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The Planar Rook Monoid

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Abstract

Pascal's triangle is a very important and useful structure in combinatorics: its entries, the binomial coefficients, are used in a number of counting-related questions. In this paper, I will define a set of functions, the Planar Rook monoid, whose structure is closely tied to Pascal's Triangle. Most of the connections between the monoid and Pascal's triangle are seen when we allow the functions to work as linear transformations on different vector spaces; I will show several examples which lead to algebraic proofs of famous binomial identities. These algebraic proofs give some insight into the deep connection between the Planar Rook monoid and Pascal's triangle.

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Chapter 1

Introduction

One of the important pictures in the representation theory of the symmetric group S_n, which is the group of permutations of $\{1, 2, \ldots, n\}$, is Young's Lattice. Young's Lattice is an infinite graph which gives us a lot of information about the structure of the Symmetric Group. In Figure 1 we show the first 5 rows. For instance:

- The vertices in the nth row are indexed by integer partitions λ of n. The irreducible modules of S_n are labeled by the same partitions $\lambda \in \Lambda_n$ and we denote them by V^λ.

- The edges between the nth row and the $(n-1)$st row of the lattice correspond to the restriction rules from S_n down to S_{n-1}. That is, if we view the S_n module as a module for S_{n-1}, it decomposes as

$$V^\lambda = \bigoplus_\mu V^\mu,$$

  where the μ runs over the partitions connected to λ by an edge.

- The dimension of V^λ, denoted f_λ, (written next to λ in the lattice in figure 1) is also the number of shortest paths from $\emptyset$ to λ.

- The sum across the nth row of the f_λ is the number of permutations of $\{1, 2, \ldots, n\}$ which are their own inverses. This is the set of involutions I_n in S_n. This is to say

$$\sum_{\lambda \vdash n} f_\lambda = |I_n| \tag{1.1}$$

  This comes from a representation of S_n on a vector space of dimension $|I_n|$ such that each V^λ appears exactly once.

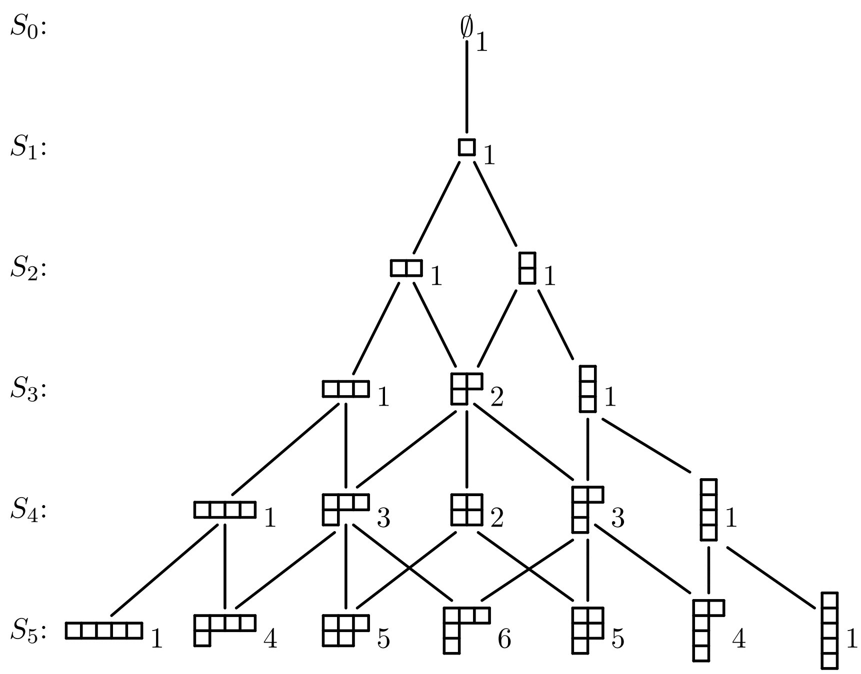


Figure 1.1: Young's Lattice, Y

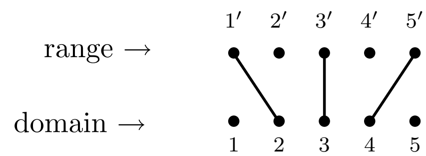
- The sum of ${f_\lambda}^2$ across the nth row is $n!$; that is

$$\sum_{\lambda \vdash n} {f_\lambda}^2 = n! \tag{1.2}$$

  This comes from a representation of S_n on a vector space of dimension $n!$ (in fact, on itself) such that each V^λ appears $\dim(V^\lambda)$ times.

We will explore these properties further in the next section. We notice that if we remove the partitions and just leave the numbers f_λ, we have a lattice which is constructed in exactly the same manner as Pascal's Triangle, only with different edges. That is, each f_λ is a sum of the f_μ connected by an edge above it. In this paper, we ask: *Is there an algebraic structure which has a similar relationship to Pascal's Triangle?*

As we will see in this paper, the Planar Rook Monoid is exactly this structure. We define the Planar Rook Monoid as the set of one-to-one functions, under function composition, from a subset of $\{1, 2, \ldots, n\}$ to a subset of $\{1, 2, \ldots, n\}$ such that, if $a < b$ then $f(a) < f(b)$ as well. If we write the elements as diagrams, then they look like this:



Th is represents the function $f : \{2, 3, 4\} \to \{1', 2', 3', 4', 5'\}$ with $f(2) = 1'$, $f(3) = 3'$ and $f(4) = 5'$.

In this paper, we will illustrate the connection between the Planar Rook Monoid and Pascal's Triangle. In Chapter 2, we will more carefully review the Symmetric Group and its relation to Young's Lattice. No proofs will be given, but we will flesh out the outline given above. The first section simply gives some basic information about the symmetric group. The second gives some information about the connection between the symmetric group and partitions, and also defines the basic terms from representation theory which we will use in this paper. In the third section, we use the information about the representation theory of the symmetric group to build Young's Lattice, and illustrate the assertions we have made about its connection to S_n. For more information about the Symmetric Group specifically, and proofs of the assertions in this chapter, see Bruce Sagan [6]. For more general information about representation theory, see James and Liebeck [4].

