance:

$$ab \cdot (v_1 + 2v_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2i \\ i \end{pmatrix} = 2iv_1 + iv_2$$

As for most algebraic structures, we call a subset W of V a submodule if W is a subspace of V and $g \cdot w \in W$ for all $w \in W$ and $g \in G$. More simply, a submodule is just a subset which is itself a module. For every module V, the zero subspace $\{0\}$ and V itself are submodules. We call a module *irreducible* if it is itself not $\{0\}$ and it has no nontrivial submodules.

But there are an infinite number of modules which differ only in the names of the variables or the basis chosen for the representation. We need a way to equate these. Thus for two FG-modules V and W, we say that a function $\vartheta : V \to W$ is a *FG-homomorphism* if ϑ is a linear transformation and

$$\vartheta(g \cdot v) = g \cdot (\vartheta(v)) \; \forall v \in V, g \in G \tag{3.1.1}$$

In particular, a function $\vartheta : V \to W$ is a FG-isomorphism if ϑ is a FG-homomorphism and ϑ is invertible. If such a function exists, then we say that V and W are isomorphic FG-modules. If two FG-modules are isomorphic, we say that their respective representations are equivalent. This is an equivalence relation, so we can consider its equivalence classes as our basic objects, disregarding the difference between equivalent representations.

Having shown how to cut down the number of modules, we next consider how to grow more. If we are given two FG-modules V and W, then $V \oplus W$ is also a FG-module. Maschke's Theorem tells us that every reducible FG-module can be so written, and in particular in the case where $F = \mathbb{C}$ and G is finite, then every non-zero FG-module can be written as $V = U_1 \oplus \ldots \oplus U_l$ where each U_i is irreducible. Also we can prove that there are only a finite number of non-isomorphic irreducible $\mathbb{C}G$-modules for a finite group G. In fact, the number of irreducible characters, and hence representations, is equal to the number of conjugacy classes in G, which must be finite if G is. Putting these results together, we get that every representation is some combination of a finite alphabet of irreducible representations, so if we can understand those, we will understand all of them.

Suppose now that $\rho : G \to GL(n, \mathbb{C})$ is a representation of the finite group G. For each matrix $\rho(g)$ we take the trace, add up all the diagonal entries, and define this complex number to be $\chi(g)$. The function $\chi : G \to \mathbb{C}$ is called the *character* of the representation ρ.

Rather than restrict ourselves to looking at representations, we wish to deal with the group more directly. Define χ to be a character of G if it is the character of some representation. Further, call χ irreducible if it is the character of an irreducible representation. The following is a fundamental property in representation theory. The interested reader should consult [JL] for a proof and other background material for this chapter.

Theorem 3.1.1.
(1) Two $\mathbb{C}G$-modules are isomorphic iff they have the same character.
(2) If x and y are conjugate in G, then $\chi(x) = \chi(y)$ for all characters χ.

Thus we can write each character as a vector on the conjugacy classes of G. So for a group, we can construct a square table containing in position $[i, j]$ the value of the i-th character on the j-th conjugacy class. The character table, despite its rather diminutive size, contain a large deal of information about the group. Whether the group is abelian or cyclic, the size of its conjugacy classes, and the normal subgroups of G can all be determined from it. Since we can label the conjugacy classes of S_n by the partitions of n, we can do the same for the irreducible characters.

S_4	$\{1^4\}$	$\{2,1^2\}$	$\{2^2\}$	$\{3,1\}$	$\{4\}$
$\chi^{\{4\}}$	1	1	1	1	1
$\chi^{\{3,1\}}$	3	1	-1	0	-1
$\chi^{\{2^2\}}$	2	0	2	-1	0
$\chi^{\{2,1^2\}}$	3	-1	-1	0	1
$\chi^{\{1^4\}}$	1	-1	1	1	-1
$\sum$	10	0	2	1	0

Figure 4.1: The character table for S_4.

The irreducible characters then form a basis for all possible characters. So if we then have the character of some representation we wish to break down into the direct sum of irreducible representations, we can find the irreducible characters which add to the character of the desired representation. It turns out to be even easier than that. We define an inner product on characters as

$$\langle \vartheta, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g)\varphi(g^{-1}) \tag{3.1.2}$$

for characters ϑ and φ. While this appears complicated at first, we then have the following powerful theorem showing that the irreducible characters are orthonormal,

Theorem 3.1.2. *If ϑ and φ are distinct irreducible characters, then*

$$\langle \vartheta, \vartheta \rangle = 1, \text{ and}$$
$$\langle \vartheta, \varphi \rangle = 0$$

Thus to find the multiplicities of the irreducibles within a given character, we merely take the inner product with each irreducible. For instance, if we wish to decompose a representation ρ having character

$$\vartheta(1) = 7,\ \vartheta((1\ 2)) = 3,\ \vartheta((1\ 2)(3\ 4)) = 3,\ \vartheta((1\ 2\ 3)) = 1,\ \vartheta((1\ 2\ 3\ 4)) = 1$$

then using the character table above, we can compute

$$\langle \vartheta, \chi_1 \rangle = \frac{1}{24}[(7*1)1 + (3*1)6 + (3*1)3 + (1*1)8 + (1*1)6] = 2$$

Similarly $\langle \vartheta, \chi_2 \rangle = \langle \vartheta, \chi_5 \rangle = 0$ and $\langle \vartheta, \chi_3 \rangle = \langle \vartheta, \chi_4 \rangle$ so by the previous theorem

$$\vartheta = 2\chi_1 + \chi_3 + \chi_4$$
$$\rho \cong 2\, U_1 \oplus U_3 \oplus U_4$$

We say that a representation ρ is the *model* of a group if it is equivalent to the direct sum of all the irreducible representations. Since the character of a model is then the sum of all the irreducible characters, we can use the character table for S_4 to compute that the character of the model

$$\chi(1) = 10,\ \chi((1\ 2)) = 0,\ \chi((1\ 2)(3\ 4)) = 2,\ \chi((1\ 2\ 3)) = 1,\ \chi((1\ 2\ 3\ 4)) = 0$$

The observant reader will note that these values are the same ones we computed as the sum of the Roichman or Saxl weights in Figure 3.1. This is not a coincidence.

