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The Geometry of Intuitions: Reconsidering Kantian Constructivism

Michael McNulty
Macalester College

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Author: Michael McNulty

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The Geometry of Intuitions

Reconsidering Kantian Constructivism

Michael Bennett McNulty
Macalester College
Submitted May 5th, 2008

Honors Thesis Committee

Professor Janet Folina, Advisor,
Department of Philosophy, Macalester College

Professor Martin Gunderson, Second Reader,
Department of Philosophy, Macalester College

Professor Roy Cook, Third Reader,
Department of Philosophy, University of Minnesota

Abstract

The role of visual methods in geometry is puzzling. Though diagrams can make a geometric theorem immediately evident, current rules of proper inference suggest that diagrams are mere heuristics—simply aiding in the psychological digestibility of a proof. Securing a justificatory role for visual methods involves describing how inference from a diagram guarantees the universality and the *a priority* of a geometric theorem. Such an analysis is provided in Kant's synthetic *a priori* account of geometry. In this paper, Kant's theory is explicated and subsequently defended from attacks related to modern advances in predicate logic, relativistic physics, non-Euclidean geometry and formalism.

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My paper is dedicated to my father, James McNulty, the source of my enthusiasm for all things knowable.

Contents

Are we then to admit that the enunciations of all the theorems with which so many volumes are filled are only indirect ways of saying that A is A?

Poincaré 1983: 394

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1 Framing the Problem

Visual evidence plays an enigmatic role in geometric proof. The proofs of Euclid's *Elements* instruct the reader to construct triangles, produce sides and draw lines. But what role do these constructions play in proving the theorems of geometry? Take for example, Euclid's Proposition 32 from Book I:

> *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*
>
> Heath 1965: 319

In the proof of this proposition, Euclid "produces" one of the sides of a triangle, then draws a line parallel to another side of the triangle as follows:

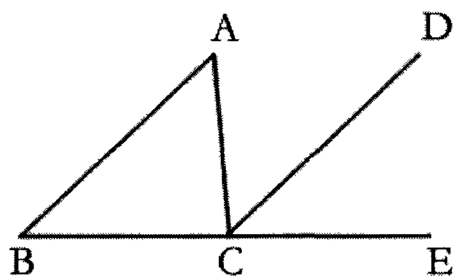

Figure 1: Diagram Accompanying Euclid's Proposition 32

After extending BC to E and drawing CD parallel to AB, Euclid appeals to propositions of angle equality to prove that $\angle ABC + \angle BCA + \angle CAB$ is equal to 180°. Is this appeal to the drawn triangle necessary for the proof? Could one simply lay down Euclidean definitions, postulates and common notions, then subsequently derive Proposition 32 without reference to a visualized triangle?

The place of visualization in mathematics is ambiguous. The drawn triangle in Figure 1 above certainly serves as a heuristic tool; one can more easily conclude that Proposition 32 is true when the diagram is before one's eyes. But can the diagram be said to serve an *epistemic* or *justificatory* role? A diagram or an image is only a single instantiation of a mathematical concept; attributing a property to a particular of a class is certainly not sufficient to conclude that the property holds for all particulars. We are at something of an impasse. Diagrams *appear* to serve a role in proof, but the machinery of valid inference would tell us otherwise.

Diagrams were not always so marginalized. Prior to the rigorization of analysis,[1] arithmetic and geometry,[2] visual techniques were accepted as valid methods

[1] In "Purely Analytic Proof", one of the key foundational texts of analysis, Bernard Bolzano *explicitly* rejects visual techniques for proving the Intermediate Zero Theorem (Bolzano 1996). This paper effectively set the tone for the acceptable proof methods in analysis—casting aside diagrams as 'misleading'.

[2] The rigorization of geometry was brought about, not only by the advent of non-Euclidean geometries (See §4), but also the introduction of imaginary points. For an exceptional anal-

of inference. Indeed, Mark Greaves pins the downfall of visual techniques not upon an increasing sophistication in mathematical proof, but rather on shifts in the accepted ontology of mathematical objects—though there is no *intrinsic* characteristic of diagrams which means they are unreliable tools of inference.

There are, however, two major obstacles to securing an epistemic or justificatory role for visual evidence: universality and *a priority*. The properties of diagrams are easily discernible—yet disentangling these properties into those which are universal and those which are particular proves to be difficult in some cases. What is required is a reliable inference from the characteristics of the token to those which are universal for the type.

However, an inference such as empirical induction, though often reliable, will not do. Through inspecting a large number of triangles, producing their sides and drawing parallel lines, one can *inductively* infer with reasonable certainty that the sum of the angles of *every* triangles adds up to 180°. Proposition 32, when justified in such a manner, becomes a contingent fact—true only insofar as empirical triangles have been seen to have the characteristic angle sum. To account for the *a priority* of geometry the inference from a diagram must *necessarily* guarantee a property to hold universally. The difficulty of describing such an inference is compounded by the empirical nature of a diagram. How can we infer a universal, *a priori* truth of geometry from an empirical object?

These perplexities draw us toward Kantian constructivism. By claiming geometry to be the paradigm of synthetic *a priori* knowledge in his *Critique of Pure Reason*, Kant appears to allow visual methods in the derivation of universal, *a priori* truths of geometry. Understanding the Kantian account of geometry may reveal a way to secure a justificatory role for diagrams generally.

One might question the motivation for reconsidering Kantian constructivism. Indeed, Michael Friedman sums up the views of many as follows: "Kant's conception [of geometry] is liable to seem quaint at best and silly at worst" (Friedman 1985: 455). The fact that Kantian constructivism is all-too-quickly discounted in the philosophy of mathematics constitutes another impetus for my project. The difficulties surrounding the interpretation of Kant's philosophy, his dependence on antiquated logic, controversy over the possibility of synthetic *a priori* knowledge and advances in pure geometry have all contributed to the commonplace dismissal of constructivism. Kant's philosophy of mathematics is frequently judged as something of an oddity—only to be studied from afar. The Marburg neo-Kantians, notably Cassirer, appear to frame subsequent approaches to Kant's philosophy in just this manner: though his motives were pure, Kant's appeal to intuition in geometry is simply untenable.[3] This reading of Kant persists. Jesse Norman, a contemporary neo-Kantian who defends the justificatory role of diagrams, similarly rejects Kant's role for *a priori* intuition (Norman 2006).

ysis of the downfall of visual techniques in geometry in the time of Poncelet and projective geometry, see Greaves 2002: 43-81.

[3]See Friedman's seminal "Philosophy and the Exact Sciences" (Friedman 1992) for characterizations of turn of the twentieth century analytic approaches to Kant's theory of mathematics.

M. Bennett McNulty
Macalester College

Yet philosophy of mathematics is currently faced with concerns akin to those which Kant addressed—namely the role of visual evidence and the relation between geometry and the world. Moreover, constructivism provides us with a desirable description of geometry. Kant's view of mathematics as synthetic *a priori* explains how geometry can be pure, while still expressing meaningful truths. An analytic description of geometry does not accord with our intuitions about geometry—it defies our experiences with geometry to think of it as a mere string of tautologies. The constructivist move to *ground* geometry in intuition provides an explanation for how geometric truths correspond to reality. This is a point which formalist philosophies of geometry tend to overlook. Furthermore, the Kantian picture of geometry provides us with a more expansive epistemic framework than alternative solutions which lapse into an appeal to a mysterious metaphysical realm, such as Platonism. Considering Kant's constructivism will prove to shed light on the precise role of visual evidence.

Our investigation of constructivism, however, will not simply be an uncritical explication of Kant's account of geometry in the *Critique*. Due to advances in geometry, mathematics and logic since Kant's time, his account *cannot* legitimately be taken at face-value. In this paper, we will explicate, rehabilitate and supplement Kant's theory of geometry while leaving the core principles intact. That is, while adhering to the central doctrines of synthetic *a priority* and pure spatio-temporal intuition we will attempt to provide a reasonable account of the justificatory role of geometric visualization in light of modern mathematical advances. We approach Kant critically but sympathetically—appreciating the motivations and advantages of constructivism as well as its significant shortcomings.

The paper will continue as follows. §2 elucidates our interpretation of Kantian constructivism. We will focus on securing universality and *a priority* for inferences from geometric intuition. After fleshing out our version of constructivism, we will attempt to defend our account from common critiques of constructivism. §3 investigates the possibility of polyadic logic supplanting the role of intuition in geometry. Lastly, §4 concerns advances in modern geometry, namely non-Euclidean geometries and Hilbert's formalist project, which seriously threaten Kant's place for intuition in mathematics.

M. Bennett McNulty
Macalester College

2 Geometric Construction: The Necessity of Intuition

In the "Doctrine of Method", Kant describes mathematical knowledge as "the knowledge gained by reason from the construction of concepts", where "to construct a concept means to exhibit a priori the intuition corresponding to the concept" (A713/B741).[4] One motive for grounding mathematical knowledge in intuition is clear from the famous Kantian adage "Thoughts without content are empty, intuitions without concepts are blind" (A51/B75). In this section, we will reveal why 'thoughts' or concepts alone are inadequate for geometry—which in turn secures the synthetic *a priority* of geometry. To begin, we must explain: (1) The manner by which intuitions and concepts are correlated, and (2) How intuitions are necessary for geometric knowledge.

2.1 Construction and Syntheticity

The characterization of mathematical cognition as that which is focused on the construction of concepts seems to directly follow from the Kant's description of the syntheticity of mathematics. Consider his example of a philosopher and a geometer attempting to derive Euclid's Proposition 32:

> *Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angles bears to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and possessing three angles. However long he meditates on this concept, he will never produce anything new. He can analyse and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any properties not already contained in these concepts. Now let the geometrician take up these questions. He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle—and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a full evident and universally valid solution of the problem.*
>
> A716-7/B744-5

[4] We will use the Norman Kemp Smith translation (Kant 1929). Further references will be cited in text: (First edition page/Second edition page).

The philosopher begins by attempting to prove the proposition purely analytically[5] from rational cognition on mathematical concepts. From the concept *triangle*,[6] one can only deduce properties which are explicitly contained in the concept, those such as *figure*, *three*, or *straight-line*. The understanding only has the capability to parse through the concepts which constitute *triangle*. *The sum of the angles of the figure add up to 180°* is not constitutive of *triangle*—hence, we cannot derive Proposition 32 from conceptual reasoning. Kant's point is that there are certain characteristics of *triangle* which *belong to* the concept, but are not *explicitly contained* in it. One cannot discern the sensible characteristics of a concept until the concept has been synthesized in intuition. Thus, the predicate (Proposition 32), lies outside the concept *triangle*, but stands in connection with it through construction—therefore, Proposition 32 is synthetic.[7]

Two questions challenge this account. First, how can sensible characteristics be said to belong *a priori* universally to a concept or class of intuitions? If the constructed intuition is empirical, any knowledge which issues from the construction would appear to be accordingly empirical. Secondly, what exactly is the connection between a concept and its construction? That is, how can one construct an intuition which corresponds to a concept? Early on in the Critique, Kant makes a sharp disconnect between understanding and sensibility, but at this point, a correlation, or a method for creating correlations, is precisely what is needed. On one hand, concepts have no sensible content, and on the other, intuitions are nothing but sensible. Hence, the exhibition of a relation between a concept and an intuition cannot take a direct route, for they have nothing in common.[8] A concept cannot be intuited nor an intuition be thought. A 'third thing' is necessary, a mediator that gives rules for this correlation between the sensibility and the understanding. Kant's schema of a concept fills this void.