In Chapter 3 we will construct the Planar Rook Monoid, and give a presentation on generators and relations. In [5], Lex E Renner studies the planar rook monoid as the ordered submonoid of the rook monoid, so he has constructed the monoid there, but without representation theory. In [2], Tom Halverson and Tim Lewandowski mention the planar rook monoid as a diagram algebra, and its Bratteli diagram appears in a combinatorial setting.

In Chapter 4 we will give several different representations of the Planar Rook Monoid, which illustrate similar identities to those given in equations 1.1 and 1.2 above. We will also build the Bratteli lattice for the monoid, which we will see is Pascal's Triangle. In [3], Halverson and Ram give general theorems about representation theory of semisimple algebras (which include monoid algebras), which could have been used in this chapter, but we explicitly construct the irreducibles without making use of these theorems.

Finally, in Chapter 5, we give a notion of conjugacy which works in the monoid, and use it to compute the irreducible characters, and then show that the character table is Pascal's triangle.

Chapter 2

The Symmetric Group, S_n

2.1 Symmetric Group Basics

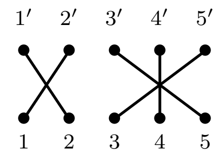
In this section, we give some of the basic properties of the Symmetric Group, S_n.

A *permutation* of $[n] = \{1, 2, \ldots, n\}$ is a one-to-one, onto function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. Therefore, every number from 1 to n in the domain corresponds to exactly one number in the image, and vice versa. Then a permutation can be viewed simply as a reordering of the numbers from 1 to n: the string $123 \cdots n$ is reordered by the permutation σ as $\sigma(1)\sigma(2) \cdots \sigma(n)$; since σ is one-to-one and onto, every number between 1 and n still appears exactly once in the reordered string. To count the number of permutations of n, we note that there are n choices for what to put in the first position, $n-1$ for the second, $n-2$ for the third, and so on, giving us $n!$ permutations of $[n]$.

In figure 2.1, we illustrate the different methods of representing a permutation. We can write a permutation in 2-line notation by writing the numbers 1 to n in order in the top line, and then below each number we write its image. Equivalently, we can write a permutation as an $n \times n$ matrix in which all entries are 0 or 1. The entry in the ith row and jth column is 1 if our permutation takes j to i, and is 0 otherwise. Therefore, the fact that our permutation is a one-to-one, onto function implies that there is *exactly one* 1 in each row and each column. As a diagram, we draw 2 rows of n vertices each, and connect each vertex in the bottom row to its image in the top row.

It is well-known that permutations can be broken down into disjoint cycles: in the 3-cycle (a, b, c) the image of a is b, the image of b is c, and the

$$x = \begin{array}{l} \text{range} \to \\ \\ \text{domain} \to \end{array}$$



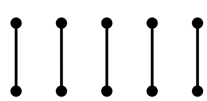
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

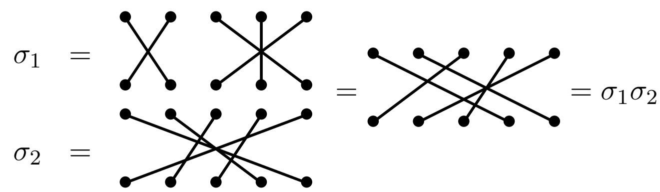
Figure 2.1: The same permutation, written in 3 different notations

image of c is a. To say that the cycles are disjoint means that no element appears in more than one cycle. It is also known that this factorization of a permutation into disjoint cycles is unique. Therefore, it makes sense to talk about the cycle type of a permutation; for instance, the permutation x in figure 2.1 is broken into disjoint cycles as $(12)(35)(4)$, and so it has cycle type $\{2, 2, 1\}$. What we are really giving here is a natural surjection from permutations of n to partitions of n. Then S_n is divided into disjoint subsets, each associated with a partition.

Under the operation of function composition, the set of permutations of n becomes a group: that is, there is an identity element and every element has an inverse. Here is the identity element in S_5.

$1 =$ 

The operation of function composition is extremely easy to perform with diagrams: we simply stack one diagram on top of the other, identify the middle row of points, and connect lines from the top row of points to the bottom row. Below, we see this in action:



Since we have invertibility, we have the action of conjugation: given permutations x and y, y is conjugate to x if and only if there is some permutation σ with $x = \sigma y \sigma^{-1}$. Note that conjugacy as defined is an equivalence relation. Therefore, S_n is divided into disjoint equivalence classes which we call conjugacy classes by this relation: each class consists of those elements which are conjugate to one another. In fact, it can be shown (though I will not show it here) that a permutation is conjugate to exactly those elements of S_n with the same cycle type. As we will see, the conjugacy classes are important in representation theory, and so we have a strong correlation, in representation theory, between the symmetric group and the partitions of n.

2.2 Representation Theory of S_n

In this section, we will first define some of the basic terms from representation theory, and then we will list, but not prove, some of the basic results in the representation theory of S_n. We will later give parallel results related to the Planar Rook Monoid, which we will prove. For proofs of any of these results, see [6]. For more information about representation theory, see [4].

A *module* for a group, in this case S_n, is a vector space V over $\mathbb{C}$ with an action $\sigma \cdot v \in V$ for all $\sigma \in S_n$, $v \in V$ such that the following three properties hold:

1. $(\sigma\tau) \cdot v = \sigma \cdot (\tau \cdot v)$
2. $1 \cdot v = v$
3. $\sigma(\alpha v + \beta w) = \alpha\sigma \cdot v + \beta\sigma \cdot w$ for $\alpha, \beta \in \mathbb{C}$

We note that σ acts on V as a linear transformation, and so we can pick a basis for V and then write the matrix of the linear transformation σ with respect to the basis. We find a basis $B = \{v_1, v_2, \ldots, v_k\}$ of V, and then we have a mapping

$$\begin{array}{ccc} S_n & \longrightarrow & GL_d(\mathbb{C}) \\ \sigma & \longmapsto & [\sigma]_B \end{array}$$

(Note that $GL_d(\mathbb{C})$ is the group of d by d invertible matrices with complex coefficients.) The first of the conditions on the module action implies that this mapping is operation-preserving, so it is a group homomorphism. This homomorphism is called a *representation*. We can build a representation from a module, and we can build a module from a representation, so sometimes the two terms are used interchangeably.