3.2 Construction of the Model for S_n

We now follow a paper of Saxl [Sxl] to give a construction for a model of S_n. Recall that we defined

$$T^f = \{t \in S_n \mid t \text{ is an involution with } f \text{ fixed points }\},$$

so define the complex vector space

$$V^f = \mathbb{C}\text{-span}\{t \in T^f\}.$$

Then the permutations $\sigma \in S_n$ act naturally on the involutions in T^f by conjugation,

$$\sigma \cdot t = \sigma t \sigma^{-1}, \qquad t \in T^f, \sigma \in S_n.$$

If k is even, then T^0 is the set of fixed-point-free involutions in S_n. The conjugation action extends to a natural representation of S_n on V^0. Call this representation W^0. It is well-known fact (see [Th, Theorem IV] or [JK, 4.1].) that W^0 decomposes in a multiplicity-free way into irreducible representations indexed by integer partitions with no odd parts. That is,

$$W^0 \cong \bigoplus_{\substack{\lambda \vdash k \\ \lambda \text{ has } 0 \text{ odd parts}}} S_n^\lambda.$$

where S_n^λ is the irreducible representation for S_n labelled by λ. For $t \in T^f$, let $C(t)$ be the centralizer of t in S_n. If $g \in C(t)$ then $gtg^{-1} = g$ and g induces a permutation on the f fixed points of t. Let $\operatorname{sign}(g,t)$ denote the sign of the permutation induced by g on the f fixed points of t, so here $\operatorname{sign}(g,t) = \operatorname{sxl}_g(t)$. Let W_t denote this signed representation of $C(t)$ on t. That is, let W_t be just the one-dimensional vector space $\mathbb{C}t$ with $g \cdot t = \operatorname{sign}(g,t)gtg^{-1} = \operatorname{sign}(g,t)t$. This is obviously a representation as most of the properties hold trivially. Given a representation on a subspace of a group it is possible to induce a representation on the whole group (see [JL] pg. 226), which allows us to define

$$W^f = \operatorname{Ind}_{C(t)}^{S_n} W_t.$$

Saxl [Sxl] and Kljăcko [Klj] independently proved the following (see also [IRS] for a short proof),

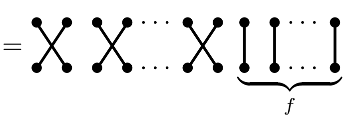
$$W^f \cong \bigoplus_{\substack{\lambda \vdash k \\ \lambda \text{ has } f \text{ odd parts}}} \chi^\lambda.$$

If we let

$$W = \bigoplus_{\ell=0}^{\lfloor k/2 \rfloor} W^{k-2\ell}, \tag{3.2.1}$$

then each irreducible representation $\chi^\lambda, \lambda \vdash n$ of S_n appears in W exactly once.

We explicitly construct W as follows. For each f, let $t^f \in T^f$ be the element,

$t^f =$ 

Let $n_f = |T^f|$ and let $g_1 = \mathbf{1}, g_2, \ldots, g_{n_f}$ be a set of minimal-length left coset representatives in $S_n/C(t^f)$. Then each g_i will have no crossings among any of the pairs of strands going to the crossed strands in t^f, nor among the strands going to the fixed points, as such crossings could be removed via multiplication with an element in $C(t^f)$. For any $g \in S_n$ let $\gamma(g), \tau(g)$ be defined as the unique decomposition

$$g = \tau(g)\gamma(g) \qquad \tau(g) \in T(g), \gamma(g) \in C(t) \tag{3.2.2}$$

Define

$$t_i^f = g_i t^f g_i^{-1}, \qquad 1 \le i \le n_f. \tag{3.2.3}$$

Then $t_1^f, t_2^f, \ldots, t_{n_f}^f$ is a basis for V^f, and $g \in S_n$ acts on the basis by

$$\begin{aligned} g \cdot t_i^f &= g \cdot (g_i t^f g_i^{-1}) \\ &= (g g_i) \cdot t^f \\ &= \tau(g g_i)\gamma(g g_i) \cdot t^f \\ &= \operatorname{sign}(\gamma(g g_i), t^f)\tau(g g_i) \cdot t^f \\ &= \operatorname{sign}(g, t_i^f) g t_i^f g^{-1} \end{aligned}$$

where the final equality comes from the fact that since g_i cannot cross any of the strands going to the fixed points of t^f, any such crossing must originate in g. But since these strands are precisely the fixed points of t_i^f, we have that $\mathrm{sign}(g, t_i^f)$ is the sign of the permutation induced on the fixed points of t_i^f by g. Note that t_i^f and $g t_i^f g^{-1}$ have the same number of fixed points and their order is permuted by g. Call this the signed-conjugation representation for S_n.

3.3 The Saxl Weight and the Signed-Conjugation Representation

We see that $\sigma \cdot t = \mathrm{sxl}_\sigma(t)t$ whenever $\sigma\, t\, \sigma^{-1} = t$. Hence the character χ_f of W^f, which we recall to be the trace of the matrixes, is given by

$$\chi_f(\sigma) = \sum_{t \in T^f} \mathrm{sxl}_\sigma(t).$$

The previous section therefore proves algebraically that

$$\sum_{t \in T^f} \mathrm{sxl}_\sigma(t) = \chi_f(\sigma) = \sum_{t \in T^f} \mathrm{rwt}_\sigma(t).$$

which can be expanded to

$$\sum_{t \in S_n^s} \mathrm{sxl}_\sigma(t) = \sum_{t \in S_n^s} \mathrm{rwt}_\sigma(t)$$

where the right hand side is known to be equal to the sum of all the irreducible characters of S_n [APR]. This is what we have just proved combinatorially in Theorem 2.2.12.

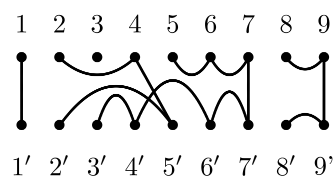
The algebraic method, however, does not work for a generalized version the symmetric group called the partition algebra, to which we will now turn our attention. There we will need to use the combinatorial approach.

Chapter 4

A Generalization

4.1 The Partition Algebra

The *partition monoid*, P_n , can be thought of as a generalization of the symmetric group. Instead of a diagram being restricted to each vertex in the top row being connected to exactly one vertex on the bottom row, we are allowed to make any connections. For example,



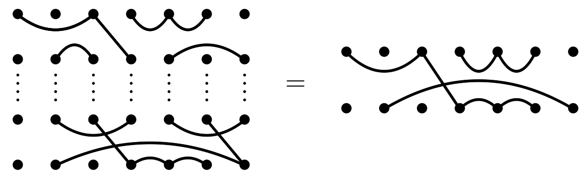
is an element of P_9 and can also be represented as the set partition

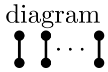
$$\{\{1,1'\},\{2,4,2',5'\},\{3\},\{5,6,7,3',4',6',7'\},\{8,8',9,9'\},\{1'\}\},$$

The graph representing d is not unique, we could connect 2 and $2'$ above, for example. We will always use the standard presentation that each vertex is connected by an edge only to the next highest vertex, counting left to right along the top row then right to left along the bottom. The diagram above is in its standard presentation.

Let $d(x)$ be the set of all vertices in the same set as x in the diagram d. The generalized version of a fixed point is a *fixed set*, a set such that for all vertices x in it $x' \in d(x)$ and call such x fixed vertices. The fixed sets of the diagram above are $\{1,1'\}$ and $\{8,9,8',9'\}$.