2.2 The Schematism of Pure Sensible Concepts

> *This schematism ... is an art concealed in the depths of the human soul.*
>
> A141/B181

Let's start with a more basic question and ask how intuitions can be subsumed by concepts. For example, when one has a plate-intuition, how can one deter-

[5]We will use Kant's distinction between analytic and synthetic given on in the Introduction to the *Critique*. Analytic: "the predicate B belongs to the subject A, as something which is (covertly) contained in the concept A," and synthetic: "B lies outside of concept A, although it does indeed stand in connection with it" (A6/B10). Though some take this definition as metaphoric, it will serve to facilitate our comprehension of constructivism. Concerns related to the analytic/synthetic distinction are beyond the scope of this paper.

[6]For the sake of clarity, concepts will appear in italics.

[7]Kant also famously used the example of '$7+5=12$' to show the syntheticity of arithmetic (B15). He argues that nowhere within the concepts of 7 or 5 is the concept 12 analytically contained. 12 lies outside of 7 and 5, accessible only through construction in intuition.

[8]Concepts and intuitions, both being necessary for knowledge, are intimately connected for Kant. I do not wish to characterize them as incommensurable—but separate and relatable.

mine that that intuition falls under the concept *circular*?[9] Kant claims that intuitions are subsumed under concepts when the two are *homogeneous*. What are we to make of this *homogeneity*? Michael Pendlebury characterizes homogeneity as follows: "An intuition, *i*, and a concept, *C*, are homogeneous if and only if *C*-ness is part of the *content* of *i*"; in our example of the plate-intuition, the plate is homogeneous with *circle* if and only if "the intuition of represents the plate as being circular" (Pendlebury 1995: 781). What matters for homogeneity is the relationship between the content of an intuition and a concept. Hence a plate-intuition as well as a wheel-intuition share a content of "circle-ness", though each intuition is quite unique. Likewise, a scalene triangle drawn on a chalkboard shares the content of "triangle-ness" with an equilateral triangle drawn in the imagination.

Schemata are rule-based procedures which serve as the vehicle for homogeneity, determining if a given intuition can be subsumed by a concept or instantiating a concept in intuition. Kant distinguishes three different types of concepts and describes their corresponding schemata. There are empirical concepts, pure sensible or mathematical concepts, and pure concepts of the understanding or the categories. For our discussion, we will focus on the schemata which are sensible. We will not dwell on the schemata for the categories, though we may depend on them for the characterization of the schemata of mathematical concepts.

In his discussion of sensible concepts, Kant claims:

> *...the schema of sensible concepts, such as of figures in space, is a product and, as it were, a monogram, of pure a priori imagination, through which, and in accordance with which, images themselves first become possible. These images can be connected with the concept only by means of the schema to which they belong.*
>
> A141-42/B181

The metaphor of a "monogram" implies that a schema is something like a structure or a mold in which an intuition can be placed or created. Empirical intuitions that are subsumed by the concept *dog* range from a miniature Daschund-intuition to a German Shepherd-intuition. What both of these intuitions have in common is that they 'fit into' the monogram: the schema of *dog*. Schemata provide rules by which the imagination can delineate the intuition of a concept. Thus, from the side of sensibility, we can take any intuition and check if it 'fits into' the schema/monogram. From understanding, one can use the imagination to create an intuition which 'fits into' the schema/monogram.

One may ask why a 'third thing' is necessary. If concepts contained not only content but also rules for application to intuitions, the Kantian system could be streamlined. For mathematical concepts, this is exactly the move that commentators such as Lisa Shabel and Paul Guyer have made. Shabel argues that "mathematical concepts come equipped with determinate conditions on

[9]Kant's consideration of this example occurs on A137/B176.

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Macalester College

and procedures for their construction" (Shabel 2006: 111), and cites the following passage from the beginning of "The Schematism": "Thus the empirical concept of a *plate* is homogeneous with the pure geometrical concept of a *circle.* The roundness which is thought in the latter can be intuited in the former" (A137/B176). Shabel here interprets Kant as arguing that direct homogeneity between pure sensible concepts and intuitions is possible; schemata are unnecessary.

Yet this position does not conform with our interpretation of homogeneity nor the divide between the understanding and the sensibility. For an intuition and a concept to be homogeneous, conceptual content in the understanding must be intuited as belonging to an object of the sensibility. However, conceptual content which is part of the understanding *strictly* cannot be intuited.

Also, schemata for empirical concepts and the categories are explicitly extra-conceptual; this gives us, at the very least, evidence that the same holds for pure sensible concepts. In the case of *dog*, one must create the concept by utilizing experience. In order to classify a given intuition of a furred four-legged animal as being subsumed by *dog*, one requires some procedure for subsuming specific intuitions (furred-intuitions, four-intuitions, leg intuitions, and so forth) under their corresponding concepts.[10] If a decision procedure is needed *before* we can have an empirical concept, schemata, being these procedures, are pre-conceptual. Furthermore, Kant unequivocally requires extra-conceptual schemata for the categories.[11] We will not dwell on this argument, since our focus is on sensible concepts, though it would be curious to hold a unique characterization for pure sensible concepts; whereas both empirical concepts and the categories require extra-conceptual schemata.

Moreover, Kant claims that "it is schemata, not images of objects which underlie our pure sensible concepts" (A142/B181). This suggests that the schemata for mathematical concepts play a distinct role from the concepts. Klaus Jørgensen also notes that combining concepts and schemata blurs what the properties of a concept are properties *of* (Jørgensen 2006). Qualities that are found to belong universally to a class of mathematical constructions cannot be said to belong to the construction procedures for a concept but rather belong to the content of the concept. If a schema were part of a concept, then qualities of the concept would only be *subsumed by*, or *apply to* one part of a concept (the content) and not another (the schema). At this point, an account which collapses schemata in concepts appears untenable. If different qualities or attributes apply to concepts and schemata, in what manner can they be said to be the *same*? At some level, the content of a concept is necessarily distinct from the concept's rules for construction/applicability (the schema of the concept).[12]

[10]See (Pendlebury 1995) for an in-depth discussion of the manner by which we create empirical concepts based on *a priori* concepts and their schemata.

[11]His chapter on schemata, "The Schematism of the Pure Concepts of Understanding", is devoted to explaining how the categories subsume/apply to intuitions through schemata.

[12]This explanation may not be entirely faithful to Kant. Guyer and Shabel argue convincingly that Kant holds pure sensible schemata to be part of concepts' content (In particular, Guyer 1987: 159-65). However, to achieve our goal of the most reasonable interpretation of

How then do schemata allow for universal reasoning? In Kant's description of the geometer proving Proposition 32 (A716-7/B744-5, quoted above), the geometer modifies the triangle-intuition, namely by extending and adding lines. Eventually, he comes to prove that for this singular intuition, the sum of angles in the triangle is 180°. To prove that all triangles have this quality, the geometer must show that no empirical characteristics of the individual triangle (whether it is scalene, isosceles, or equilateral, for example) are necessary for the conclusion. This is done through the schema of the concept *triangle*, described by Kant in "Axioms of Intuition":

> *If I assert that through three lines, two of which taken together are greater than the third, a triangle can be described, I have expressed merely the function of productive imagination whereby the lines can be drawn larger or smaller, and so can be made to meet at every possible angle.*
>
> A165-6/B206

This is the procedure which is utilized to both construct a triangle and also to judge whether an intuition falls under the concept *triangle*. In the former sense, one can construct three lines in intuition (in accordance with the schema for *line*) such that two of which taken together are greater than the third, then the line-intuitions together to form a triangle-intuition. In the latter sense, if one can judge an intuition to have been constructed in this manner, then the intuition can be said to be subsumed by the concept triangle.

However, schemata do not only reconcile intuitions and concepts—they also hold the key for universal reasoning from singular intuitions. Visual techniques applied to the triangle only proves a particular case of Proposition 32—however, the geometer can use the schema of *triangle* to imagine making each angle and each line of the triangle as large or as small as possible:

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constructivism, we will refer to schemata as extra-conceptual. If the *content* of a concept can be differentiated from its *construction procedures* then referring to them as distinct is the most cogent interpretation.

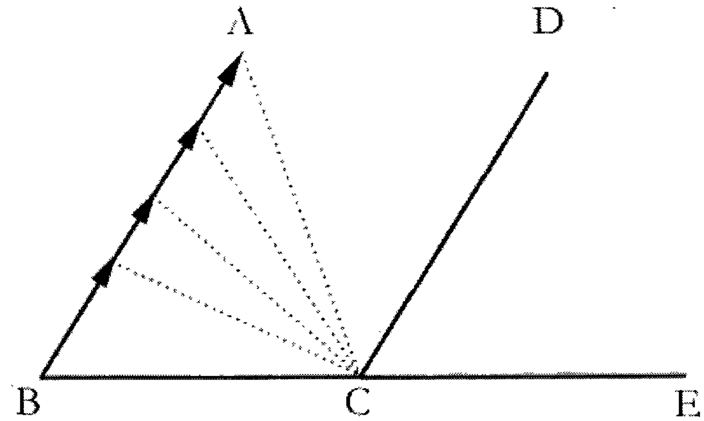

Figure 2: Function of Productive Imagination in the Proof of Proposition 32

Figure 2 shows us how this function of productive imagination works: $\angle ACB$ can be made to be any size (less than 180°) by lengthening and shortening AB. Hence, the geometer can imagine various triangles with different angle and side measures. In this way, he can imagine constructing every possible triangle—and at all times, the extended and added lines remain to show that the angles of each triangle add up to 180°. What matters for the inference are not the particular characteristics of the triangle but rather the procedures by which the triangle is constructed. Thus, since the truth of the proposition for the individual empirical intuition is only dependent on properties which issue from the schema of *triangle*, the geometer knows *universally* that the sum of the angles of a triangle is equal to two right angles.

2.3 The Construction of *A Priori* Intuitions

In this manner, the Kantian account of mathematics appears to expand the role of visual techniques in mathematics. Indeed, many advocates of visualization similarly ground the universality of a property on a diagram's construction procedures. However, the question of *a priority* is troublesome. Inference via the schema of a concept may guarantee universality, though this inference does not *prima facie* deny that the knowledge of Proposition 32 could be empirical.[13] In fact, the procedure we previously examined is grounded upon empirical objects, which suggests that resultant knowledge must be empirical. The moment that we intuit our construction empirically, all succeeding cognition appears in someway tainted, unable to achieve *a priority.* Consider our earlier description of the proof of Proposition 32—to an extent, our proof appears dependent upon experience. That is, one must sense the properties of what appears to be an empirical object.[14] Kant certainly recognized that the idea of intuitions grounding *a priori*

[13]That is, unless one equates universality with *a priority*. We will approach the problem more conservatively, requiring an *a priori* foundation for geometric properties.

[14]At this point, I would be remiss if I did not mention the imagination. A shrewd reader would note that Kant claims that intuitions in the imagination can be *a priori*. We should

inferences is worrisome. Consequently in his characterization of mathematical construction, Kant requires that "For the construction of a [mathematical] concept, we therefore need a *non-empirical* intuition" (A713/B742). The intuited tokens of a geometric concept, however, appear to be empirical.

There are two ways to explain the role of particular intuitions in mathematical cognition: either particular intuitions are empirical but represent or symbolize a non-empirical intuition (which can be utilized *a priori* in mathematical reasoning), or are themselves, in some manner, pure. The former is a view, defended by Jørgensen (Jørgensen 2006), which characterizes constructions as symbols for the pure intuitions of space and time (the symbolic interpretation).[15] The *a priority* of mathematical knowledge is therefore grounded on space and time, rather than the instantiations of a concept in space and time. The latter account claims that singular intuitions are sufficient for the attainment of *a priori* mathematical knowledge (the ostensive interpretation).