From here, we must define a few more terms. A subspace of a module V which is also a module is called a *submodule*. A module which has no submodules is called *irreducible*. Further, whenever we find one submodule W in a module V, there is another submodule Z in V such that every element of V can be written as a sum of an element from W and an element from Z. In this case, we write $V \cong W + Z$. Another way to say this is that a basis for W and a basis for Z together span V. If the intersection of W and Z is 0 only, then we say that *V is the direct sum of W and Z*, and we write $V \cong W \oplus Z$. Further, if $V \cong W \oplus Z$ and W consists only of copies of a single irreducible submodule, then we call W an *isotypic component* of V.

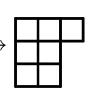
There is an important result in group representation theory (Maschke's Theorem) which says that any module reduces completely into irreducible submodules: we can write any module as the direct sum of irreducible submodules. Further, it can be shown that this direct sum is unique up to isomorphism; it does not matter how we start reducing, we will always end up with the same thing.

The irreducible representations of the Symmetric Group are known, and have been thoroughly studied. We summarize the main results here:

We define an *integer partition* $\lambda \vdash n$ to be a set of integers

$$\{l_1, l_2, \dots, l_k\}, l_1 \geq l_2 \geq \cdots \geq l_k \geq 1$$

and $\sum_{i=1}^{k} l_k = n$. We can represent partitions graphically: each of the l_i corresponds to a row of l_i boxes, and we start with l_1 on top and stack the next rows under this row so that the left sides of the rows line up, like this:

$\{3, 2, 2\} \rightarrow$ 

Therefore, each partition can be thought of simply as a shape.

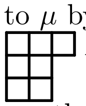
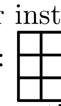
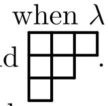
Corresponding to each $\lambda \vdash n$, there is an irreducible S_n which we call V^λ; these modules are nonisomorphic (they are all different) and, in fact, these are all of the irreducible S_n modules. Further, we will define

$$\dim(V^\lambda) = f_\lambda$$

and we will soon show how to calculate this value.

First, we will view $S_{n-1} \subseteq S_n$ as the set of permutations of n which fix n. Then all of our irreducible S_n modules are also S_{n-1} modules, but they may be reducible as S_{n-1} modules. We call this process of treating a group module as a module for some subgroup restriction. We find that restriction from S_n to S_{n-1} works as follows:

$$\mathrm{Res}\downarrow^{S_n}_{S_{n-1}} (V^\lambda) = \bigoplus_{\substack{\mu\subseteq\lambda \\ \lambda-\mu=\square}} V^\mu \tag{2.1}$$

We have $\lambda \vdash n$ and $\mu \vdash (n-1)$ such that if we take away all of the boxes of μ from λ we are left with one box in λ. We can also think of this, since μ is a partition (and therefore never has a box below or to the right of empty space) as seeing all of the partitions we can get by removing one box from the right side of a row and the bottom of a column in λ; if we can get from λ to μ by such a removal of a box, then V^μ is in V^λ. For instance, when λ is  then there are only 2 corresponding partitions μ:  and . Any other boxes which we remove will not give us a valid partition, because they are not at the right side of a row, and the bottom of a column of boxes. Since we have a direct sum (so the bases of the components are disjoint) we can in fact get the dimension of V^λ as the sum of the dimensions of the V^μ:

$$V^\lambda = \bigoplus_{\substack{\mu\subseteq\lambda \\ \lambda-\mu=\square}} V^\mu \tag{2.2}$$

As a direct consequence of equation 2.2, we get a recursive formula for f_λ:

$$f_\lambda = \sum_{\substack{\mu\subseteq\lambda \\ \lambda-\mu=\square}} f_\mu \tag{2.3}$$

2.3 Building Young's Lattice

In this section we will use the information we have gotten about the irreducible S_n modules to build up Young's Lattice. The elements in the nth row of the lattice are the partitions of n (corresponding to the irreducible modules of S_n). We draw edges between the nth row and the $(n-1)$st row according to the restriction rules (that is, between $\lambda \vdash n$ and $\mu \vdash (n-1)$ as we discussed above): if V^μ is a component of V^λ restricted from S_n to S_{n-1}, then there is an edge between λ and μ in the diagram. In figure 2.2

we show the first five rows of the lattice, with f_λ written to the right of each λ:

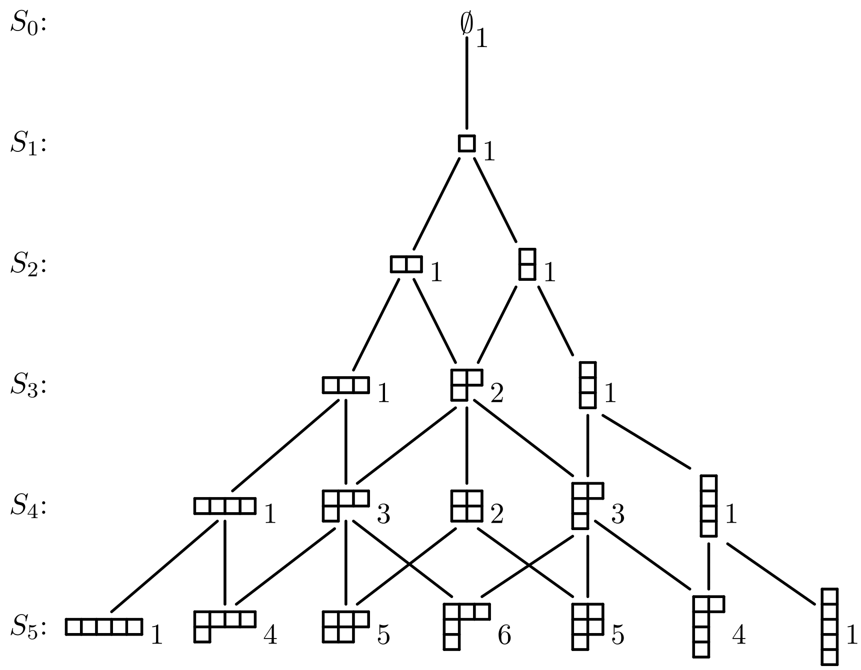


Figure 2.2: Young's Lattice

We find, if we sum the number of paths to λ in the diagram, that it is equal to f_λ again. In fact, we could prove that this is true by induction on the row, since it is true for the 0th row, and the recursion must hold.