Define the composition $d_1 \circ d_2$ of partition diagrams $d_1, d_2 \in P_n$ to be the set partition $d_1 \circ d_2 \in P_n$ obtained by placing d_1 above d_2, identifying the bottom vertices of d_1 with the top vertices of d_2, and removing any connected components that live entirely in the middle row. For example,

$d_1 \circ d_2 =$ 

The choice of diagram does not affect this, so diagram composition makes P_n into an associative monoid with identity, $\mathbf{1} =$ , the same identity we had for the symmetric group. Importantly, we do not always have inverses as it is easy to see that we cannot compose anything with the empty diagram, the diagram with no connections, to get the identity. In fact the only invertible diagrams are the permutations. When we define conjugation therefore, we only say that two diagrams d_1 and d_2 are conjugate if there is a permutation σ such that $\sigma \circ d_1 \circ \sigma^{-1} = d_2$ rather than allowing conjugation by any diagram.

We define the *propagating number* of a diagram, $\text{pn}(d)$, to be the number of sets which include elements from both the top and bottom rows. Hence the propagating number of the first example above would be three. It is an easy fact to see that the propagating number satisfies $\text{pn}(d_1 \circ d_2) \leq \min(\text{pn}(d_1), \text{pn}(d_2))$. Notice that the symmetric group S_n lives inside P_n and in fact $S_n = \{d \in P_n | \ \text{pn}(d) = n\}$.

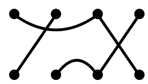
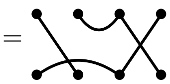
For $n \in \mathbb{Z}_{>0}$ and $n \in \mathbb{C}$, the *partition algebra* $\mathbb{C}P_n(r) = \mathbb{C}\,\text{span-}\{d \in P_n\}$ is an associative algebra over $\mathbb{C}$ with basis P_n. Multiplication in $\mathbb{C}P_n(r)$ is defined by

$$d_1 d_2 = r^{\ell}(d_1 \circ d_2),$$

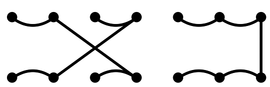
where ℓ is the number of blocks removed from the middle row when constructing the composition $d_1 \circ d_2$ and r is a constant. In the example above $d_1 d_2 = r^2 d_1 \circ d_2$. If r takes on certain small 'bad' values, however, the algebra becomes more difficult to analyze. For instance $\mathbb{C}P_n(r)$ is semi-simple whenever r is not an integer in $[0, 2n-1]$ [Ha]. Thus for this paper we will always assume that $r = 2n$. We often use the same terminology to refer to objects in $P_n(r)$ as we do for P_n, but the reader should assume that we are referring to the algebra unless it is explicitly stated otherwise.

For each $n \in \mathbb{Z}_{>0}$, the partition monoid P_n has several important submonoids including: S_n the symmetric group, B_n the Brauer monoid, R_n the rook monoid, and T_n the Temperley-Lieb monoid. We will use D_n when we discuss properties common to each of these. For each monoid, we can make an associative algebra in the same way that we construct the partition algebra $\mathbb{C}P_n(r)$ from the partition monoid P_n. For example, we obtain the Brauer algebra $\mathbb{C}B_n(r)$, the Temperley-Lieb algebra $\mathbb{C}T_n(r)$, and the group algebra of the symmetric group $\mathbb{C}S_n$ in this way(see [HR1]).

Since P_n and most other diagram monoids do not have inverses, their role is played by the transpose. Define an involution $\theta : P_n \to P_n$ given by flipping or transposing each diagram over the horizontal axis through its middle. For example

$d =$  $\Leftrightarrow$ $d^T =$ 

Borrowing terminology from the symmetric group, we say a diagram in P_n is *symmetric* if $d = d^T$ and that such a diagram is an involution. Let P_n^s be the set of all symmetric diagrams in P_n. This will play the same role as S_n^s did before, a base which we sum our functions over. The diagram above is not symmetric, while the following is

 $= \{\{1, 2, 3', 4'\}, \{3, 4, 1', 2'\}, \{5, 6, 7, 7', 6', 5'\}\}$

We can then define an analogue of conjugation as $d_1 \bullet d_2 = d_1 \circ d_2 \circ d_1^T$. Then if t is symmetric, so is $t \bullet d$ for any d.

4.2 The Generalized Saxl Weight and Conjecture

Given a symmetric diagram $t \in P_n^s$ and a permutation $\sigma \in S_n$, we can naively try to generalize the concept of the Saxl weight

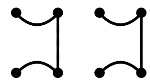
$$\mathrm{sxl}_\sigma(t) = \begin{cases} \mathrm{sign}_{\downarrow t}(\sigma) & \text{if } \sigma \circ t \circ \sigma^{-1} = t \\ 0 & \text{if } \sigma \circ t \circ \sigma^{-1} \neq t \end{cases}$$

It is not, however, clear that the definition of $\mathrm{sign}_{\downarrow t}(\sigma)$ as given before, the sign of the permutation induced by σ on the fixed sets of t is even well defined. For this we will need the a better understanding of when $\sigma t \sigma^{-1} = t$.

Lemma 4.2.1. *Suppose $\sigma t \sigma^{-1} = t$. Then two vertices x and y both in the top row are connected in t iff $\sigma^{-1}(x)$ and $\sigma^{-1}(y)$ are. Also a vertex x is fixed by t iff $\sigma^{-1}(x)$ is.*

Proof. The first part comes from simply examining the picture for $\sigma t \sigma^{-1}$, while for the second part, note that $x' \in t(x) \iff x' \in (\sigma \circ t \circ \sigma^{-1})(x) \iff \sigma^{-1}(x) \in (\sigma^{-1} \circ t)(x)$. □

Combining these facts, we see that σ must take each fixed set of t to another fixed set of the same size. Thus simply labelling the fixed sets $1, \ldots, m$, σ must act on them as a permutation in S_m so we can indeed compute its sign. If

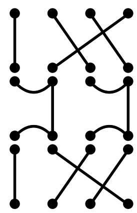
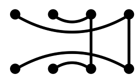
$t =$  $= \{\{1,2,1',2'\},\{3,4,3',4'\}\}$

Both sets are fixed and of the same size, so any permutation which fixes t under conjugation must act on these sets as a permutation of S_2. Examples of Saxl weight calculations are

σ

t

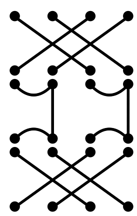
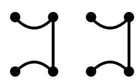
σ^{-1}

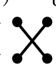
 $=$  $\neq t \implies \mathrm{sxl}_\sigma(t) = 0$

σ

t

σ^{-1}

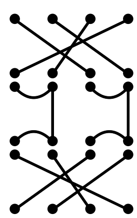
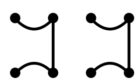
 $=$  $= t$

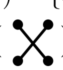
$\sigma(\{1,2,1',2'\}) = \{3,4,3',4'\}$ and $\sigma(\{3,4,3',4'\}) = \{1,2,1',2'\}$, hence $\mathrm{sxl}_\sigma(t) = \mathrm{sign}_{\downarrow t}(\sigma) = \mathrm{sign}($  $) = -1$

σ

t

σ^{-1}

 $=$  $= t$

$\sigma(\{1,2,1',2'\}) = \{3,4,3',4'\}$ and $\sigma(\{3,4,3',4'\}) = \{1,2,1',2'\}$, hence $\mathrm{sxl}_\sigma(t) = \mathrm{sign}_{\downarrow t}(\sigma) = \mathrm{sign}($  $) = -1$

These last two examples show that two different permutations may look the same when restricted to the fixed sets.