Let us first consider the symbolic account. Recall Kant's definition of construction: "to construct a concept means to exhibit a priori the intuition corresponding to the concept" (A713/B741). Interestingly, in this definition, Kant states that construction involves a correspondence between the concept and ***the*** intuition which relates to it. Perhaps then, our inference from the particular to the general achieves *a priority* through symbolizing a special, *single* intuition which corresponds to the concept. Indeed, the symbolic interpretation seems consistent with Kant's introduction of the *object of the concept*:

> *...mathematical knowledge [considers] the universal in the particular, or even in the single instance, though still always a priori and by means of reason. Accordingly, just as this single object is determined by certain universal conditions of construction, so* ***the object of the concept****, to which the single object corresponds merely as its schema, must likewise be thought as universally determined.*
>
> A714/B742, emphasis added

At first glance, the introduction of this "object of the concept" may give one pause. What can a "universally determined" object, to which each singular intuition relates, be? Jørgensen notes that here Kant appears to be characterizing

be skeptical of this contention and its utility for securing *a priority*. It is curious what makes imagined tokens so unique from drawn intuitions. Indeed, imagined constructions, similarly to drawn constructions, have *particular* properties. Asserting that imagined triangles are *a priori* while drawn triangles are empirical is incongruous. Our route is a conservative one: since imagined triangles have particular qualities, we will treat them as empirical. Again, this may be a divergence from Kant—however, if our concern is deriving *a priori*, universal properties from individuals, we should be skeptical of *any* intuition with particular properties.

[15]For Jørgensen, empirical figures are drawn or instantiated in outer intuition, where the construction in the imagination is pure. So our manipulations of a drawn triangle correspond to analogous manipulations upon the imagined triangle. As stated earlier, in our account, we will not distinguish the imagined and the outer instances of a concept. Hence, in our critique of the symbolic account, we do not mean to characterize Jørgensen as a pseudo-Platonist, for that which is symbolized in his account *is* intuitable (in the imagination). His views are here to introduce a more radical version of the symbolic account.

an arbitrary object to allow for an inference similar to universal introduction in predicate logic. This object of the concept is an arbitrary intuition subsumed by the concept: sensible but pure. We are now faced with a formidable difficulty—explaining the possibility of an arbitrary intuition.

Since space and time are the only pure intuitions, the object of a concept must be a representation of space and/or time, though it cannot be a singular empirical intuition. The object of the concept *triangle* must have sensible properties (so that geometry is not analytic), but *only* those which belong to the concept. In this way, the object of the concept *triangle* is an *a priori* triangle which relates to all triangles in virtue of its representing the necessary properties of every triangle. Hence, our non-empirical triangle-intuition is a triangle-shaped part of space, but one which has *no* empirical content. Yet certainly, we cannot hold such an arbitrary triangle in the imagination or instantiate it in intuition—each triangle which we intuit certainly has properties particular to *that* triangle. Moreover, the manner by which the object of the concept imparts *a priority* to construction or drawn figures is still unclear.

Jørgensen claims that "the empirical intuition functions as a symbol which refers by analogy to the pure intuition" (Jørgensen 2006: 13). Empirical figures hence serve as a cognitive stimulus, a basis for the intuition of sensible properties which correspond to *a priori* properties of the object of the concept. For example, extending the base of the empirical triangle in the proof of Proposition 32 symbolizes what can be done to the pure construction of *triangle*. The qualities which we derive from an empirical intuition, so long as they are dependent only on the schema of the concept triangle, correspond by analogy to properties of the universally determined object of the concept *triangle*. As alluded to previously, in mathematical cognition a pure intuition serves as the arbitrary object for our universal introduction. Hence an empirical intuition, in combination with the schema of the concept, serves to illustrate the sensible characteristics of the arbitrary object of the concept which we universalize. This account purports to thereby guarantee the *a priority* of mathematical knowledge achieved through construction.

However, under the symbolic account, the object of the concept takes on a mysterious role—though it guarantees the *a priority* of inferences from particular intuitions, we cannot access the pure intuition. The arcane character of the object of the concept under this account should be familiar to anyone acquainted with Platonism. The pure intuition of *triangle* certainly resembles the Platonic *form* of triangle. Indeed the object of the concept in this account inherits similar epistemological worries—our epistemological relation to the object of the concept is nebulous at best.

Indeed, given that we have no access to the pure intuition, the 'by analogy' relation appears too weak to guarantee *a priority*. Moreover, the relation between the object of the concept and the concept or its schema is quite vague. The ability of an intuitively inaccessible object to guarantee the universality and *a priority* of geometric inference is questionable. Interpreting the object of the concept as an inaccessible arbitrary intuition muddies our account of construction, introduces a mystical realm of intuition and construes Kant as a pseudo-Platonist.

The ostensive interpretation, on the other hand, denies the need for an *independent* object of the concept, taking empirical intuitions as sufficient for *a priori* inference. Shabel claims that there is no worry about the *a priority* of empirical *mathematical* intuitions, for such constructions *display the content of their concepts.* Empirical triangle constructions are *a priori* for they inherit their spatial properties from *a priori* space. If a sensible quality is based only on the procedure of delimiting a figure from the singular manifold of space, then concluding that that quality holds *a priori* for all so produced intuitions is justified. So for Shabel, even "...if [geometric] figures *are* rendered empirically, the apriority of the reasoning they support is not surrendered" (Shabel 2006: 109).

When inferring that an intuited property holds universally for a concept, one uses the schema of the concept to guarantee that the property is not dependent upon any particular, empirical qualities of the intuited token. The properties that are left are the pure sensible properties of the concept. When one strips an intuition of its empirical qualities, what is left is *the sensible content of the concept as it is instantiated in the pure intuition of space.* Universality is attained because the intuited construction displays the general content of the concept—meaning that the inferred property belongs to the concept and can be found in all subsumed intuitions. *A priority* is achieved because the inferred property belongs to the token in virtue of being a specific type of intuition carved out of the *a priori* intuition of space. Shabel defends the ostensive account:

> *...the shapes we construct in a mathematical context are not abstracted from our sensible impressions of shaped objects, such as plates or tables. Rather, on Kant's view, our empirical intuitions of shaped objects borrow* their *patterns from our pure intuitions of shapes in space. So, in constructing the intuition that corresponds to a mathematical concept, we attend not to the particular features of the resultant figure but to the act that produced it. So, in constructing the concept* triangle *one might produce a scalene or an equilateral figure; either way, one has produced a representation of all possible triangles by producing a single paradigm triangle. That one figure has unequal and another equal sides is irrelevant: one abstracts from the particular magnitudes of the sides and angles in order to recognize the relevant feature of the figure, namely three-sidedness.*
>
> Shabel 2006: 109-10

Intuitions, framed out of the *a priori* intuitions of space and time, inherit certain *a priori* properties. When we conclude that a property belongs to, but is not contained in a concept, this can be said to be synthetic knowledge. This knowledge is only *a priori* when the intuition is constructed by delimitation of the all-encompassing space. The difference between geometric and empirical inferences however is ambiguous at this point. If certain properties of a triangle-intuition inherit *a priority* in virtue of being carved out of the pure intuition of space, then a dog-intuition, framed by space, should likewise have *a priori* characteristics that can be universalized. The problem can be restated: what is unique about geometric intuitions, as opposed to empirical intuitions, that allows us to derive synthetic *a priori* knowledge?

For Shabel, geometric intuitions are unique in that their patterns are derived from the pure intuition of space; whereas the patterns of empirical intuitions are derived from sensible impressions. Hence, the properties which we can find in the patterns of geometric intuitions achieve *a priority*, where the properties of empirical intuitions remain empirical. This distinction between geometric intuitions and empirical intuitions is best illustrated by the act which instantiates each. In imagining an instantiation of an empirical concept, such as *dog*, we visualize any number of dog-intuitions, which fixes some intuitive content—that is, all dog-intuitions have some commonalities. *However*, the *spatial content* of an empirical intuition is *not* fixed. For example, a two-legged short-haired Chihuahua has little spatially in common with Lassie. What these intuitions have in common is that we have learned, in some way, that each fall under the concept *dog*—that is, the pattern of intuitions which are subsumed by the concept is derived from empirical impressions of shapes.

Geometric concepts, however, fix their spatial content. In constructing a triangle-intuition, we attend to an act of the productive imagination (as was outlined previously), which *determines* some *spatial* content of the intuition. A triangle-intuition, for example, is a three-sided spatial figure in virtue of our fixed construction procedure in intuition. Geometric concepts are characterized by the fixed construction procedure for carving instantiations our of the pure intuition of space, determining their spatial content. With the schema of a concept, we can ascertain the universal, *a priori* properties of intuitions which constitute geometry. In this manner, a singular intuition of geometry, in some sense, can be stripped of its empirical qualities and serve as an arbitrary object of the concept, effectively providing universality and *a priority*.

3 The Priority of Singular Space

Kantian constructivism is most commonly rejected for two reasons. The first of these stems from the advent of polyadic quantification, which allows concepts to force infinite extensions. Secondly, modern advances in geometry, specifically non-Euclidean geometries, question the *a priority* of Kant's geometry. We will consider the latter in §4.

The ability of concepts to force such extensions suggests that a general concept of space could cognitively precede the singular intuition of space. That is, instead of the singular manifold of space being necessary to delimit all particular spaces, a polyadically defined general concept of space could subsume the infinite parts of space. If a general concept of space were so prior to the all-encompassing space, the appeal to construction would be unnecessary: geometry could be practiced purely analytically and intuitions would be mere heuristics.

Many commentators, most notably Michael Friedman, deem Kant's utilization of intuition as something of a logical stopgap. Kant's syllogistic logic, lacking the capabilities of polyadic quantification, is clearly augmented by the use of intuition. Hence, though concepts such as denseness or continuity could not be *logically* represented in Kant's time, the indefinite iterability of intuitive constructions does the work of polyadic quantifiers. Despite the importance of this role for intuition for the Kantian enterprise, this section will establish that one cannot reasonably hold, *qua* Friedman, logical gap-filling as the *sole* function of intuition. In his analysis, Friedman focuses on the nature of space, claiming that each of Kant's justifications for the intuitive nature of space depend upon the limitations of monadic logic. In our analysis, we will first explicate Kant's description of space, then consider and subsequently evaluate Friedman's interpretation of Kant.

Our considerations, however, are not meant to characterize Friedman as anti-Kantian. Indeed, in "Kant's Theory of Geometry," Friedman's motive was positive: to reconsider and shed new light on Kant's account of geometry. While many reject constructivism outright in the face of contemporary logic and mathematics, Friedman showed how Kant's appeal to intuition was sensible in his time—necessitated by his inadequate logic. Though the account may look strange, Kant's motives were pure: to give an sufficient account of mathematics with the tools he was given. Friedman's departure from the canonical rejection of Kant in the analytic tradition in fact laid the foundation for contemporary reconsiderations such as this paper.

However, where Friedman set out to simply validate the appeal to intuition as a logical tool, we aim to account for intuition's philosophical role for Kant. Constructivism, for us, is not a mere historical oddity, but rather makes important claims about mathematical thought and the interface between geometry and the world.

3.1 The Metaphysical Exposition: Space as Intuition

Kant justifies the intuitive nature of space in the Metaphysical Exposition. He gives two arguments in each edition of the *Critique*, though only the first (The Part-Whole Relation) is common to both.

The Part-Whole Relation

On A24-5/B39, Kant claims the relation between space and its constituents differs from that between a concept an its instances. Space is a singular manifold—there is one overarching, infinite space. However, we can carve up this representation and conceive of all manner of distinct spaces, though they all *belong to*, or are *parts of*, the single overarching space. These parts of space are only represented through the introduction of "limitations", i.e. these specific spaces do not precede the one all-inclusive space, but rather are related to it via a *part-whole relation*. By delimiting the singular manifold of space we can refer to its constituents; singular space precedes these parts.