If we sum the numbers across the nth row, we get the number of permutations of n which are there own inverses, called the *involutions*. Written as diagrams, these elements look the same if we flip them over top to bottom. For instance, row 3 of Young's Lattice sums to 4, and here are the 4 involutions in S_3:

For more information about involutions and a representation-theoretic interpretation, see Michael Decker [1]. Here is a small summary. There should be a vector space N with a basis made of the involutions from I_n. Then S_n acts on n by "signed conjugation" (see [1] for a more thorough description) and

$$N \cong \bigoplus_{\lambda \vdash n} V^\lambda \tag{2.4}$$

Since each irreducible appears exactly once, we have

$$|I_n| = \sum_{\lambda \vdash n} f_\lambda \tag{2.5}$$

(which is equation 1.1 from the introduction) as a direct consequence.

Now, we note that if we sum the squares of the entries in the nth row, we get

$$\sum {f_\lambda}^2 = n!$$

To see why this is the case, we must know a little bit about a representation known as The Regular Representation of S_n. This representation involves creating a vector space with S_n as a basis, which we will call V_{S_n}. Then, we let S_n act on V_{S_n} by left multiplication. As it turns out, all of the irreducibles of a group will show up in the Regular Representation, with multiplicity equal to their dimension. In the Regular Representation of S_n, this means that V^λ will appear f_λ times in the decomposition of V_{S_n}. Therefore, we have

$$V_{S_n} = \bigoplus_{\lambda \vdash n} f_\lambda V^\lambda$$

Again, we will sum the dimensions of the components since this is a direct sum:

$$n! = \sum_{\lambda \vdash n} {f_\lambda}^2 \tag{2.6}$$

which is equation 1.2 in the introduction.

Chapter 3

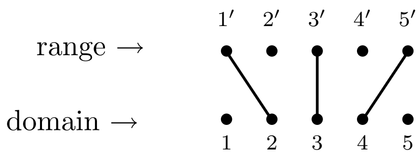
The Planar Rook Monoid, PR_n: Basics

3.1 Introducing the Planar Rook Monoid

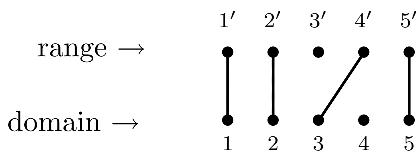
First, we define the *Rook Monoid* as the set of one-to-one functions from a subset of $\{1, 2, \ldots, n\}$ to a subset of $\{1, 2, \ldots, n\}$. We can write such functions as matrices much like permutation matrices, except that there need only be *at most* one 1 in each row and each column, rather than exactly one. Therefore, since rooks can attack any square in their row or in their column, if we visualize such a matrix as an n by n chessboard, then we can place a rook on the position of each 1 without any two rooks attacking each other. The representation theory of the rook monoid is studied by Solomon [7].

The *Planar Rook Monoid*, denoted PR_n, is the set of *strictly increasing* one-to-one functions from a subset of $\{1, 2, \ldots, n\}$ to a subset of $\{1, 2, \ldots, n\}$. By strictly increasing, we mean that if $a > b$ are in the domain of our function f, then $f(a) > f(b)$. These functions can be written as diagrams, in 2-line function notation, and as matrices. As diagrams, we make 2 rows of n dots, and we connect the ith dot on bottom to the jth dot on top if the function sends i to j. As diagrams, we note that the condition that our functions be strictly increasing requires exactly that our diagrams have no crossings (where edges between vertices are straight lines): they are planar. In 2-line function notation, we simply write those elements in the domain in the top line, and their images in the bottom line. In matrix form, there is a 1 in the ith row and the jth column if our function takes j to i, and a zero otherwise. If we visualize these matrices as n by n chessboards, as with the rook monoid, we note that they are still non-attacking, and that any rook to the right of any other must also be below.

We say that the *rank* of a diagram in PR_n is the number of edges that diagram has. Here are two functions in PR_5 in each of these forms; the first has rank 3, the second has rank 4:

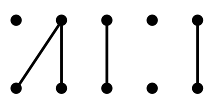


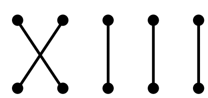
$$= \begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

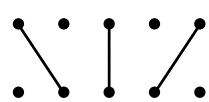
The following diagrams are not in PR_5. The first is not one-to-one, the second is not planar:





To calculate the size of PR_n, we note that once we have chosen a domain and range (which must be the same size), there is exactly one way to connect the elements so that our function is strictly increasing, so that choosing a domain and range completely determines the function. In order to choose a set out of $\{1, 2, \cdots, n\}$ and then choose a set of the same size from $\{1', 2', \cdots, n'\}$, we can choose a single subset of size n out of $\{1, 2, \cdots, n, 1', 2', \cdots, n'\}$, and then take those elements of $\{1, 2, \cdots, n\}$ which we chose to be the domain, and those elements of $\{1', 2', \cdots, n'\}$ which we *didn't* choose to be the range.

For example, if we choose $\{2, 3, 4, 2', 4'\}$ from $\{1, 2, 3, 4, 5, 1', 2', 3', 4', 5'\}$ (which is a subset of size 5 taken from a set of size 10), this will correspond to the diagram



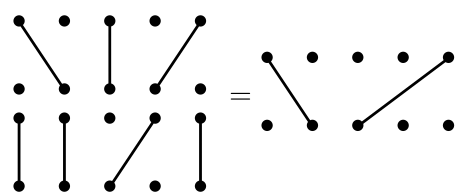
These set of points we chose from $\{1, 2, \ldots, n\}$ and the set of points we didn't choose from $\{1', 2', \ldots, n'\}$ must be the same size if we choose a subset of size n, and we get all possible pairs of domain and range in this way. The overall set which we are choosing a subset of size n from has size $2n$, and so we have

$$|PR_n| = \binom{2n}{n} \tag{3.1}$$

3.1.1 Monoid Operation

The Planar Rook Monoid is a *monoid* under the operation of function composition. That is, the composition is associative and there is a multiplicative identity, but elements are not necessarily invertible (if they were, it would be a group). In fact, in PR_n, the only invertible element (unit) is the identity element.