As before we conjecture that sum of the Saxl weight with respect to a permutation σ over all symmetric diagrams is equal to the sum of the irreducible characters of the partition algebra evaluated on σ. Of course, we have immediate problems because $P_n(r)$ is an algebra, while we have defined representation theory only for groups. It is, however, possible to define it on the partition algebra in a way that many analogous results hold. An analogue of the Roichman weight does not exist, so in order to calculate the sum of the irreducibles though, we must find some other way of calculating the irreducibles. We will first need to introduce some additional concepts, both as they relate to the symmetric group and to the partition algebra.

4.3 Young Tableaux

The Young (or Ferrers) diagram of a partition λ is the left-justified array of boxes with λ_i boxes in row i. Given two partitions λ, μ we say that $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all i. Given $\mu \subset \lambda$, let λ/μ denote the skew shape given by removing μ from λ, that is, $\{\lambda_1 - \mu_1, \ldots, \lambda_l - \mu_l\}$. For instance,

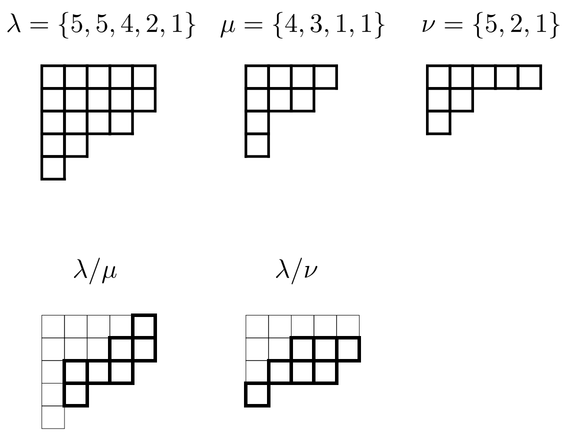


Figure 4.1: Young diagrams and skew shapes

We say that the skew shape λ/μ is a border strip if all the boxes are connected and it does not contain any 2 by 2 blocks. Thus the first skew shape above is a border strip but the second is not as it both contains a 2 by 2 box and is not connected. We say that the height of a border strip is equal to the number of rows occupied by the strip minus 1. Thus the first border strip above has height 3.

For a partition $\lambda \vdash n$, a standard Young tableau, T_λ, of shape λ is a filling of the boxes of the Young diagram with the integers $\{1, \ldots, n\}$ in such a way that the entries increase from left to right in the rows and top to bottom in the columns. While there are other kinds of tableau, we will only be considering standard ones, and so we will usually drop the qualifier. for example, there are five standard tableaux of shape $\{3, 2\}$:

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

Lemma 4.3.1. *In any standard tableau, $(i+1)$ must always be either north-east or south-west of i. In particular it cannot be directly north or west, and if it is directly south or east, it must be adjacent to i.*

Proof. Whenever we move one box to the right or down in the diagram, we must increase by at least one. Hence anything to the NW would be smaller than i and anything to the SE would be at least 2 more.

□

The Roichman weight can be, and in fact was first, defined on a tableau. It is taken with respect to a second partition of the same size as the tableau, so it is a two variable function, but we will most often write $\mathrm{rwt}_\mu(T_\lambda)$ where T_λ is the tableau and μ the partition. We define it recursively, starting with the full partition,

$$\mathrm{rwt}_{\{n\}}(T_\lambda) = \prod_{i=1}^{n-1} r_i \quad \text{where } r_i = \begin{cases} +1 & \text{if i+1 is to the NE of i in } \lambda \\ -1 & \text{if i+1 is to the SW of i} \end{cases} \tag{4.3.1}$$

If, however, we ever get a positive term immediately followed by a negative one, the whole product becomes 0. We then define

$$\mathrm{rwt}_{\{\mu_1,\ldots,\mu_l\}}(T_\lambda) = \mathrm{rwt}_{\{\mu_l\}}(T_{\lambda^{\mu_l}}) * \mathrm{rwt}_{\{\mu_1,\ldots,\mu_{l-1}\}}(T_\lambda / T_{\lambda^{\mu_l}}) \tag{4.3.2}$$

where $T_{\lambda^{\mu_l}}$ is the tableau formed by picking out only the boxes in T_λ having entries from $n - \mu_l + 1$ to n. While the preceding definition of the Roichman weight isn't defined here, it is obvious how to extend it. For example,

Lemma 4.3.2. *If λ/μ is a border strip of size k then there is one and only one way to fill λ/μ with a set of k distinct integers so that $\mathrm{rwt}_{\{k\}}(T_\lambda/\mu) \neq 0$ and in that case $\mathrm{rwt}_{\{k\}}(\lambda/\mu) = (-1)^{ht(\lambda/\mu)}$.*

Proof. Since the Roichman only deals with the relative sizes of the numbers, we may assume that they are $\{1, \ldots, k\}$. We must increase down the columns and increase to the right, but we must never have consecutive moves up then down. Thus once we start moving up, we can never come back down, so we must first fill down the columns, the come back up filling in the rows. □

$$\lambda \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \mathrm{rwt}_{\{7\}}(\lambda) = (-1)(-1)(-1)(+1)(0)(+1)(+1) = 0$$

$$\mu \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & 10 & 11 \\ \hline 7 & 9 & \multicolumn{2}{c}{} \\ \cline{1-2} 8 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \mathrm{rwt}_{\{5,4,2\}}(\mu) = \mathrm{rwt}_{\{5\}}\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\right) * \mathrm{rwt}_{\{4\}}\left(\begin{array}{cc} & \multicolumn{1}{|c|}{6} \\ \hline \multicolumn{1}{|c|}{7} & \multicolumn{1}{c|}{9} \\ \hline \multicolumn{1}{|c|}{8} & \\ \cline{1-1}\end{array}\right)$$

$$* \,\mathrm{rwt}_{\{2\}}\left(\begin{array}{|c|c|}\hline 10 & 11 \\ \hline\end{array}\right)$$

$$= (-1) * (-1)^2 * (+1) = -1$$

Figure 4.2: Computing Roichman weights

$$\lambda/\mu \qquad \begin{array}{cccc} & & & \multicolumn{1}{|c|}{1} \\ \cline{4-4} & & & \multicolumn{1}{|c|}{2} \\ \cline{4-4} & & & \multicolumn{1}{|c|}{3} \\ \cline{3-4} & & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{11} \\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{10} \\ \hline \multicolumn{1}{|c|}{6} & & & \\ \cline{1-1} \multicolumn{1}{|c|}{7} & & & \\ \cline{1-1}\end{array} \qquad \mathrm{rwt}_{\{11\}}(\lambda/\mu) = (-1)(-1)(-1)(-1)(-1)(-1)(-1)(+1)(+1)(+1)(+1) = -1$$