However, a concept and its instances are not related in this manner. Consider the concept *dog*, and an instance of a dog, Fido. According to Kant, the relation between *dog* and Fido is one of *homogeneity* or *subsumption*: the "dog-ness" of the former is instantiated in the latter. One could hardly say that this relation is between a part and a whole. If Fido is a constituent of *dog*, then consider how we would come to the concept *dog*. Each individual instance of a dog (the parts) would have to be put together to make *dog* (the whole). Yet, a writhing pile of Great Danes and French Poodles is hardly the concept *dog*. For Kant, the relation between an instance of a concept and the concept is not one of a part to the whole. Only within intuition can the whole precede the parts. As Norman Kemp-Smith notes, "Intuition stands for multiplicity in unity, conception for unity in multiplicity" (Kemp-Smith 1923: 105)—that is, only in intuition can we begin with a manifold which can be parsed into its constituents. Since distinct spaces are merely parts of the all-embracing space, as such, singular space be an intuition.

The Magnitude Problem

Kant also argues for the intuitive nature of space via appealing to its infinitude. There are two versions of this argument. In the first edition of the *Critique*, Kant argues that a general concept of space cannot represent space, for a general concept of space, being that which is common to all space, common both to "a foot and an ell", cannot refer to magnitude, since each of the spaces to which it refers have different magnitudes (A25). That is, if a general concept of space has instances which are of all possible magnitudes, then the intension of the concept cannot contain magnitude.

Let us take an example to clarify this point. The concept *x is a foot* does not subsume an ell-intuition, and *x is an ell* does not subsume a foot-intuition—x

is a foot or x is an ell, however, subsumes both. To create a general concept of space, we require a concept which will subsume *all* magnitudes. To create such a concept, we may naturally want to use a concept which refers to all magnitudes, such as *x is a picometer or x is a nautical mile or x is a light-year or...* and so on. This concept, though referring to all magnitudes, has no monadic logical form. We have no way to logically represent the intension of such a concept.[16] Therefore, the general concept of space, unable to represent infinite magnitude, must be cognitively preceded by that which is infinite: the singular intuition of space. Kemp-Smith elucidates:

> *Such infinity must be derived from limitlessness in the progression of intuition. Our conceptual representations of infinite magnitude must be derivative products, acquired from this intuitive source.*
>
> Kemp-Smith 1923: 108

Reference to magnitude can only be achieved in the sensibility; so again, space must be an intuition.[17]

The *Under/Within* Distinction

On B39-40, Kant gives the second edition version of the infinity argument argument. Besides an appeal to the infinitude of space, these arguments seem to have little in common. Kant claims:

> *Space is represented as an infinite* given *magnitude. Now every concept must be thought as a representation which is contained in an infinite number of different possible representations (as their common character), and which therefore contains these* under *itself; but no concept, as such can be thought as containing an infinite number of representations* within *itself. It is in this latter way, however, that space is thought; for all the parts of space exist* ad infinitum.
>
> B39-40

This quote is somewhat dense; parsing the passage will prove difficult, yet quite fruitful. A streamlined version of this argument is as follows: (1) Space contains an infinite number of representations *within* itself, (2) A concept may only contain a finite number of representations *within* itself, (3) Therefore, space

[16] This point will be considered in our explication of the *under/within* distinction. The ability to define an infinite intension, a logical reader may note, is provided by polyadic logic. This is precisely the quality of polyadic logic which threatens the Kantian account (See §3.2).

[17] One may find a distinct similarity between the magnitude problem and Kant's argument that 'A straight line is the shortest distance between two points' is a synthetic proposition (See B15-17).

is not a concept. (1) and (2) clearly imply (3), though there are certainly ambiguities surrounding Kant's premises.

The first concern is what containment *within* means in contrast with containment *under*. Consider this passage from the Jäsche Logic:

> *Every concept,* as a partial concept, *is contained in the representation of things; as a* ground of cognition, i.e. as a mark , *these things are contained* under *it. In the former respect every concept has a* content, *in the other* ***an extension****. The content and extension of a concept stand in inverse relation to one another. The more a concept contains* under *itself, namely, the less it contains* in *itself, and conversely.*
>
> Kant 1992: 593, emphasis added

Those representations falling *under* a concept are the extension of the concept. For example, the concept *animal* contains *under* itself intuitions of Yellow Labradors and Duck-Billed Platypuses. On the other hand, the concepts which are *within* a concept constitute its content or intension. Again, to illustrate, *animal* contains *within* itself concepts such as *life* or *substance*, for each of these concepts subsumes *animal.*

A vital concern is why concepts cannot contain *within* themselves an infinitude of representations. Suppose that a concept *could*, in principal, subsume an infinite number of possible intuitions. There could be an infinite number of intuitions corresponding to *dog* that I construct in the imagination. Why then, can a concept not contain *within* itself an infinite number of representations? What makes containment *under* so distinct from containment *within*? The answer lies in the fact that a concept can contain an intuition *under* itself, but not *within* itself.

Given a concept and its schemata, one has a procedure by which one can represent an instance of the concept in the imagination. Imagine that one is constructing intuitions of the concept *dog* in an attempt to intuit every dog. Consider, for any dog-intuition one represents in the imagination, there is another distinct dog-intuition (perhaps one with pink hair, a gold tooth, a bit-off ear, and so on). Hence, the set of dog-intuitions is possibly infinite (for you can always construct *one more* dog-intuition). As such, the number of possible representations *under dog*, by virtue of the its schema, is infinite.

On the other hand, no such construction procedures exist for representations *within* a given concept. While, as we just illustrated, the schema of a concept allows us to construct intuitions *under* a concept, by its very nature the schema cannot produce representations *within* the concept. Kant gives us no method to produce concepts which constitute the intension of a concept. For the concept *dog*, I can start counting concepts that are contained *within* it: *animal*, *substance*, *furred animal*, etc. But without a procedure by which I can parse-out these sub-concepts, I cannot *guarantee* that there exist an infinite number of representations *within dog*. As Michael Friedman notes, without such a construction procedure, one would have to conceive of an infinite number of concepts, but the finite powers of the mind preclude this possibility.

This distinction alone does not rule out the possibility that the singular manifold of space could be a concept. What rules out this possibility is that, as Kant claims above, each particular space contains *within* itself an infinite number of other particular spaces. How are we to make sense of this claim? Kant takes the statement "all the parts of space coexist *ad infinitum*" to justify his argument. Each part of space coexists *ad infinitum* because there exist, not only the infinite particular spaces inside the part of space, but also there exists an infinite number of spaces that *contain* the part of space. Through the limitlessness of the introduction of limitations, one can demarcate an infinite number of parts of space that contain any given part of space. Hence, each part of space recognizably contains an infinite number of representations (other parts of space) *within* itself. Therefore, space must be an intuition.

3.2 Friedman's Redundancy of Intuition

Thus, for Kant a concept cannot contain *within* itself an infinite number of representations; however, there clearly exist representations which require infinite extensions [e.g. *denseness*, *continuity* or the natural numbers]. Given this conceptual limitation, representing infinity and infinite extensions is difficult for constructivists. As we have seen, Kant must appeal to the infinite iterability of construction procedures in intuition in order to achieve such representations: for example, in order to represent the singular manifold of space or universalize Euclid's Proposition 32, Kant requires intuition. One may query why Kant advances a recondite system of pure intuitions, objects of the concept, and schemata, when it seems that the entire project could be streamlined by simply permitting space to be a concept, or permitting concepts to contain an infinite number of representations *within* themselves analytically. Friedman argues Kant's stipulation that concepts cannot force infinite extensions is a product of his antiquated logic and that advances since his time [namely predicate logic and polyadic quantification] show us that both the stipulation and Kant's appeal to intuition are unnecessary. In fact, he takes this contention further, claiming that Kant's inadequate logic constitutes the *only* impetus for the utilization of intuition: all arguments for the intuitive nature of geometry and space are motivated by the insufficiency of Kant's weak monadic logic.

The focus of Friedman's argument is the intuitive nature of space. Kant freely admits that we can have a general concept of space such as *x is a space*; however, the singular manifold of space in intuition must cognitively precede such a concept. Friedman interrogates this account. Friedman sees Kant's move as one necessitated by his logic, but now unnecessary given the enhanced power of polyadic logic. Friedman concentrates his efforts on demonstrating the possibility for a general concept of space to precede the all-encompassing space—a possibility which issues from the power of polyadic quantification to allow concepts to *analytically contain* reference to infinity.

We will consider Friedman's argument as follows. First, we will examine

M. Bennett McNulty
Macalester College

his claim that polyadic quantification yields the necessary logical machinery to represent infinite extensions conceptually. Secondly, Friedman argues that the magnitude argument rests upon Kant's stipulation that concepts cannot contain *within* themselves an infinite number of representations and hence collapses to logical inadequacy. Finally, Friedman's argument that the part-whole relation similarly rests upon obsolete conceptions of geometry and logic will be presented.

Failure of the *Under/Within* Argument

The allure of intuition for Kant is clear. Intuition allows a constructivist to essentially bolster a weak logic. Take for example, an intuitive description of denseness. A monadic concept of denseness cannot guarantee that for any two points, there will always exist a third between them for a concept cannot *force* an infinite extension. Logical limitations underpin the impossibility of making infinity analytic of a concept; Kant, however, uses intuition as a stopgap. Denseness and the existence of the third point is represented as a faculty of construction: "whenever I can represent (construct) two distinct points a and b on a line, I can represent (construct) a third point c between them" (Friedman 1985: 467). Intuition thus bolsters the extensional power of mathematical concepts, allowing us a rigorous definition of denseness. Denseness, or more specifically, the infinity implicitly contained therein, cannot be analytic of a monadic concept, but rather must be a function of the infinite constructive possibilities of intuition.

However, polyadic logic gives us another method for ensuring the existence of c without an appeal to constructions. We can define denseness as $\forall a \forall b \exists c | a < b \rightarrow a < c < b$—guaranteeing that the third point, c, exists without dependence upon intuition. Given a set of k predicates $\{F_1, F_2, ..., F_k\}$ in monadic logic, we can only have extensions with 2^k objects: specifically objects that are F_1, F_2,..., F_k, those that are $\neg F_1$, F_2,..., F_k, those that are F_1, $\neg F_2$,..., F_k and so on.

Emily Carson states the superiority of polyadic quantification as follows: "by allowing us to bypass infinite conjunction, [polyadic quantification] allows finite intellects to grasp infinite many representations in one concept and thus to 'describe' infinitely many objects by presenting formulas with only infinite models" (Carson 1997: 493). Monadic logic precludes concepts from *forcing* infinite extensions. Thus, given a concept which contains *within* itself k predicates, we can know with certainty that 2^k objects *must* fall under the concept. Though the actual extension may be larger, the concept can only *force* its extension to be 2^k, leaving us well short of infinity. The only way to *force* an *infinite* extension is to have a concept which contains *within* itself an *infinite* number of concepts. As we saw previously (§3.1), the reason that a concept cannot contain *within* itself an infinite number of representations is that the human mind cannot hold an infinite number of concepts at a given time and has no procedure by which these representations can progressively established. As Carson claims, polyadic logic releases us from this constraint; a finite mind can logically represent forced infinite extensions with polyadic quantification. Our question now becomes: if the elements of geometry can be modeled polyadically, must geometry be an intuitive endeavor?

Friedman shows that polyadic quantification gives us the infinite intensions—meaning that concepts with such intensions analytically have infinite extensions. Hence, a general concept of space can be defined polyadically to contain *within* itself an infinite number of instances.[18] Such a concept would force an infinite extension. The singular manifold of space does not necessarily precede the general concept, for the extension of the concept, in virtue of being infinite, includes *all possible spaces.*

However, Friedman's above contention that intuition is merely a logical stopgap only addresses the *under/within* distinction. Though polyadic logic may allow a concept to contain *within* itself an infinite number of representations, Kant provided two *other* arguments for the intuitive nature of space: the magnitude problem and the part-whole relation. Friedman now sets his sights on describing how these other arguments are similarly derived from the limitations of a weak logic.