In diagram form, function composition is very simple: we put the leftmost diagram on top, identify the two middle rows of dots, and connect edges from the very upper row to the very bottom row. As an example, here is the composition of our two example elements:



In two-line function notation, we compose our functions just as we would any other functions in two-line function notation, just noting that the only elements of the domain of the first function which have any images are those which are sent to something in the domain of the second function. In reality, this is the least convenient way to consider our elements, as the easiest way to compose two elements is to convert to one of the other two forms, compose in that form, and then convert back to 2-line notation. Here is the same example as above, in 2-line function notation:

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 3 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$$

In matrix form, we simply use regular matrix multiplication to compose our elements. Here, again, is the same example, in matrix form:

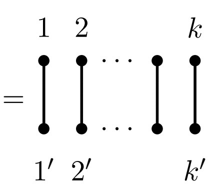
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

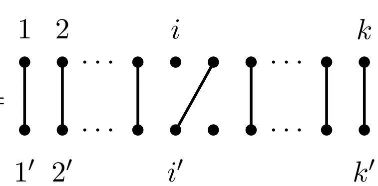
3.2 Presentation on Generators and Relations

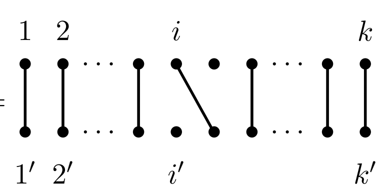
Here, we are going to give a presentation on generators and relations for PR_n. That is to say, we are going to give a list of elements of PR_n such that, when we multiply them together, we get all of the elements of the monoid, along with all of the multiplication rules which apply to these elements. This gives us a way to give formal multiplication rules for all of the elements of the monoid, without having to give a multiplication table which is $\binom{2n}{n}$ by $\binom{2n}{n}$; we simply treat arbitrary elements of PR_n as products of the generators, and simplify according to the rules.

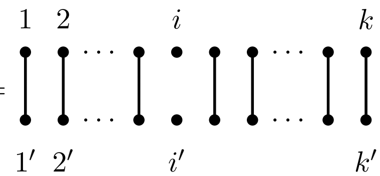
In this section, we define our generators, and show that they do in fact generate all of PR_n. Then, we list the relations satisfied by these generators, and in the next section, we prove that this set of generators and relations give us PR_n.

We define a set of generators for the Planar Rook Monoid as follows:

$1 =$ 

$\ell_i =$ 

$r_i =$ 

$e_i =$ 

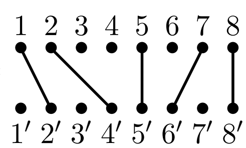
Then for ℓ_i and $r_i \in PR_n$, $1 \le i \le n-1$ are the subscripts for which ℓ_i and r_i are defined.

Note that

$$\begin{aligned} e_1 &= \ell_1 r_1, \\ e_i &= \ell_i r_i = r_{i-1}\ell_{i-1}, \quad 2 \le i \le k-1 \\ e_k &= r_{k-1}\ell_{k-1}. \end{aligned}$$

Lemma 1. *Every diagram can be written as a product of ℓ_i and r_i.*

Proof. To any diagram $d \in PR_k$, we associate $S \subseteq \{1, 2, \ldots, k\}$ and $T \subseteq \{1', 2', \ldots, k'\}$ with $|S| = |T|$, where S is the domain of d and T is the codomain. In other words, S is the set of top vertices which are adjacent to an edge and T is the set of bottom vertices that are adjacent to an edge. As mentioned above, the sets S and T completely determine the diagram d. For example, if

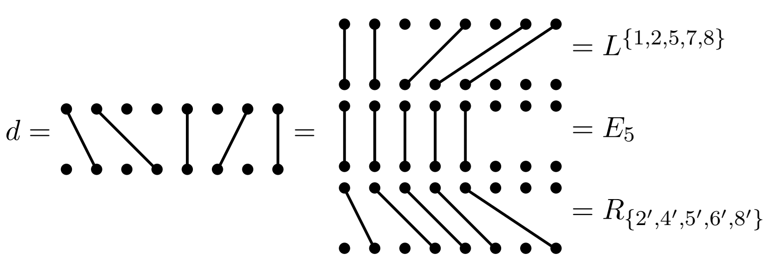
$d =$ 

then $\text{rank}(d) = 5$, $S = \{1, 2, 5, 7, 8\}$ and $T = \{2', 4', 5', 6', 8'\}$.

For any subset $S \subseteq \{1, 2, \ldots, k\}$, let L^S be the diagram with domain S and codomain $\{1, 2, \ldots, \text{rank}(d)\}$. For any subset $T \subseteq \{1', 2', \ldots, k'\}$ let R_T be the diagram with domain $\{1, 2, \ldots, \text{rank}(d)\}$ and codomain T. For any $0 \le t \le k$, let E_k be the diagram with domain $\{1, \ldots, j\}$ and codomain $\{1', 2' \ldots, j'\}$. Then if d has rank j, domain S, and codomain T, we can write

$$d = L^S E_j R_T.$$

In our example above, we have



Define $L^{a,a} = R_{a,a} = 1$ and define

$$L^{a,b} = \ell_{a-1}\ell_{a-2}\cdots\ell_b, \qquad 1 \leq b < a \leq k,$$
$$R_{a,b} = r_a r_{a+1}\cdots r_{b-1}, \qquad 1 \leq a < b \leq k.$$

Note that $L^{a,b}$ and $R_{a,b}$ each give an edge which goes from a in the domain to b' in the codomain. Then if $S = \{s_1 < s_2 < \cdots < s_j\}$ and $T = \{t_1 < t_2 < \cdots < t_j\}$, we have