Figure 4.3: The only way to fill this border strip having non-zero Roichman weight

4.4 Bratteli Diagrams

The *Bratteli diagram*, Λ_{D_n} is a figure which shows the relationship between irreducible characters of a diagram algebras of successive sizes. Just as we can induce a representation from a subgroup to the whole group, we can restrict a representation on the whole group to a subgroup. In particular, each symmetric group contains a copy of each smaller symmetric group within it, i.e. $S_1 \subseteq \ldots \subseteq S_{n-1} \subseteq S_n$, and similarly for the partition algebra. When we restrict an irreducible representation from S_n to S_{n-1}, we may not, however, get another irreducible, but we must get some combination of the irreducibles. Recalling that we can label the irreducibles by cycle types which can be viewed as partitions, we construct the Bratteli diagram, Λ_{S_n}, by putting in the $n-th$ row the tableaux which correspond to these partitions and by connecting vertices in successive rows wherever the irreducible on top appears in the restriction of the one on the bottom. There turns out to be a simple interpretation of these connections for S_n. The irreducibles are indexed by the partitions of n and two tableau are linked if the one can be obtained from the second by adding a single box. If we count the number of paths from an irreducible to the first row, this tells us the dimension of the irreducible, which is just the value of the character evaluated on the identity.

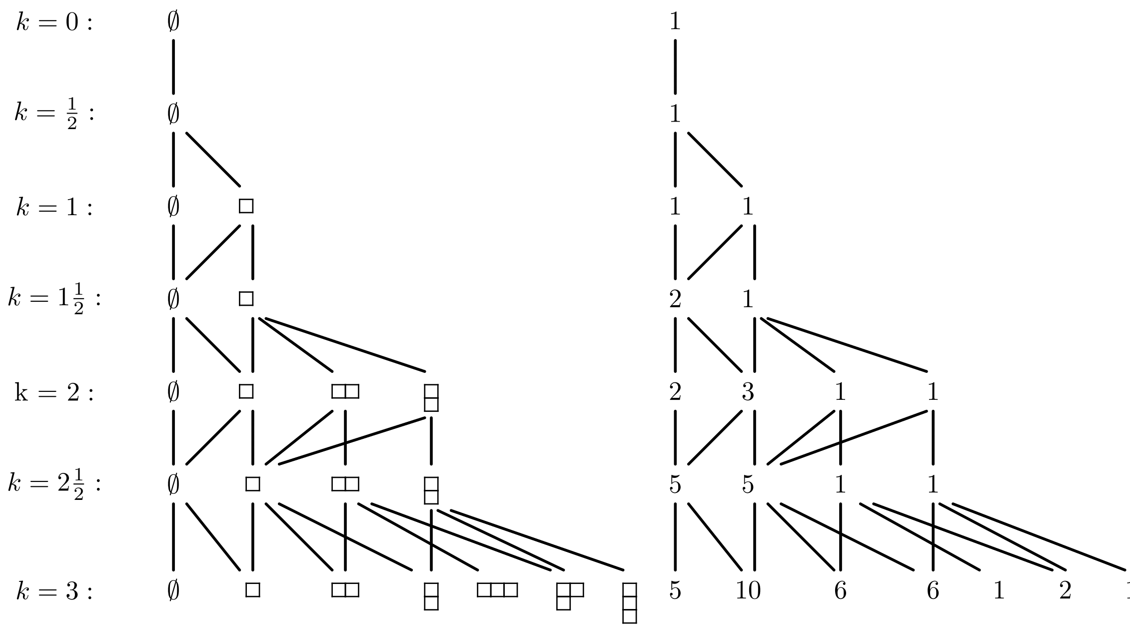


Figure 5.5

The Bratteli diagram for P_n is slightly more complicated. The irreducibles in row n here are indexed by all tableaux with size less than or equal to n. We have two steps between each level. In the first we either remove a box or do nothing and in the second we can either add a box or do nothing. The Bratteli diagram $\Gamma_{P_n(r)}$ is given in Figure 5.6.