The Magnitude Problem and the *Under/Within* Argument

Friedman believes A25 and B40 give identical arguments for the intuitive nature of space, though that B40 is more refined. That is, since B40 was inserted in place of A25, the *under/within* argument is merely a polished, general version of the argument from the infinitude of space. In A25, Kant argues that space cannot be a concept because infinite magnitude is not analytic of a general concept of space. This is meant to show that a general concept such as *x is a space* is an insufficient foundation for *a priori* space. However, Friedman points out that one could instead represent space by a concept which *does* refer to magnitude, such as *x is a greater space than y* or *x is a cubic foot.* Magnitude is analytic of such a concept, suggesting that it may avoid the magnitude problem.

This possibility is expressly ruled out by the *under/within* argument of B40. Friedman notes that "The second edition passage at B40 is clearer, for Kant is more explicit that the problem is not with the general concept 'x is a space' in particular but with *all general concepts as such*" (Friedman 1985: 473, emphasis added). Hence a concept such as *x is a greater space than y* can avoid the magnitude problem; however, *no* concept can circumvent the *under/within* problem. For Friedman, the problem lies in the fact that *any* concept, whether it refers to magnitude [*x is a greater space than y*] or not [*x is a space*], cannot *force* its extension to be infinite. Where A25 suggests that this is a unique defect of the concept *x is a space*, B40 rightfully establishes this problem as one for *all* concepts. No concept may force its extension to be infinite; whereas

[18] Given the capabilities of polyadic quantification, we can thus logically represent our earlier general concept of space with an infinite extension (*x is a picometer or x is a nautical mile or x is a light-year or...*).

the indefinite iterability of construction yields the infinitude of space. Concepts such as *x is a space* or *x is a part of y* are consequently cognitively preceded by the intuitive act of delimiting parts from the singular manifold of space. The magnitude problem, in virtue of being a *specific case* of the *under/within* argument, also stems from the inadequacy of Kant's monadic logic.

The Inadequacy of the Part-Whole Argument

In §3.1, we saw that concepts do not have the right relations to their instances in order for space to be a concept. The relation between concepts and their instances is one of subsumption, whereas the relation between individual spaces and the singular space is one of the part to the whole. Friedman, however, argues that the part-whole relation can be analytic of a concept, for example, the concept *x is a part of y*. Thus, if concepts can have infinite extensions [as was shown above] and can relate as parts and wholes, why can't concepts logically represent space?

Kant rejects this possibility. Space exists as the whole from which its constituents can be delimited; furthermore, *the whole must be prior to the parts.* Kant claims that a "general concept of relations" [presumably a concept such as *x is a part of y*] cannot precede the all-encompassing space: "on the contrary, [the parts] can be thought only as *in it*" (A25/B39). For Kant, the whole can only precede the parts in intuition; it is worth restating Kemp-Smith's quote on the matter: "Intuition stands for multiplicity in unity, conception for unity in multiplicity" (Kemp-Smith 1923:105). A general concept of space puts the cart in front of the horse; that is, a general concept of space takes the parts as cognitively prior to the whole: an antecedence which Kant expressly denies. Indeed, "these concepts are themselves only possible via the intuitive act of 'cutting out' parts of space from the singular intuition *space*" (Friedman 1985: 472).

Friedman interrogates this response, questioning the cognitive priority of the singular intuition. Why, he asks, must the all-encompassing space precede the parts or the general concept? The answer is pinpointed as issuing from our experiences with geometry. Directly following Kant's presentation of the part-whole argument, we find an appeal to geometry:

> *So too are all principles of geometry—for example, that in a triangle two sides together are greater than the third—derived: never from general concepts of line and triangle, but only from intuition, and this indeed* a priori, *with apodeictic certainty.*
>
> A25/B39

A similar reference to geometry is made by Kant after his presentation of the part-whole argument in the *Inaugural Dissertation*:

> *Geometry does not demonstrate its own general propositions by thinking an object by means of general concepts as happens with things*

> *rational, but by subjecting it to the eyes by means of a singular intuition as happens with things sensitive.*
>
> Kant 1894: §15.C

For Friedman, Kant justifies the cognitive priority of the singular intuition of space by these geometric appeals which, in turn, support the part-whole argument. The reason that the all-encompassing space precedes the general concept is that the theorems of geometry require all variety of singular intuitions.

However, Friedman claims that the appeal to intuition in geometry is again merely a product of Kant's monadic logic. Consider how a constructivist would intuitively represent infinity: given a constructed point a, one can construct a point b such that $a > b$. Such an iterative construction procedure can represent infinity, but only as a *possible* infinity. For Kant, as with any constructivist, we can never get to the 'end' of this procedure, point to our product and say, "this is infinity". Similarly, space cannot be the product of a progressive iteration in intuition. Since space is infinite, there is no progression of greater spaces such that we can get to the 'end' of the procedure, point to the result and say, "this is space". Accordingly, positing the precedence of the singular infinite intuition—the whole before the parts—is the only option. Let us call this the 'top-down' approach. If we take the whole as primordial, construction procedures can produce any particular space similarly to our earlier description of embedding *denseness* in intuition [between any two constructed points, we can construct a third]. Given an all-encompassing space, we can delimit the single intuition into any particular space, no matter how small or large.

Polyadic quantification, frees us from the top-down approach. Infinity can be analytic of a concept through the form $\forall a \exists b | a < b$. The priority of the singular manifold of space again appears to merely be a by-product of Kant's simplistic logic; with monadic logic, positing space as the preeminent whole from which the parts are delimited is the only option. Given the tools of polyadic logic, however, *we can analytically express the infinitude of space from its constituents.* We shall refer to this as the 'bottom-up' approach.

Therefore, for Friedman, the Kantian description of space is avoidable; polyadic logic's bottom-up approach undercuts the priority of the singular intuition of space. Moreover, the power of polyadic logic appears to allow us to do geometry analytically.[19] Hence, constructive procedures in intuition are *logically* unnecessary. For Friedman, construction is a mere artifact of an impotent logic.

3.3 Defense of Construction

The main point where we disagree with Friedman is in his claim that intuition serves *solely* as a logical stopgap. Friedman is correct in arguing that intuition allows Kant to force infinite extensions where none would have been possible otherwise; he errs in assuming that this logical role is the *only* function for

[19] A point which will be considered in greater detail in §3.3 and §4.4.

intuition. To reject construction on purely logical grounds is to ignore the epistemological status of intuition in geometry: the syntheticity of geometry gets thrown out with monadic logic.

The most pressing concern, however, is that Friedman's account overlooks the motives and advantages of constructivism in mathematics. That is, the philosophical underpinnings which motivate Kantian constructivism are ignored in reducing intuition to a mere logical gap-filler. Moreover, we should be skeptical when Friedman collapses the three main arguments for the intuitive nature of space: Kant presented these *as separate arguments*. Hence, in this section, we will illuminate the *philosophical motives* behind construction and focus on the unique character of each argument, concluding that the part-whole argument is wholly independent of the *under/within* argument, and furthermore, that collapsing the part-whole argument to the failures of monadic logic misses the philosophical character of this argument and the overall Kantian enterprise.

For Kant, one of the roles of construction is to constrain geometric concepts to those which are *possible* in intuition. Without this check on conceptual representation, we would be in danger of creating empty concepts in geometry. One can easily conceive of a meaningless geometric concept, such as a *square-circle*. However, when one attempts to construct a square-circle in intuition in order to acquire knowledge of the concept, it becomes apparent that *square-circle* does not correspond to any possible intuition, hence is a mere empty concept. This example demonstrates both the necessity of construction regardless of the capabilities of logic and the motive of the Kantian epistemological requirement that knowledge requires both concepts and intuition.[20] As Kant claims: "Although all these [Euclidean] principles, and the representation of the object with which this science occupies itself, are generated in the mind completely *a priori*, they would mean **nothing**, were we not always able to present their meaning in appearances, that is, in empirical objects" (A239-40/B299, emphasis added).

This quotation illuminates another, perhaps more subtle, role of construction in geometry. Not only does construction prohibit meaningless concepts; it explains the applicability of geometric concepts to experience. In constructivism, intuited tokens are necessary for the derivation of geometric theorems. Hence, since geometry is inferred from tokens, we get applicability to these tokens for free. This point will be further expounded in §4; for now, we simply note the ease with which constructivism deals with a historically formidable problem in the philosophy of mathematics. Eliminating the role of intuition in geometry would merely reintroduce the question.

Besides these advantages of constructivism, we can also find fault in Friedman's argument. He contends that geometry serves as Kant's motivation for claiming the priority of the intuited space. That is, since we can delimit infinitely in geometry, and since Kant lacks the logical machinery to produce this infinitude conceptually, space must be given as a cognitively prior intuition.

Friedman takes the proximity of Kant's arguments for the priority of space

[20] Carson gives an excellent analysis of this role for construction in the third section of "Kant on Intuition in Geometry" (Carson 1997: 501-11).

with geometric considerations to show that the former is justified by the latter.[21] This, however, is not enough to prove the priority of geometric concerns over the nature of space. In fact, the references to geometry appear to be mere examples of analogous situations that serve to illuminate the nature of space. Doubt is hence cast on Friedman's account.

Kant's explicit *rejection* of the priority of geometric considerations, however, casts much more than doubt:

> *For the representation of space (together with that of time) has a* peculiarity *found in no other concept; viz., that all spaces are only possible and thinkable as parts of one single space, so that the representation of parts already presupposes that of the whole. Now, if the geometer says that a straight line, no matter how far it has been extended, can still be extended further, this does not mean the same as what is said in arithmetic concerning numbers, viz., that they can be continuously and endlessly increased through the addition of other units or numbers. In that case the numbers to be added and the magnitudes generated through this addition are possible* for themselves, *without having to belong, together with the previous ones, as parts of a magnitude. To say, however, that a straight line can be continued infinitely means that* the space in which I describe the line is greater than any line which I might describe in it. *Thus the geometrician expressly grounds the possibility of his task of infinitely increasing a space (of which there are many) on the original representation of a single, infinite, subjectively given space.*
>
> Allison 1973: 175-6[22]

The possibility for the geometer to extend a line *depends upon the singularity of space in intuition.* The only way that the geometer can continuously extend a line is if there is a preexistent infinitude of space into which the line can be extended. Kant here is presenting something of a transcendental argument. Since we can continuously iterate extension of a line, the infinitude of space must be prior to the extension. This is where Friedman's account fails. For him, the particular characteristics of geometry may justify the nature of space—for Kant, however, the nature of space comes first as the grounds for geometry. The fact that geometry is even *possible* is a product of the prior, intuitive nature of space. We may take the truths of geometry to *affirm* features of space but not to *justify* them.

Though Friedman legitimately illustrates the logical function of construction in Kant's theory of geometry, he is wrong to claim that this is the only role for

[21] Namely in the 'Metaphysical Exposition of Space' in the *Critique* and §15 in *Inaugural Dissertation.*

[22] This passage is one of some note. Emily Carson depends heavily upon it in her analysis of Kant's theory of geometry and her arguments against Friedman's interpretation (Carson 1997: 497-8).

intuition. Construction constrains geometric concepts to the realm of possible intuitions while also reasonably describing the connection between intuitions and concepts. Moreover, specific geometric concerns do not justify the features of the all-encompassing space; rather, the possibility for geometry requires a certain nature for the prior singular manifold of space. Consequently, intuitive space and construction are necessary elements of Kant's philosophy; mere logical advances do not eliminate the indispensable role of intuition in geometry.