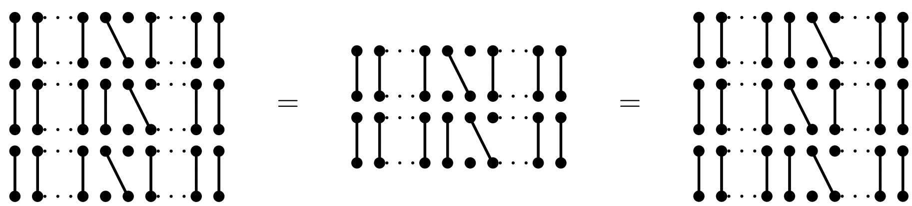
$$\begin{aligned} L^S &= L^{s_1,1}L^{s_2,2}\cdots L^{s_j,j}, \\ E_j &= e_{j+1}e_{j+2}\cdots e_k, \\ R_T &= R_{j,t_j}R_{j-1,t_{j-1}}\cdots R_{1,t_1}. \end{aligned}$$

This shows that any diagram can be written as a product of the generators ℓ_i, r_i and e_i. Furthermore as each e_i is a product of ℓ_i and r_i, every diagram is a product of ℓ_i and r_i. □

Theorem 2. *The Planar Rook Monoid is generated by ℓ_i and r_i subject to the following relations (where i is such that all terms in the relations are defined):*

1. $\ell_i^3 = \ell_i^2 = r_i^2 = r_i^3$
2. a) $r_i r_{i+1} r_i = r_i r_{i+1} = r_{i+1} r_i r_{i+1}$
   b) $\ell_i \ell_{i+1} \ell_i = \ell_{i+1}\ell_i = \ell_{i+1}\ell_i\ell_{i+1}$
3. a) $r_i \ell_i r_i = r_i$
   b) $\ell_i r_i \ell_i = \ell_i$
4. a) $r_{i+1}\ell_i r_i = r_{i+1}\ell_i$
   b) $\ell_{i-1} r_i \ell_i = \ell_{i-1} r_i$
   c) $\ell_i r_i \ell_{i+1} = r_i \ell_{i+1}$
   d) $r_i \ell_i r_{i-1} = \ell_i r_{i-1}$
5. $r_i \ell_i = \ell_{i+1} r_{i+1}$
6. *if* $|i-j| \geq 2$, *then* $r_i \ell_j = \ell_j r_i$, $r_i r_j = r_j r_i$, $\ell_i \ell_j = \ell_j \ell_i$

We will not go through and show that the diagrams satisfy all of these relations, as it is tedious, but it can done. For example, the diagram for 2a looks like



3.3 An Alternate Definition of the Monoid

In this section, we want to prove theorem 2. To go about this, we first will define a set of formal words which are subject exactly to these relations. Then, we want to prove that there is an isomorphism between this new set and PR_n as we defined it in 3.1.

We define $\widehat{PR}_k$ to be the monoid generated by $\widehat{1}, \widehat{\ell}_i, \widehat{r}_i$ subject to the same relations as above:

1. $\widehat{\ell}_i^3 = \widehat{r}_i^2 = \widehat{\ell}_i^2 = \widehat{r}_i^3$
2. a) $\widehat{r}_i\widehat{r}_{i+1}\widehat{r}_i = \widehat{r}_i\widehat{r}_{i+1} = \widehat{r}_{i+1}\widehat{r}_i\widehat{r}_{i+1}$
   b) $\widehat{\ell}_i\widehat{\ell}_{i+1}\widehat{\ell}_i = \widehat{\ell}_{i+1}\widehat{\ell}_i = \widehat{\ell}_{i+1}\widehat{\ell}_i\widehat{\ell}_{i+1}$
3. a) $\widehat{r}_i\widehat{\ell}_i\widehat{r}_i = \widehat{r}_i$
   b) $\widehat{\ell}_i\widehat{r}_i\widehat{\ell}_i = \widehat{\ell}_i$
4. a) $\widehat{r}_{i+1}\widehat{\ell}_i\widehat{r}_i = \widehat{r}_{i+1}\widehat{\ell}_i$
   b) $\widehat{\ell}_{i-1}\widehat{r}_i\widehat{\ell}_i = \widehat{\ell}_{i-1}\widehat{r}_i$
   c) $\widehat{\ell}_i\widehat{r}_i\widehat{\ell}_{i+1} = \widehat{r}_i\widehat{\ell}_{i+1}$
   d) $\widehat{r}_i\widehat{\ell}_i\widehat{r}_{i-1} = \widehat{\ell}_i\widehat{r}_{i-1}$
5. $\widehat{r}_i\widehat{\ell}_i = \widehat{\ell}_{i+1}\widehat{r}_{i+1}$
6. if $|i-j| \geq 2$, then $\widehat{r}_i\widehat{\ell}_j = \widehat{\ell}_j\widehat{r}_i$, $\widehat{r}_i\widehat{r}_j = \widehat{r}_j\widehat{r}_i$, $\widehat{\ell}_i\widehat{\ell}_j = \widehat{\ell}_j\widehat{\ell}_i$

Similarly, we define an algebra on $\widehat{PR}_k$ which we denote $\mathbb{C}\widehat{PR}_k$.

Then we define the function $\phi : \widehat{PR}_k \to PR_k$ given by

$$\widehat{\ell}_i \mapsto \ell_i$$

$$\widehat{r}_i \mapsto r_i.$$

Since the diagrams ℓ_i, r_i satisfy the relations of $\widehat{PR}_k$, this is a monoid homomorphism: that is, it is operation-preserving.

In order to show that ϕ is in fact an isomorphism, we will show that $|\widehat{PR}_k| \geq |PR_k| = \binom{2k}{k}$ and that $|\widehat{PR}_k| \leq |PR_k|$.