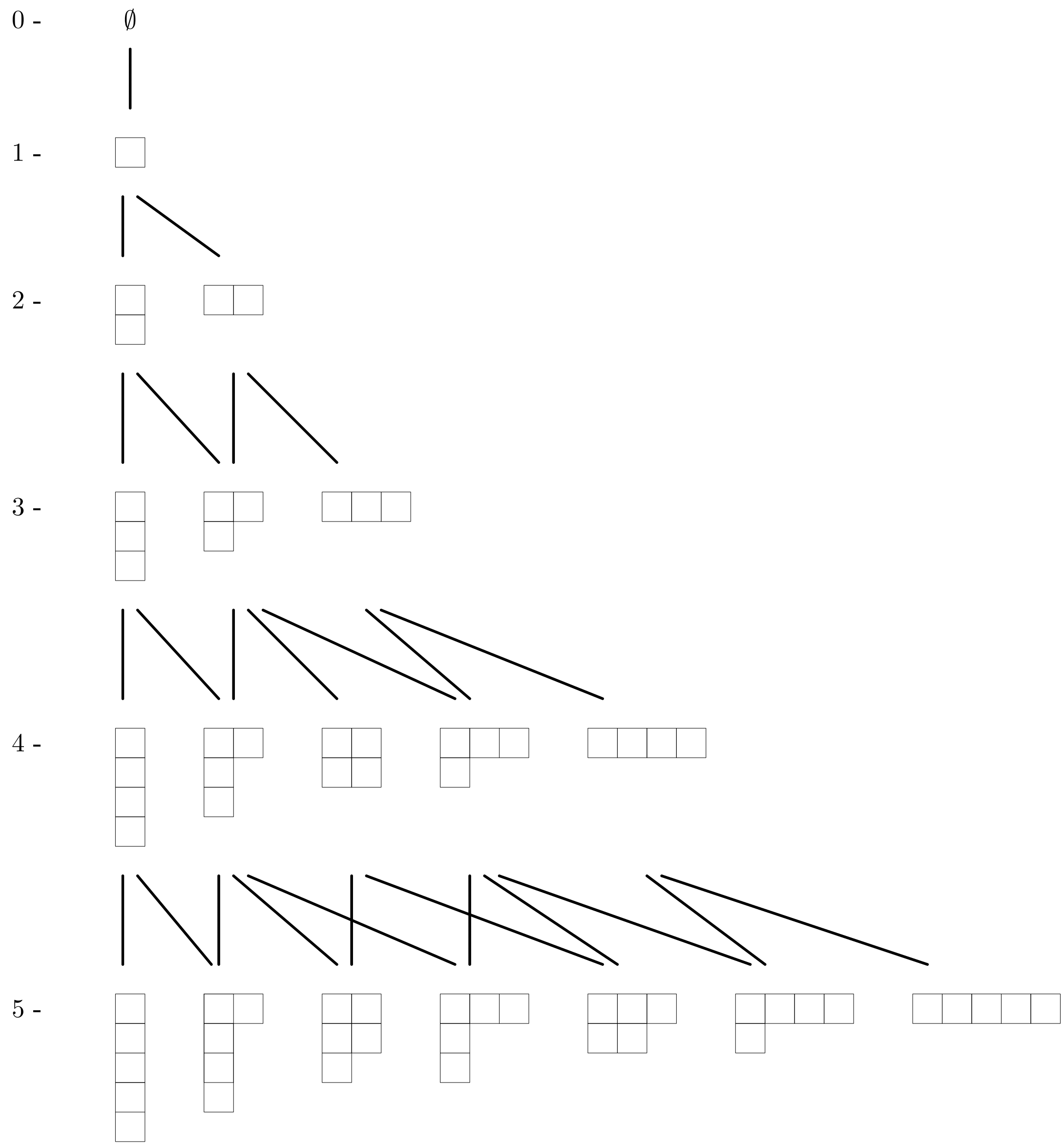


Figure 4.4: The Bratteli diagram for S_n

4.5 RSK Insertion

While tableaux are studied on their own for many reasons, their importance in this paper is due to the fact that they can be linked with symmetric diagrams in a 'nice' way. This method is known as Robinson-Schensted-Knuth Insertion, or simply RSK. It gives a bijective correspondence between elements of S_n and pairs of standard Young tableaux of the same shape. Formally

$$\begin{aligned} S_n &\iff \bigsqcup_{\lambda \in \Lambda_{S_n}} \mathcal{T}^\lambda \times \mathcal{T}^\lambda \\ d &\stackrel{\text{RSK}}{\longleftrightarrow} (P_\lambda, Q_\lambda). \end{aligned} \tag{4.5.1}$$

where $\mathcal{T}^\lambda$ is the set of all tableaux having shape λ.

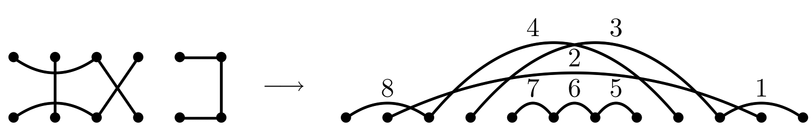
Alternatively, it can be thought of as giving a pair of paths in the Bratteli diagram for that algebra from top to the $n-th$ row and back. Generally, we will only care about the terminal node and say that a diagram inserts to this tableau, though this is of course ignoring much information.

In the symmetric group case, the RSK simply involves applying the following algorithm to the numbers $1, \ldots, n$ in the order they appear in the permutation and then removing them in reverse order $n, \ldots 1$.

Bump algorithm:
Given a partial Young tableau on $[1, \ldots, n]$, we insert the element k by finding the position in the first column where k can sit. Since the tableau is standard, there will be only one such position with the property that the element above is smaller than k and the element below is larger. This may be at the bottom of the column. If it is not at the bottom, then put k in this position and insert whatever number k has replaced into the second column, etc.

At each step we will have a standard tableau since we start with one, the empty tableau, and at each step the bumping algorithm ensures it remains standard. When we remove a box, it will always be at the outside of the tableau, so this too creates no problems. Thus for instance:

RSK can be generalized to the other diagram algebras. To transform a diagram, we first 'open' it, so that all the vertices are in a single row, beginning with the upper left to upper right then lower right to lower left, and connect each vertex to the next highest one which it is connected to in the diagram. We then enumerate the edges in the order they appear from right to left:



Starting now from the leftmost vertex, we construct the tableau by first inserting the number of the edge coming into that vertex from the left, if any, and then removing the number of the edge leaving to the right, if any. As we perform these actions we must always have a standard tableau for the same reasons as above, so this too can be seen as a path through the Bratteli diagram. To continue our example, omitting steps where the diagram does not change, we

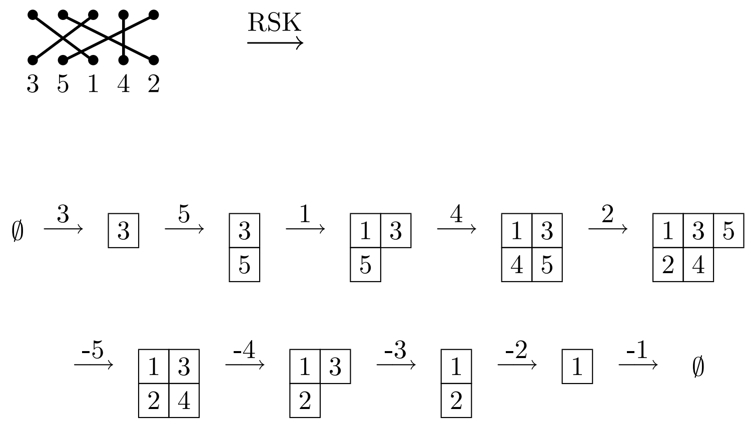
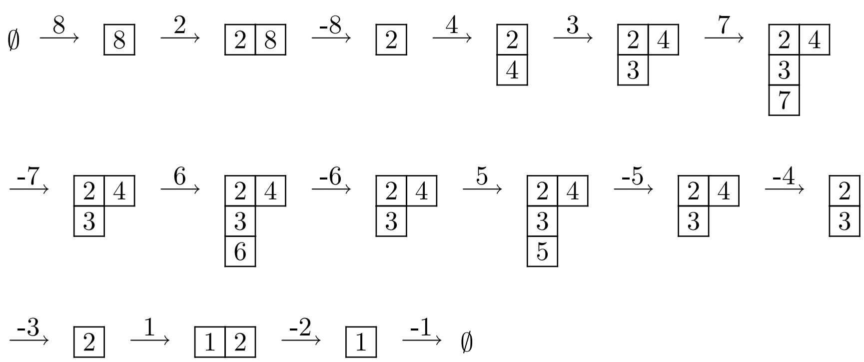


Figure 4.5: RSK Insertion for the permutation 35142. The corresponding Bratteli diagram path is highlighted in Figure 4.4

then have



Lemma 4.5.1. *The terminal node, the one on the bottom row of the Bratteli diagram, will be the same after performing RSK on all strands of a diagram versus just doing it on the propagating strands.*

Proof. At the middle node, we will not have yet introduced any of the edges which go between bottom vertices and so all their existence means is that the other edges are numbered higher, but since insertion only cares about relative size, this doesn't affect it. Correspondingly the top edges will have been deleted off so they do not directly affect the outcome and they wouldn't changed how the insertion proceeded at any intermediate steps since they are all numbered higher than all propagating strands, so they will sit at the bottom of the tableau until it is time for them to be removed. □

A corollary is that this definition of RSK matches the one given for the symmetric group.