M. Bennett McNulty
Macalester College

4 Non-Euclidean Geometry, Constructibility and Modality

> *The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned to stone and trees turn into men, where the drowning haul themselves up out of swamps by their own topknots—all these remain, so long as they remain intuitable, still subject to the axioms of* [Euclidean] *geometry.*
>
> Frege 1950: 20

Kant took Euclidean geometry to impose itself necessarily upon space; however, modern developments suggest otherwise. Lobachevsky, Bolyai, and Riemann developed theories of geometry in which not all of Euclid's axioms hold. The existence of these consistent non-Euclidean geometries casts doubt on the *a priority* of Euclidean geometry. If space can possibly be interpreted as non-Euclidean, in what manner is space *a priori* Euclidean? Alternate theories of geometry suggest that we arbitrarily impose Euclidean axioms onto appearances, whereas we might just as easily impose Lobachevsky-Bolyai or Riemannian geometry. If we can *construct* non-Euclidean geometries, the *a priority* of Euclidean geometry is seriously threatened.

Moreover, science suggests that Euclidean geometry is strictly *false* of physical space. The application of relativity to astronomy suggests that space is curved and that lines are 'warped' by gravity; consequently, the Euclidean interpretation of a line as being 'straight' may not be consistent with the true nature of space. The properties of physical space seem contrary to Kant's all-encompassing space. The concern is clear: the *a priori* manifold of space may *not* be Euclidean.

Similar concerns are elicited by David Hilbert's work on geometry. Hilbert's *The Foundations of Geometry* proved appeal to intuition to be unnecessary for geometric proof; by setting down axioms, the truths of geometry can be derived by logical analysis. The Hilbertian project thus questions the role of intuition in geometry. If Euclidean geometry can be shown to be analytically derived from a set of axioms, the necessity of construction is questioned. This threat is somewhat akin to Friedman's thesis in the preceding chapter; however, there is a key difference. Where Friedman attempts to prove a general concept of space to be prior to the singular manifold of space, Hilbert is mute on the topic. He rather proves that geometry can be done conceptually without any appeal to eitherspace or intuition. The priority of space or spatial concepts does not matter for Hilbert, for the theorems of geometry can be derived without *any* appeal to objects or intuitions.

In this chapter, we will present the major developments in geometry since the time of Kant and address their effects upon constructivism. We will begin with a description of non-Euclidean geometries and physical space, then a discussion of the constructibility of non-Euclidean geometries. Finally, we will consider Hilbert's formalist project, which will isolate the role of intuition and construction in contemporary geometry.

4.1 Describing Non-Euclidean Geometry

The history of Euclid's fifth postulate is one of contention. Euclid began his *Elements* with definitions, common notions, and postulates [axioms] meant to serve as the foundation for his investigations of geometry. The fifth and final postulate has historically been viewed as an anomaly. Whereas the first four postulates are simple and seemingly self-evident,[23] the 'parallel postulate' takes a complicated and less obvious form:

> *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*
>
> Heath 1956: 155

Geometers realized the aberrancy of this postulate and consequently spent centuries attempting to prove the parallel postulate from the 4 other axioms. These efforts were fruitful, allowing geometers to derive many propositions which are logically equivalent to the parallel postulate, though not successful.[24] However, in the early 1700s, Giovanni Girolamo Saccheri approached the parallel postulate in a new, promising manner, utilizing a *reductio ad absurdum* method of proof in his ambitiously titled *Euclides ab Omni Naevo Vindicatus* ["Euclid Freed of Every Flaw"]. Saccheri's method was to show that the negation of the parallel postulate, in conjunction with the other 4 axioms, would derive a contradiction. Instead of using Euclid's version of the parallel postulate, Saccheri chose a logically equivalent proposition, based on his work with *Saccheri quadrilaterals*. Such quadrilaterals have two adjacent right angles. Let these angles be 'base angles' and the line which they contain be the 'base'. The line opposite the base will be the 'summit' and the angles containing the summit will be 'summit angles'. Hence the logically equivalent proposition is 'that in a Saccheri quadrilateral, the summit angles are right'.

Saccheri first considered the case when the summit angles are obtuse. A contradiction was not hard to find; Saccheri simply proved that in such a system of geometry, lines are finite, contradicting Euclid's second postulate.[25] The 'hypothesis of the acute angle', however, led to no readily apparent contradiction.

[23] Take for example, postulate 4: "That all right angles are equal to one another" (Heath 1956: 154).

[24] Geometers have shown that there are over a dozen propositions which are equivalent to the parallel postulate. The formulation which will prove most important in our discussion will be the equivalent proposition "that the angles of a triangle sum to 180°".

[25] However, Riemannian geometry was later developed by rejecting *both* the second and fifth postulates.

After rejecting the obtuse hypothesis with such ease, Saccheri becomes somewhat frustrated in his attempt to 'free Euclid of every flaw', defiantly claiming at the end of book I that "the hypothesis of the acute angle is absolutely false, being repugnant to the nature of the straight line" (Saccheri 1920: 208). In proposition 37 of book II, Saccheri presents three proofs of contradictions derived from the acute hypothesis— though all have been since proven spurious. Saccheri is in apparent duress when, in proposition 38, he states that "the hypothesis of the acute angle is absolutely false because it destroys itself" (Saccheri 1920: 225).

The acute hypothesis, however, does not 'destroy itself'; in fact, as Lobachevski and Bolyai recognized in the 1830s, adopting the acute hypothesis in lieu of the parallel postulate yields a consistent form of geometry. In Lobachevski-Bolyai geometry, the summit angles of a Saccheri quadrilateral are acute and there exist more than 1 line incident with a point which does not intersect a given line.

Commonly, Lobachevski-Bolyai geometry is modeled as the geometry of hyperbolae or lines drawn on the contours of an infinitely extended saddle-shape.[26] As such, the lines of hyperbolic geometry appear everywhere curved outwards:

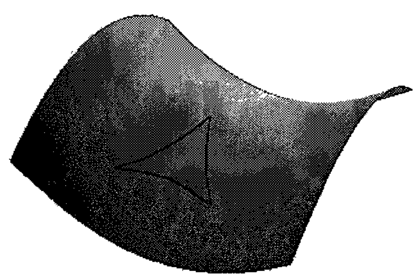

Figure 3: Saddle Model of Hyperbolic Geometry

In the above figure, we can see that, when modeled on the contours of a saddle, the angles of a triangle sum to less than 180°.

Another useful model of Lobachevski-Bolyai geometry was given by Klein in 1871. In this model, we confine our geometric universe to the inside of a circle, and the straight lines of Lobachevski-Bolyai geometry are the interior line segments or the 'chords' of the circle. As we can see from the figure below, this model fulfills the 'many parallel lines' theorem:

[26]Though there are *many* models of Lobachevski-Bolyai geometry, including the Lorentz model and Poincaré's disc and half-plane models [which are each isomorphic to the Saddle and Klein models].

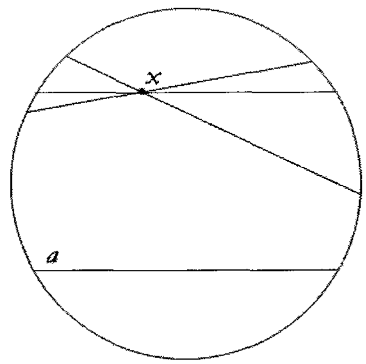

Figure 4: Klein model of Hyperbolic Geometry

In this model, we can see that there are an infinite number of lines through point x which do not intersect a.[27]

In 1854, Riemann investigated the possibility for a consistent geometry grounded in the obtuse hypothesis and the rejection of Euclid's second postulate [that lines are infinite]. He thus invented Riemannian or elliptic geometry. As the name may suggest, the lines of this geometry are commonly construed as ellipses or great arcs of a spheroid. Whereas are many methods for visually modeling hyperbolic geometry, elliptic geometry is nearly always modeled as the geometry of great arcs on the surface of a sphere or spheroid:

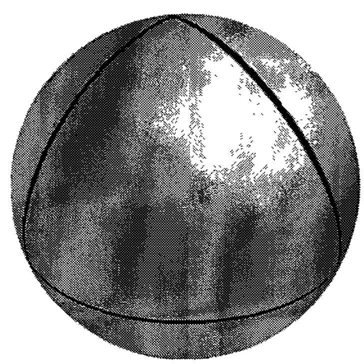

Figure 5: Spherical Model of Elliptic Geometry

Here we see a visual representation of a theorem of elliptic geometry: that the angles of a triangle sum to greater than 180°.

The simple existence of these non-Euclidean geometries is troublesome for the Kantian account of space, especially when coupled with the work of Eugenio Beltrami and Henri Poincaré. In 1868, Beltrami proved the *relative consistency* of hyperbolic geometry with Euclidean geometry; that is, hyperbolic geometry is consistent if and only if Euclidean geometry is consistent. This work was followed by Poincaré's proof that Euclidean geometry is *also* relatively consistent

[27] This proposition, named Playfair's theorem, is logically equivalent to the parallel postulate.

with elliptic geometry. With no trouble, a constructivist can plausibly assert that space is Euclidean when there are no other options. However, given alternative geometries, it is not clear which geometry or which set of geometric theorems holds of space. Even more troublesome, perhaps, are the results of Einstein's application of his General Theory of Relativity, which suggest that Euclidean geometry is in fact *false* of physical space.

4.2 Relativity and Applied Geometry

> *As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*
>
> Albert Einstein

Einstein's application of the theory of relativity yields a persuasive argument against the Euclidean nature of physical space. Assuming Euclidean geometry it appears that, in certain situations, there exists local but uniform forces which act upon all surrounding space. Hence, all measurements of the local space are uniformly deformed. For example, due to the force of gravity, measurements surrounding large heavenly bodies are warped. We can see this effect when light beams bend around a quasar or the edge of a black hole.[28] Hence, our measurements surrounding these bodies are deformed; a straight edge would curve around the source of gravity. Instead of taking this uniform force into consideration each time the physicist does a calculation, he can choose to set the strength equal to zero and treat *the surrounding space itself* as curved.

Einstein's move was to reject the convention of using rods and lines for measurement. For Einstein, the shortest distances in space are curves—the paths of light rays around these bodies. If space is considered Riemannian, scientists can more simply represent and calculate the relativistic physics.[29] This conversion merely involves a uniform recalibration of the physicists' measuring instruments. This translation is straightforward and permittable; Hans Reichenbach notes:

[28] The famous experiment of Sir Arthur Eddington confirms this result of relativity. Relativity implies that during the daytime, light reaching Earth from other stars would be pulled by the gravity of the sun. Consequently, Eddington and a team of scientists measured the position of stars during a total solar eclipse, and compared the findings with the positions of the same stars at nighttime. The predicted deviation in position was confirmed. This is one of Karl Popper's favorite examples of a crucial experiment (Popper 1963: 33-9).

[29] In physics, Riemannian and Euclidean geometries only diverge at large scales: the levels of astrophysics. Hence, at smaller scales, Euclidean and Riemannian geometries give us (practically) identical results. This fact allows the physicist to treat *all* space as Riemannian while retaining an adequate model of space.

M. Bennett McNulty
Macalester College

> *Just as we can measure the temperature with a Fahrenheit thermometer and then convert the results into Celsius, measurements can be started under the assumption of Euclidean geometry and later be converted into non-Euclidean measurements. There is no logical objection to this procedure.*
>
> Reichenbach 1958: 30

However, it is misleading to claim that Einstein's geometry is more true of space than a wholly Euclidean geometry. Certainly for the bodies with which Einstein is concerned, his theory of geometry is simpler. Using a Riemannian method of measurement is more manageable than taking into account a uniform force in every calculation. This approach is significantly more successful than considering local space as Euclidean and warped by gravity. But the hasty physicist may be willing to conclude that physical space itself is non-Euclidean (at least in specific circumstances).