3.3.1 First Direction

Theorem 3. *$\phi : \widehat{PR}_k \to PR_k$ is surjective, so $|\widehat{PR}_k| \geq |PR_k|$.*

Proof. As we noted before, ℓ_i and r_i generate all of PR_k (together with 1). Given an arbitrary element $x \in PR_k$, we can write x as some product of ℓ_i and r_i, say $x = \prod_{a=1}^{n} x_{i_a}$, where each of the x_i is either ℓ_i or r_i, and then we have that x is the image of $\widehat{x} = \prod_{a=1}^{n} \widehat{x_{i_a}}$: that is $\phi(\widehat{x}) = x$. This is rather cumbersome, but just states that given an arbitrary element x in PR_k, it can be written as a product of generators, and then the corresponding product of generators in $\widehat{PR}_k$ gives us x as an image. Then we have that ϕ is a surjection, and therefore we must have that $|\widehat{PR}_k| \geq |PR_k|$. □

3.3.2 Second Direction

Theorem 4. $|\widehat{PR}_k| \leq |PR_k|$

We now want to show that $\widehat{PR}_k$ has at most as many distinct elements as there are diagrams in PR_k. We begin by defining a *Standard Word* in $\widehat{PR}_k$. We define S and T as above, and then define $\widehat{L}^S, \widehat{R}_T$ and $\widehat{E}_j$ formally, exactly like we did for PR_k. To write these words, we will first define $\widehat{e}_i = \widehat{\ell}_i\widehat{r}_i = \widehat{r}_{i+1}\widehat{\ell}_{i+1}$, except for $i = 1$ and $i = k$, where the only applicable form of $\widehat{e}_i$ is the one for which the r and ℓ elements exist; these $\widehat{e}_i$ correspond exactly to the e_i as defined in PR_k. As before, we define $\widehat{L}^{a,a} = \widehat{R}_{a,a} = \widehat{1}$ and define

$$\begin{aligned} \widehat{L}^{a,b} &= \widehat{\ell}_{a-1}\widehat{\ell}_{a-2}\cdots\widehat{\ell}_b, \qquad 1 \leq b < a \leq k, \\ \widehat{R}_{a,b} &= \widehat{r}_a\widehat{r}_{a+1}\cdots\widehat{r}_{b-1}, \qquad 1 \leq a < b \leq k. \end{aligned}$$

From there, we define $\widehat{L}^S, \widehat{R}_T$ and $\widehat{E}_j$ as we defined L^S, R_T and E_j: that is, if $S = \{s_1 < s_2 < \cdots < s_j\}$ and $T = \{t_1 < t_2 < \cdots < t_j\}$.

$$\begin{aligned} \widehat{L}^S &= \widehat{L}^{s_1,1}\widehat{L}^{s_2,2}\cdots\widehat{L}^{s_j,j}, \\ \widehat{E}_j &= \widehat{e}_{j+1}\widehat{e}_{j+2}\cdots\widehat{e}_k, \\ \widehat{R}_T &= \widehat{R}_{j,t_j}\widehat{R}_{j-1,t_{j-1}}\cdots\widehat{R}_{1,t_1}. \end{aligned}$$

Finally, for each pair (S, T) with $|S| = |T| = j$, define

$$\widehat{W}_T^S = \widehat{L}^S\widehat{E}_j\widehat{R}_T$$

Given sets S and T of size j, $\widehat{W}_T^S$ is defined to be the Standard Word corresponding to the sets. There are, then, $\binom{2n}{n}$ Standard Words in $\widehat{PR}_n$. We see that, by definition of ϕ, W_T^S and $\widehat{W}_T^S$, ϕ maps $\widehat{W}_T^S$ to W_T^S. Then if we show that we can standardize each formal word in $\widehat{PR}_n$, we will have proved theorem 4.

The following relations involving $\widehat{\ell_i}, \widehat{r_i}, \widehat{e_i}$ can be derived from the relations of $\widehat{PR}_k$ with just $\widehat{r_i}$ and $\widehat{\ell_i}$, but are useful for our purposes:

- $\widehat{e_i}\widehat{e_j} = \widehat{e_j}\widehat{e_i}$ for all i, j
- $\widehat{e}_{i+1}\widehat{r_i} = \widehat{r_i}$
- $\widehat{e_i}\widehat{r_i} = \widehat{e_i}\widehat{e}_{i+1}$
- $\widehat{e}_{i+1}\widehat{\ell_i} = \widehat{e_i}\widehat{e}_{i+1}$
- $\widehat{e_i}\widehat{\ell_i} = \widehat{\ell_i}$
- $\widehat{e_i}\widehat{r_j} = \widehat{r_j}\widehat{e_i}$ for $i \neq j+1$ or j

We will now show that every element of $\widehat{PR}_k$ is equivalent (under the relations) to one of the standard words. To do this, we will first observe that the identity element is a standard word, and then we will show that the product of an arbitrary standard word and a generator (either $\widehat{r_i}$ or $\widehat{\ell_i}$) can be standardized using the relations.

First, we must note that the identity is, itself, already a standard word: for the identity, the relevent sets are $S = T = \{1, 2, \ldots, k\}$, and that $\text{rank}(\widehat{1}) = k$. Therefore, we have that

$$\begin{aligned}
\widehat{L}^S &= \widehat{L}^{1,1}\widehat{L}^{2,2}\cdots\widehat{L}^{k,k} = \widehat{1}\,\widehat{1}\cdots\widehat{1}, \\
\widehat{E}_j &= \widehat{1}, \\
\widehat{R}_T &= \widehat{R}_{j,t_j}\widehat{R}_{j-1,t_{j-1}}\cdots\widehat{R}_{1,t_1} = \widehat{1}\,\widehat{1}\cdots\widehat{1}.
\end{aligned}$$

Therefore $\widehat{1} = \widehat{L}^S\widehat{E}_k\widehat{R}_T = \widehat{1}\ \widehat{1}\ \widehat{1} = \widehat{1}$, and so the identity is already in standard form.

Next we want to show that given a standard word W_T^S and a generator $x_i \in \{\widehat{\ell_i}, \widehat{r_i}\}$, the word $W_T^S\, x_i$ can be standardized using the relations. There are several cases to consider with relation to i, $i+1$ and T: we have different situations if $i,\ i+1 \in T$, if only one of the two is in T, or if neither are.

Lemma 5. *1.* $w_T^S r_i = w_{T'}^{S'}$ *where*

$$(S', T') = \begin{cases} (S, T), & \text{if } i, i+1 \notin T \quad (i) \\ (S \setminus \{b\}, T \setminus \{i\}), & \text{if } i, i+1 \in T \quad (ii) \\ (S, (T \setminus \{i\}) \cup \{i+1\}), & \text{if } i \in T, i+1 \notin T\, (iii) \\ (S \setminus \{b\}, T \setminus \{i+1\}), & \text{if } i+1 \in T, i \notin T\, (iv) \end{cases}$$

where b is connected to $i+1$ in w_T^S.