4.6 Properties of Insertion

In [HL] it is shown that if $d \xrightarrow{\text{RSK}} (P_\lambda, Q_\lambda)$, then

1. the propagating number of d, $\text{pn}(d) = l$ if and only if $|\lambda| = l$.
2. d is planar if and only if each vertex in the pair of paths (P_λ, Q_λ) is a single column (or $\emptyset$).
3. $d \xrightarrow{\text{RSK}} (P_\lambda, Q_\lambda)$ if and only if $d^T \xrightarrow{\text{RSK}} (Q_\lambda, P_\lambda)$.
4. d is symmetric ($d = d^T$) if and only if $Q_\lambda = P_\lambda$.

The last item shows that for each of our diagram algebras D_k,

$$|\{d \in D_k \mid d = d^T \}| = \sum_{\lambda \in \Lambda_{D_k}} \dim(V_{D_k}^\lambda) \tag{4.6.1}$$

When d is a symmetric diagram, we can write $d \xrightarrow{\text{RSK}} Q_\lambda$ in place of $d \xrightarrow{\text{RSK}} (Q_\lambda, Q_\lambda)$. In fact we will often use the diagram and the tableau interchangeably.

Proposition 4.6.1. *If d is a symmetric diagram such that $d \xrightarrow{\text{RSK}} Q_\lambda$, then Q_λ is a single column (or $\emptyset$) iff the propagating strands of d are planar.*

Proof. Applying lemma 4.5.1, it is only the boxes representing propagating edges which will remain. A crossing between these edges will produce a second column in Q_λ, while the lack of any such crossing will mean that the edges are inserted in order, resulting in a single column. □

Conjecture 4.6.2. *Define the bump path of a number k with respect to d to be the sequence of entries which get bumped when k is inserted into the growing tableau of RSK for the diagram d. If d is symmetric, then for each disjoint transposition $(ij), i < j$ of propagating strands in d then when j is inserted its bump path terminates at i.*

This, unproved conjecture, would have a corollary which will help us refine the sum from over all symmetric diagram to being equal over all those with the same number of fixed points.

Proposition 4.6.3. *If d is a symmetric diagram such that $d \xrightarrow{\text{RSK}} Q_\lambda$, then*

$$(d \text{ has } f \text{ fixed points}) \quad \Leftrightarrow \quad (\lambda \text{ has } f \text{ odd parts})$$

Proof. Proof 1:
We can compute the number of rows with odd length by taking the alternating sum of the sizes of the columns. Now suppose t is a symmetric permutation in S_n. Then examine how n appears within the growing tableau for t. If n is switched with another, $t(n)$, then it is first inserted on the $t(n)$-th step. Since it is larger than all other entries it will always stay at the bottom of whatever column it's in, not affecting the insertion of any other terms. When we finally insert $t(n)$ at the last step, we must bump n, which will then land in the next column. So comparing the shape of t to the shape of the permutation we would have had if we had not inserted n and $t(n)$ we see that the first has an extra box in two adjacent columns. But then the alternating sums are equal, while the number of fixed points are also equal. If n is fixed, then removing must change the alternating sum by one even as it decreases the number of fixed points by one. Inducting, we see that the two amounts must be equal.

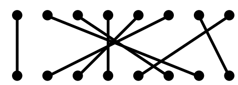
Proof 2: Fortunately we do not have to rely on this unproven assumption, but can use a different formulation of RSK insertion to prove this. It follows immediately from Viennot's Geometric construction RSK given in [Sa], pg. 112. □

Proposition 4.6.4. *If we use RSK as a bijection between symmetric diagrams and tableaux, then this definition of the Roichman weight is equivalent to the one given earlier. For a diagrams* $t \xrightarrow{\text{RSK}} T_\lambda$ *and* σ*,*

$$\text{rwt}_\sigma(t) = \text{rwt}_{\rho(\sigma)}(T_\lambda) \tag{4.6.2}$$

Proof. Computing the Roichman weight of a permutation which has been inserted into a tableau, $r_i = +1$ iff $\sigma(i) > \sigma(i+1)$If i is inserted before $i+1$ then i will always be to the NE of $i+1$ as if they were ever in the same column, then any term which would bump $i+1$ will bump i instead, so $i+1$ cannot move past i and so will always be to the SW. Conversely if $i+1$ is inserted before i then i can never be in the same column as $i+1$ as it would always bump it instead. Thus i will always be to the SW of $i+1$. □

For example:

$t =$ 

$$\mathrm{rwt}_{\{5,3\}}(t) = \mathrm{rwt}_{\{5\}}(15432) * \mathrm{rwt}_{\{3\}}(687)$$

$$= [(-1)(1)(1)(1)(1)] * [(-1)(1)] = 1$$

$$= \mathrm{rwt}_{\{5,3\}}\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 8 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \right)$$

Figure 4.6: Computing the Roichman weight of a diagram and its associated tableau.

Unfortunately, while it makes sense to define the Roichman weight in this way for the partition algebra, the sum is not what we want.

Chapter 5

Future Directions

5.1 The Murnaghan-Nakayama Rule

The Murnaghan-Nakayama rule is a recursive derivation for the value of the irreducible characters of S_n. It is given by

$$\chi^{\lambda}_{S_n}(\gamma_\mu) = \sum_{\lambda^{-k}\subset\lambda} (-1)^{ht(\lambda/\lambda^{-k})} \chi^{\lambda^{-k}}_{S_{n-k}}(\gamma_{\overline{\mu}}) \tag{5.1.1}$$

where $\lambda, \mu \vdash n, \overline{\mu}$ is obtained by removing a part of size k, and the sum is over all partitions such that λ/λ^{-k} is a border strip of size k.

The Murnaghan-Nakayama weight is then defined by noticing that at each step in the above process there is exactly one way that each border strip could have been filled, so summing over all ways to remove a border strip from a Young diagram is equivalent to considering all the ways to remove a border strip from a tableau. This leads to a definition which is similar to how we defined the Roichman weight, but that at each step the block we remove we must be border strip. In fact, historically this was done in the opposite order, the Roichman weight was derived from the definition of the Murnaghan-Nakayama weight. Importantly for $MN_\mu(T_\lambda)$, the weight of a tableau with respect to a partition

$$\sum_{T_\lambda} \mathrm{MN}_\mu(T_\lambda) = \chi^{\lambda}_{S_n}(\gamma_\mu)$$

which immediately leads to the fact that the sum over all tableau is equal to the sum of all the irreducible characters. If the Murnaghan-Nakayama weight is non-zero it is equal to the Roichman weight and it turns out that over symmetric diagrams all the terms with Murnaghan-Nakayama weight of zero but non-zero Roichman weight can be paired off and cancelled, hence for all σ

$$\sum_{t\in S_n^s} \mathrm{rwt}_\sigma(t) = \sum_{t\in S_n^s} \mathrm{MN}_\sigma(t) = \sum_{\chi_i \textit{ is irreducible}} \chi_i \tag{5.1.2}$$

And while we don't have these more convenient definitions for the sum of the irreducibles, Halverson [Ha] proved an analogous Murnaghan-Nakayama rule for the partition algebra.