We will consider two refutations of this argument which share a similar character. First, the physicist's chief concern when determining which geometry to choose is simplicity. Euclidean and non-Euclidean measurements yield identical results; it just happens that in this situation, regarding space as non-Euclidean makes calculation easier. However, simplicity does not entail truth.

Though I neither want to open the Pandora's box of scientific realism nor endorse his constructive empiricism, Bas van Fraassen sums up our point nicely:

> *...some writings on the subject of induction suggest that simple theories are more likely to be true. But it is surely absurd to think that the world is more likely to be simple than complicated (unless one has certain metaphysical or theological views not usually accepted as legitimate factors in scientific inference).*
>
> van Fraassen 1980: 90

Reichenbach himself states, "...we can no more say that Einstein's geometry is "truer" than Euclidean geometry, than we can say that the meter is a "truer" unit of length than the yard" (Reichenbach 1958: 35). This point is all-too-quickly ignored by anti-Kantians. Ease for the physicist is hardly the *summa bonum* of geometry—simplicity does not imply a specific nature of space. As we will see, in determining the geometry of the singular manifold of space, we are concerned with which geometry is *intuitable*, not simpler for a physical account.

Our second defense is something of a more fleshed-out version of the first. For this argument, we must first dwell on the distinction between pure and applied geometry. Pure geometry is solely concerned with the derivation of theorems from a given set of axioms; its objects are devoid of an interpretation. Hilbert's *The Foundations of Geometry* is the quintessential example of this approach to geometry. The non-interpretive nature of the Hilbertian enterprise is exemplified by his definitions: his formalist project defines its objects merely

M. Bennett McNulty
Macalester College

based on axioms and relations. Indeed, Hilbert defines the domain of geometry as follows:

> *Let us consider three distinct systems of things. The things composing the first system we will call* points *and designate them by the letters* A, B, C,....; *those of the second, we will call* straight lines *and designate them by the letters* a, b, c,....; *and those of the third system, we will call* planes *and designate them by the Greek letters* α, β, γ,....
>
> Hilbert 1902: 3

Hilbert defines the objects of geometry without appeal to an interpretation. His endeavor [apparently] showed that geometry can be a purely analytical process.[30] From sets of objects, defined merely by axioms and relations to other objects, one can deduce various properties of these objects [theorems]. For Hilbert, nothing hangs upon the Euclidean interpretation of a point as 'that which has no part', or a line as 'an breadthless length'; indeed, he once famously proclaimed, "One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs and beer mugs".[31]

Applied geometry, on the other hand, is ultimately concerned with the correct interpretation of geometry. A physical theory is input into pure geometry which fills in the content of the geometric concepts. It is in conjoining a pure geometry with a physical theory which allows us to test the empirical 'truth' of the geometry. Carl Hempel notes:

> *Thus, the physical interpretation transforms a given pure geometrical theory—Euclidean or non-Euclidean—into a system of physical hypotheses which, if true, might be said to constitute a theory of the structure of physical space. But the question whether a given geometrical theory in physical interpretation is factually correct represents a problem not of pure mathematics but of empirical science; it has to be settled on the basis of suitable experiments or systematic observations.*
>
> Hempel 1945: 30

Applied geometry is an empirical endeavor, which depends upon physical measurements. As such, the findings of the physicist, whether confirming or denying the physical truth of Euclidean geometry, do not concern our investigations. What matters for the nature of the singular manifold of space is whether a particular geometry is imposed upon *appearances* or *intuitions.* In §4.3, we will consider the possibility of intuiting non-Euclidean space.

However, the cleavage of pure and and interpreted geometry is a double-edged sword. Though this distinction shows that the space of physics is not the

[30]We will analyze the implications of this claim in §4.4.

[31]Uttered by Hilbert as he waited for a train in Berlin (Reid 1970: 57).

all-encompassing space with which we are concerned, it will also raise concerns regarding the syntheticity as well as the *a priority* of geometry. As we will see in §4.4, our rejection of possible non-Euclidean intuitions will place constructivism in a precarious position. If we can lie down a set of axioms and inference rules from which geometry proceeds *analytically*, then what is the place of construction? We will conclude by considering the implications of divorcing pure and applied geometry.

4.3 Kant Vindicatus: Constructing Geometry

We have hence determined that physical space is not the space of our interest: scientific simplicity is no reason to accept that the *a priori* manifold of space is non-Euclidean. We must take a step back if we are to adequately characterize the all-encompassing space. We look first to the role of intuition in geometry; the role of these constructions will help to illuminate the character of the *a priori* framework of these intuitions: space.

Our investigations will begin with a curious passage from the *Critique*:

> *Thus there is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of two straight lines and of their coming together contain no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its construction in space, that is, from the conditions of space and of its determination.*
>
> A220-1/B268

Kant here appears to be allowing for the logical possibility of non-Euclidean geometry. An evident theorem of Euclidean geometry is 'that no figure can be enclosed in two lines'. However, such a figure is possible in Riemannian geometry.[32] This should strike the reader as bizarre at first glance. Kant takes a proposition which is, at face-value, *inconsistent* with Euclidean geometry as, in some manner, 'consistent'. How can Kant justify such a possibility as *anything but* a serious threat to his theory of space and geometry?

Kant's answer hangs on a distinction between *logical* and *constructive* possibility. We can admit that there is no contradiction in the concept of a figure enclosed within two straight lines because the concepts of two straight lines contain nothing of figure. That is to say, neither the concept of *figure* nor the concept of *not-figure* is *analytically contained within the concepts of two lines.* Such a property is only accessible in intuition, much as the quality of being the shortest distance between two points can only be accessed by constructing the concept *line* in intuition. Geometry is synthetic a priori: underdetermined by logical, analytic processes. Hence, that which constrains or limits the domain of geometry is constructive possibility *in addition to* logical possibility. For Kant,

[32] Imagine two great arcs of a sphere which intersect on the poles. These 'lines' of elliptical geometry enclose a space on each side of the sphere.

the concepts of geometry are only those which can possibly be constructed in intuition.

Our task is therefore to show that non-Euclidean geometry strictly cannot be intuited. If this can be shown, then the *a priori* space is necessarily Euclidean. Otherwise imposing Euclidean theorems upon space is unjustifiable. Since constructions inherit their *a priority* in virtue of symbolizing intuitions carved out of space, if non-Euclidean geometry is constructible, then, for example, our *Euclidean* construction of a triangle is no longer universal [rather, it is particular in that it is a *Euclidean* triangle]; consequently, properties of Euclidean triangles cannot be universalized to all triangles. The potential constructibility of non-Euclidean geometries constitutes a serious threat to the *a priority* of synthetic geometric judgments.

In non-Euclidean geometries, there exist more than one line between two points, and each line is of minimal distance. Such a possibility, however, is impossible to visualize:

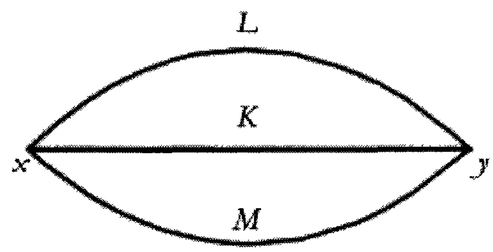

FIGURE 6: Visualizing Lines

The existence of multiple lines between points x and y cannot be reconciled with the minimal distance requirement. One can imagine L becoming less and less curved, and getting closer to K, but in order for L to be of minimum distance, it must be identical with K. But if L is identical to K, then there exists only one distinct line from x to y. Holding both requirements to be true is obstructed by the sensibility.

Perhaps if we simply *treat* L and M as lines of minimal magnitude between x and y or *assume* that a rigid body measures L to be the same size as K, then we can truly construct non-Euclidean geometry. As Reichenbach says. "one can forget that from the viewpoint of Euclidean geometry these distances are different in length" (Reichenbach 1958: 56). He refers to this ignorance of the Euclidean viewpoint as an "emancipation from Euclidean congruence". Reichenbach takes this emancipation as the necessary and sufficient condition for visualizing non-Euclidean geometry: "so long as we cannot emancipate ourselves from Euclidean congruence...non-Euclidean relations can only be mapped upon the visualized Euclidean space" (Reichenbach 1958: 57) and "Whoever has successfully adjusted himself to a different congruence is able to visualize non-Euclidean structures as easily as Euclidean" (Reichenbach 1958: 55).

James Hopkins likens this emancipation to the well-known duck-rabbit drawing (Hopkins 1982: 45). There is a set stock of lines drawn on a piece of paper; one could look at it and choose to see a duck, then decide to see a rabbit. Analogously, one could look at Figure 6 and choose to see K as the unique minimal distance between x and y, then look at it again and decide to see K and L to be of the same minimal distance. This is what would constitute Reichenbach's emancipation of Euclidean congruence.

However, in the duck-rabbit drawing, both the duck and the rabbit are natural interpretations of the lines on the page. Judging K and L to be of equal length, on the other hand, involves deceiving oneself of appearances. For Reichenbach, one must 'forget' Euclidean congruence, and perhaps convince oneself that the lines have any quality. Yet this does not change the fact that *intuition*, the guide of geometry, represents lines L and M as *curved* lines of *non-minimal* length.

Hopkins postulates a different method for constructing non-Euclidean geometries. As we know from physics, the divergence between non-Euclidean and Euclidean geometries is minimal in small regions. Calculations in each geometry only deviate at astronomical levels. One will also note that all intuitable lines are not, strictly speaking, lines; that is, they exist three-dimensionally. Each 'line' one sees has thickness, where the lines of geometry are purely one-dimensional constructions. Hopkins takes the two preceding points to suggest that intuited lines have an amount of 'wiggle-room'. That is, since the naked eye could not possibly note a visible difference between a Euclidean and non-Euclidean line at human-levels and all intuited lines have thickness, visible lines are not strictly Euclidean *or* non-Euclidean. Within the ink which represents K between x and y, there are possible theoretically indistinguishable Euclidean and non-Euclidean lines. Hence, empirical intuition does not *determine* the geometry of appearances. One could choose to interpret K as one Euclidean line or as two non-Euclidean lines nestled together.

Hopkins here appears to be presenting a refined version of Reichenbach's emancipation from Euclidean congruence. To construct non-Euclidean geometries, one simply must look at K and forget that it is one line, just as for Reichenbach one would look at L and M and forget that K is the shortest distance between x and y. Both possible avenues for construction involve ignoring that which we *actually* intuit. Hopkins does, however, avoid the duck-rabbit problem: whereas Reichenbach's move was to read non-Euclidean qualities into a Euclidean intuition, Hopkins claims that intuitions are geometrically indeterminate and that assigning *any* geometry is reading a quality into the intuition.

This raises an interesting question: is congruity imposed by intuition, or can we interpret intuitions as congruent at our own discretion? In §13 of the *Prolegomena*, Kant argues for the former. This argument will prove to have import on the constructibility of non-Euclidean geometry.[33] In arguing for the ideality of space, Kant describes two scenarios in which the independent reality of space is interrogated. The second of these scenarios secures the Euclidean character of intuitions.

[33] Kant's use of seemingly non-Euclidean examples both in the *Critique* (A220-1/B268) and in the *Prolegomena* (§13) have led some commentators, namely Leonard Nelson, to postulate that not only is Kant's theory of geometry unthreatened by non-Euclidean geometry but is moreover *vindicated* by the existence of these geometries.

The first of these is Kant's [in]famous 'glove argument':

> *What can be more similar in every respect and in every part more alike to my hand and to my ear than their images in a mirror? And yet I cannot put such a hand as is seen in the mirror in the place of its original, for if this is a right hand, that in the mirror is a left one, and the image or reflection of the right ear is a left one, which never can serve as a substitute for the other. There are in this case no internal differences which our understanding could determine by thinking alone. Yet the differences are internal as the senses teach, for, notwithstanding their complete equality and similarity, the left hand cannot be enclosed in the same bounds as the right one (they are not congruent); the glove of one hand cannot be used for the other.*
>
> Kant 1977: §13

As Norman Kemp-Smith notes, Kant is here arguing that congruence and similar relational properties of intuitions are dependent upon the possibility of distinguishing how and where objects lie in relation to our body: "The three dimensions of space are primarily distinguishable by us only through the relation in which they stand to our body.... Through these distinctions we are enabled to define differences which cannot be expressed in any other manner" (Kemp-Smith 1923: 162).[34] The example of the gloves is meant for us to merely reinforce the fact that the relation of congruence, and the possible emancipation therefrom, is brought to space by distinguishing the ways in which outer intuitions are related to one's body.[35]

Moreover, the glove argument is similar to the preceding argument given by Kant in the *Prolegomena*, which will serve as the foundation for our rejection of constructible non-Euclidean Geometry. At the beginning of §13, Kant presents another scenario which is meant to show that congruence is preceded by the relations of the subject to space. The reader is asked to imagine "two spherical triangles on opposite hemispheres which have an arc of the equator as their common base" (Kant 1977: §13). These triangles are assumed to have identical angles and sides, though placed on opposite hemispheres of the globe. Though the triangles are clearly quite similar, they cannot be said to be congruent, for by simple Euclidean displacement the triangles cannot occupy the same space, just as a right-hand cannot occupy a left-hand glove. If we move the triangles on the surface of the sphere, we have the correct curvature, but the vertices do not match up; on the other hand, if we move one off the face of the sphere and flip it, the triangles' vertices will coincide, but the curvature will not match. Hence,

[34] For an in-depth discussion of incongruous counterparts in Kant, see "The Paradox of Incongruous Counterparts" in Kemp Smith 1923: 161-6.

[35] If extra dimensions were visualizable, Wittgenstein's counter-argument (Wittgenstein, 1922: 6.36111) would succeed [that is, that fourth-dimensional manipulations to right- and left-handed gloves would make them congruent]. Since our concern is with that which is *intuitable* for humans, we will move past the fourth-dimensional argument.

though the triangles are *internally* identical [having the same measurements], the *external* relations of the triangles to *a priori* space differentiate them as incongruous.

Congruence is therefore imposed upon our intuitions. An intuition, framed by the singular manifold of space, has external relations which strictly defines with what it is congruous. Contrary to the opinions of Reichenbach and Hopkins, we cannot simply *choose* to ignore the congruity of intuitions. Emancipation from Euclidean congruence while *logically* possible, *constructively* is not.

Finally, I will argue that the impetus still remains upon those who argue against Kant's theory of geometry to provide a method for constructing non-Euclidean geometry which does not merely describe a specific Euclidean space. The disc model describes the Euclidean geometry of the chords of a circle. The spheroid model describes the Euclidean geometry of the great arcs of a spheroid and moreover is only a model of two-dimensional Riemannian space. The saddle model is only elliptic near the center of the saddle and there is no center of Lobachevski-Bolyai space; moreover, the model again is merely describing the Euclidean geometry of the contours of the saddle. To secure the constructibility of non-Euclidean geometry, one needs to provide a model in which the lines are straight, the planes are straight, and the geometry abides by the non-Euclidean theorems.[36] As Bertrand Russell notes: "Unless non-Euclideans can prove, what they have certainly failed to prove to this point, that we can frame an *intuition* of non-Euclidean spaces, Kant's position cannot be upset by Metageometry alone, but must also be attacked, if it is to be successfully attacked on its purely philosophical side" (Russell 1897: 56-7).

4.4 Formalism, Logic and Constructive Possibility

We now return to the threat of formalism upon constructivism. The worry is that Hilbert's *The Foundations of Geometry* show that geometry can be done analytically. The advent of formalism, however, does not refute Kant's claim that geometric truths "are not conclusions from some general notion of space, but only *discernible* in space as in the concrete" (Kant 1984: §15.C). Modern advances merely demonstrate an enhanced role for a "general notion of space". The Hilbertian enterprise, in developing an enhanced role of logic, is forever tied to logic; indeed, in showing that geometry can be done purely analytically, without reference to an interpretation, Hilbert is merely proving the *logical possibility of geometric concepts.* Hilbert's project is not at odds with

[36]There is another possibility, however. Helmholtz provided a thought-experiment which seems to suggest that there is a possible world in which beings with the same cognitive structure to humans could visualize in a non-Euclidean manner; consequently, the intuitive necessity of Euclidean geometry is lost (Helmholtz 1977). However, as Robert Hanna notes, Helmholtz's argument at best yields an isomorphism between Euclidean and non-Euclidean geometry, which is not enough to avoid the *intuitive* necessity of Euclidean geometry. For a discussion of Helmholtz's argument and another defense of Kant in light of the challenge from non-Euclidean geometry, see Hanna 2001: 270-9.

the Kantian account, for Kant would claim it merely to be a work of *logic*, not geometry.

Here, we again focus on the distinction between logical and constructive possibility. The Hilbertian enterprise is certainly useful in showing the concepts of geometry to be logically consistent. However, for Kant, the concepts of geometry must also be shown to be constructively possible. Intuition plays the critical role in confining geometry to those concepts which refer to possible intuitions, that is, non-empty concepts. As Emily Carson notes in her discussion of Friedman,

> *...geometry is constrained by pure intuition because only intuition makes the representation of mathematical concepts possible. If a purely conceptual representation were possible, it seems, there would be no such constraint. I am suggesting that mathematics is constrained by pure intuition because only that can provide its concepts with objective content, thereby ensuring that it is not a mere play of the imagination.*
>
> Carson 1997: 510

We should take a concept's constructibility in intuition to be a property worth noting. Construction achieves two desirable characteristics for geometry: the meaningfulness of geometric propositions and the applicability of geometric concepts to intuitions. These two qualities are certainly not achieved by logical proof. By declaring geometry to be synthetic *a priori*, Kant achieves both. The intimate relation between geometry and the world is achieved because the theorems of geometry themselves are *derived* from intuitions.

'Geometry' comes from the Greek $\gamma\epsilon\omega\mu\epsilon\tau\rho\iota\alpha$, which means 'earth-measure'. This material origin should not be ignored for the sake of placing geometry on a 'more primordial' foundation. The utility of geometry, and I would venture to say all of mathematics, comes from applicability to intuitions. Geometry is not 'mere play of the imagination' nor a long string of empty tautologies. Rather, through construction, the emptiness of the analytic is filled by the content of intuitions and this play of the imagination is made into reality. Without a foundation in intuition, the objects of geometry are mere logical possibilities; that is, they do not contradict themselves. This is not an adequate description. To claim *triangle* is merely a way of saying 'A is A' deprives geometry of its essential applicability and contentful nature.

5 Afterword: Geometric Truth

In his widely influential "Mathematical Truth," Paul Benacerraf isolates two objectives for an account of mathematical truth (Benacerraf 1983). The first is for seamless semantics—that a description of the semantics for mathematics mirrors the semantics for the rest of language. That is, a proposition such as '17 is a prime number is true in the same manner as 'the flower is pink' is true. Secondly, an explanation of mathematical truth should mesh with a 'reasonable epistemology'. In accounting for the truth of mathematical propositions, we must leave open a reasonable route by which we can come to know what is true. Benacerraf contends that one of these objectives can be achieved only at the expense of the other. We can illustrate this point well by briefly considering naïve formulations of Platonism and formalism.

Platonism is a realist philosophy of mathematical objects: such objects independently exist of humans. Through some sort of faculty of the mind or perception, we *discover* these objects and the facts about them. Platonism easily achieves the goal of seamless semantics. Just as "the flower is pink" is true because it corresponds to independently existent reality, "17 is a prime number" is true because the object 17 has certain independent qualities, one of these being its prime-ness. On the other hand Platonism, classically construed, presents a capricious epistemology. How exactly we come to discover these independent objects and facts is vague—as Benacerraf says, "a typical [Platonism] will depict truth conditions in terms of conditions on objects whose nature, as normally conceived, places them beyond the reach of the better understood means of human cognition (e.g. sense perception and the like)" (Benacerraf 1983: 409). Where one can point to the flower and observe its color, the nature of 17 and the medium of our acquaintance with 17 are nebulous. Rarely do some Platonists do much better than Plato's own maligned theory of recollection. This is not to say that some versions of Platonism do not fare better than others. Our point is that the preeminent problem for Platonists is one of epistemology.

On the other hand, formalism appears to achieve a reasonable epistemology at the expense of seamless semantics. Under formalism, mathematics is the result of lying down axioms and rules for what can be derived from these axioms. For example, given the Peano axioms and logic, we can derive arithmetic. For the formalist, mathematics is the process of manipulating our axioms under certain constraints—the products of these manipulations are the theorems of mathematics. The truth of these theorems issues from the truth of the axioms and appropriate application of inference rules. Formalism hence deals well with epistemology—a proposition of mathematics is true based on its derivability from the axioms or its syntax. The ease with which formalism deals with epistemology however comes at the cost of seamless semantics. Where propositions of the world are true in virtue of corresponding to actual facts, the statements of mathematics are true in virtue of their form. The truth of the statement statement "17 is a prime number", does not mean that the object 17 has the quality of being prime; rather, it means that the proposition itself is therefore a product of our mathematical 'game' of setting up axioms and inference rules.

Those who prioritize one of Benacerraf's two objectives generally choose to defend either Platonism or formalism. Admittedly, Platonism easily secures seamless semantics and formalism clearly achieves a reasonable epistemology. I contend, however, that there is a third motivation for an account of mathematical truth: the account must explain the applicability of mathematics to the world. An adequate account of mathematics must not make the success of mathematics a divine coincidence.

Constructivism provides the best such explanation for two reasons.[37] First, constructive geometry is concerned with determining the *a priori* qualities of objects delimited from the singular intuition of space. When we determine such a quality, it attains universality in experience for, since space is the *a priori* form of outer intuition, the properties of space are imposed upon *all* outer intuitions. A properly derived geometric theorem holds for all associated intuitions in virtue of each intuition's being framed by the spatial form of intuition. Secondly, construction in intuition is a necessary part of proof. In a geometric proof, a specific property of an intuition is universalized through the schema of the corresponding concept—embedding the property in experience is *necessary* for the proof. Hence, it is no divine coincidence that a property applies to all intuitions of a class; a necessary step in the proof is universalizing the property for all such intuitions.

On the other hand, Kantian constructivism gives no account for that which is not intuitively possible. A Kantian is hard-pressed to give an account for the 'truth' of non-Euclidean geometry, segments of set theory, or even the calculus. Many objects and concepts of such domains are strictly *not constructible.* Yet simply denouncing these areas as *not-math* is clearly unacceptable. There is much more work to be done by the constructivist.[38]

This said, our objective was never to provide an account of Kantian constructivism which avoids this critique. Rather, we meant to show that constructivism is more resilient than many have thought. Contrary to being "quaint" or "silly", constructivism deals well with many problems in philosophy of mathematics. When we recognize the applicability of mathematics to the world as a serious question, constructivism is as much a player in philosophy of mathematics as Platonism or formalism. Indeed, we should look for the best account perhaps not in rehabilitating Platonism or formalism, but rather in giving constructivism the honest consideration it deserves.

[37]Platonism, however, seems to do markedly better than formalism in accounting for applicability to intuitions.

[38]A *possible* (but not well thought-out) move for the Kantian may be to expound a hybrid account of mathematics. A classical constructivist account could be coupled with a formalism for non-constructible domains of mathematics. I see two possible problems with this account. First, the fact that *some* areas of mathematics are constructible seems to devalue the areas which are the results of formal manipulation. Secondly, the hybrid theory inherits the shortcomings of formalism at the fringes where it is invoked.

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