Theorem 5.1.1. *Let $\lambda \vdash r$ with $|\lambda^*| \le n$. Let μ be a partition with $0 \le |\mu| \le n$, and let μ be the partition obtained from μ by removing a block of size k. Then*

$$\chi^{\lambda}_{P_n(r)}(\gamma_\mu) = \sum_{d|k} \sum_{\substack{\delta=(\lambda^{-k})^{+k} \\ |\delta^*|\le n-k}} (-1)^{ht(\lambda/\lambda^{-d})}(-1)^{ht(\delta/\lambda^{-d})}\chi^{\delta}_{P_{n-k}(r)}(d_{\overline{\mu}})$$

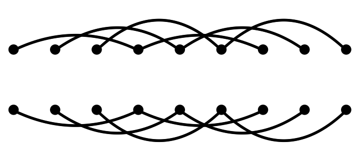
where the outer sum is over all divisors of d of k and the inner sum is over all $\delta \vdash r$ with $|\delta^*| \le n-k$ such that δ is obtained from λ by removing a border strip of size d to obtain λ^{-1} and then adding a border strip of size d back on, possibly in the same location.

While it is cumbersome, the Murnaghan-Nakayama rule allows us to start proving the conjecture for certain cases. Just as we did in S_n, we start by characterizing the instances when $\mathrm{sxl}_\sigma(t)$ is nonzero, beginning with the simple, full cycles and then trying build upwards.

Proposition 5.1.2. *If d is a cycle of length k, then*

$$\sum_{t\in P_n^s(r)} \mathrm{sxl}_n(\sigma) = \sum_{d|n} \chi_{S_d}(d) + 1 \tag{5.1.3}$$

Proof. Let σ be a symmetric diagram which fixes $d$and $c(i)$ be the number of vertices connected to i. We have that $c(x) = c(\sigma(x)) = c(x+1)$ and because of symmetry, this means that they are connected to the same number of vertices in the top row. Inducting, we see that each vertex must be connected to the same number of vertices, so there must be an integer number of these sets. In particular, if we suppose there are i such sets, we must be able to divide the top and bottom rows into sets of the form $\{\frac{k}{i} \cdot n + j | 1 \le n \le i\}$



If any one of these sets is connected from top to bottom, then they all must be. In that case we are simply dealing with a full cycle on of length i in S_i. Hence the sum of the terms in this form will simply be $\chi_{S_i}(i)$ but we will have one additional positive term from the case when none of the sets are connected. □

Theorem 5.1.3. *If d is a full cycle of length n in $P_n(r)$ where $r \geq 2n$, then*

$$\sum_{\lambda} \chi^{\lambda}_{P_n(r)}(d) = \sum_{t \in P^s_n(r)} \mathrm{sxl}_{\{d\}}(d) \tag{5.1.4}$$

Proof. Applying Theorem 6.1.1, we have that:

$$\begin{aligned}\sum_{\lambda} \chi^{\lambda}_{P_n(r)}(d) &= \sum_{\lambda \in P_n(r)} \Bigg(\sum_{d|n} \sum_{\substack{\delta=(\lambda^{-d})^{+d} \\ \delta\,=\,\{r\}}} (-1)^{ht(\lambda/\lambda^{-d})} \Bigg) \\ &= \sum_{d|n} \Bigg(\sum_{\lambda \in P_n(r)} \sum_{\substack{\delta=(\lambda^{-d})^{+d} \\ \delta\,=\,\{r\}}} (-1)^{ht(\lambda/\lambda^{-d})} \Bigg) \\ &= \sum_{d|n} \Bigg(1 + \sum_{t \in S^s_n} \mathrm{rwt}_{\{n\}}(t) \Bigg) \\ &= \sum_{d|n} \chi_{S_d}(d) + 1\end{aligned}$$

where $\delta = (\lambda^{-d})^{+d}$ means that δ is obtained from λ by first removing a border strip of size d then replacing a possibly different one of the same size. The extra 1 in each case comes from the situation where $\lambda = \{r\}$ and so we are just removing and replacing the final d boxes. □

Define

$$f(\mu,\nu) = (2\mu)^{\nu-1} + \begin{cases} \mu^{\nu-1} & \text{if } \mu \text{ is odd} \\ 0 & \text{if } \mu \text{ is even} \end{cases}$$

and let

$$g(\mu,\alpha) = \sum_{\nu \vdash \alpha} \prod_i f(\mu, |\nu_i|)$$

where the sum is over not partitions but rather set partitions of α, i.e. $\{\{1\},\{2,3\}\} \neq \{\{1,2\},\{3\}\}$.

Conjecture 5.1.4.

$$\sum_{t \in P^s_n(r)} \mathrm{sxl}_{\sigma}(t) = \sum_{\mu|\rho(\sigma)} \prod_i g(\mu_i, m(\mu_i)) \tag{5.1.5}$$

where in the first sum $\mu|\rho(\sigma)$ means that each μ_i is a divisor of $\rho(\sigma)_i$ and $m(\mu_i)$ is multiplicity of μ_i.

Proving that the sum of the irreducibles is equal to this as well has thus far proven intractable. Computing, this formula works at least up to $P_5(r)$, but I have been unable to verify it past that point. Indeed, one would rather prove esoterically that the two sums must be equal and then use just the one to compute with. Unfortunately, this is not possible at the present time.

5.2 Conclusion

In S_n the Murnaghan-Nakayama weight, and hence Roichman weight, are derived from a Murnaghan-Nakayama formula on shapes similar to that in Theorem 6.1.1. This is done by noticing that at each step the border strip removed could only have been filled in a single way. Thus, we have that the sum of the MN-weight over tableau of shape λ is equal to using the MN-rule to compute χ^λ. We can perform a similar operation here, but unfortunately in most cases the weight will not be +1, 0, or -1. Thus we cannot run the same kind of proof, since it relies on pairing off opposite entries, though some modification may be possible. Alternatively, we could sum the Saxl weight over all tableau of a given shape and prove that this sum obeys the same inductive rule as the Murnaghan-Nakayama rule for the irreducible characters. Finally, in [HL] a formula is given which relates the irreducible characters in $P_n(r)$ to those in S_r. However, the sum of the irreducibles in one will not be immediately related to the sum of the irreducibles in the others as certain terms will be missing.

Even if we can prove that the sum of the Saxl weights is equal to the sum of the irreducibles, we will not immediately have a model. In S_n we were able to extend the definition of the sign from the points which are fix an involution to those that do not. We did this by constructing the coset relative to the centralizer of a standard element. But it is unclear how to do this in the partition algebra where we cannot as easily break the set into cosets.

Nevertheless, the paper shows the strength of representation theory and the ability to use combinatorics to prove algebraic statements.

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