2. $w_T^S \ell_i = w_{T'}^{S'}$ *where*

$$(S', T') = \begin{cases} (S,T), & \text{if } i, i+1 \notin T \\ (S \setminus \{a\}, T \setminus \{i+1\}), & \text{if } i, i+1 \in T \\ (S \setminus \{a\}, T \setminus \{i\}), & \text{if } i \in T, i+1 \notin T \\ (S, (T \setminus \{i+1\}) \cup \{i\}), & \text{if } i+1 \in T, i \notin T \end{cases}$$

where a is connected to i in w_T^S.

Before beginning the proof of this, figures 3.1 through 3.4, at the end of this section, give diagrams in standard form, showing the four cases for $w_T^S \widehat{r}_i$.

Proof. The statement in part 2 is the same as part 1 except with r_i replacing ℓ_i and with all i and $i+1$ interchanged. The relations involving r_i and ℓ_i have this same symmetry. Thus, we only prove part 1, since part 2 follows by symmetry.

First, we will consider the situation when $i, i+1 \notin T$. Then there are 2 subcases to consider:

- if $i > t_j$ then for $1 \leq a \leq t_j - 1, |a - i| \geq 2$.
  We note that for every $\widehat{r}_a$ in $\widehat{R}_T$, we have that $1 \leq a \leq t_j - 1$ and so we have $\widehat{r}_a \widehat{r}_i = \widehat{r}_i \widehat{r}_a$. Repeating this process, we have

$$\begin{aligned} \widehat{L}^S \widehat{E}_j (\widehat{r}_j \cdots \widehat{r}_{t_j - 1}) \quad \cdots \quad & (\widehat{r}_1 \cdots \widehat{r}_{t_1 - 1}) x_i = \widehat{L}^S \widehat{E}_j x_i \widehat{R}_T \\ = \quad & \widehat{L}^S \widehat{e}_{j+1} \cdots \widehat{e}_i \widehat{e}_{i+1} x_i \widehat{e}_{i+2} \cdots \widehat{e}_k \widehat{R}_T \\ = \quad & \widehat{L}^S \widehat{E}_j \widehat{R}_S \\ = \quad & W_T^S \end{aligned}$$

- if $1 \leq i < t_j$ then let $c > 0$ be the smallest integer such that $t_c > i+1$.
  We have that for $1 \leq a \leq t_{c-1} - 1, \widehat{r}_a \widehat{r}_i = \widehat{r}_i \widehat{r}_a$, and so we have

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c,t_c} \cdots \widehat{R}_{1,t_1} \widehat{r}_i \\ &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_c \cdots \widehat{r}_i \widehat{r}_{i+1} \cdots \widehat{r}_{t_c - 1}) \widehat{r}_i \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_c \cdots \widehat{r}_i \widehat{r}_{i+1} \widehat{r}_i \cdots \widehat{r}_{t_c - 1}) \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c,t_c} \cdots \widehat{R}_{1,t_1} \\ &= W_T^S \end{aligned}$$

Next, we have the case in which both $i, i+1 \in T$. We let $t_c = i$, and then $t_{c+1} = i+1$ since $i+1 \in T$ also. For $k < c$, all of the indices of terms in $\widehat{R}_{k,t_k}$ are at most $i-2$, so they commute with $\widehat{r}_i$. We have, then

$$\begin{aligned}
W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,i+1} \widehat{R}_{c,i} \cdots \widehat{R}_{1,t_1} \widehat{r}_i \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,i+1} \widehat{R}_{c,i} \widehat{r}_i \cdots \widehat{R}_{1,t_1} \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,i+1} (\widehat{r}_c \cdots \widehat{r}_{i-2} \widehat{r}_{i-1}) \widehat{r}_i \cdots \widehat{R}_{1,t_1}
\end{aligned}$$

Now, if we look at the $\widehat{r}_i$ in $\widehat{R}_{c+1,i+1}$, we note that it will commute past all of the terms of $\widehat{R}_{c,i}$ until $\widehat{r}_{i-1}$, then we simplify using $\widehat{r}_{i+1}\widehat{r}_i\widehat{r}_{i+1} = \widehat{r}_i\widehat{r}_{i+1}$. We do the same with the $\widehat{r}_{i-1}$ which is then the rightmost element of the remains of $\widehat{R}_{c+1,i+1}$, and so forth with each of the elements of $\widehat{R}_{c+1,i+1}$, as shown:

$$\begin{aligned}
W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_{c+1} \cdots \widehat{r}_{i-1} \widehat{r}_i)(\widehat{r}_c \cdots \widehat{r}_{i-2} \widehat{r}_{i-1} \widehat{r}_i) \cdots \widehat{R}_{1,t_1} \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_{c+1} \cdots \widehat{r}_{i-2} \widehat{r}_{i-1})(\widehat{r}_c \cdots \widehat{r}_{i-2} \widehat{r}_i \widehat{r}_{i-1} \widehat{r}_i) \cdots \widehat{R}_{1,t_1} \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_{c+1} \cdots \widehat{r}_{i-2} \widehat{r}_{i-1})(\widehat{r}_c \cdots \widehat{r}_{i-2} \widehat{r}_{i-1} \widehat{r}_i) \cdots \widehat{R}_{1,t_1} \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_{c+1} \cdots \widehat{r}_{i-2})(\widehat{r}_c \cdots \widehat{r}_{i-1} \widehat{r}_{i-2} \widehat{r}_{i-1} \widehat{r}_i) \cdots \widehat{R}_{1,t_1} \\
&\vdots \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+2,t_{c+2}} \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1}
\end{aligned}$$

We now have a product which looks almost like a standard word, but with a gap between c and $c+2$ on the $\widehat{R}$ side. Intuitively, we want to add this gap in the $\widehat{L}$ side of the word as well, then fill in the gap by "pulling" the middles of all of the strings to the left (see figure 3.2; this is what we have done between (a) and (b)). We make use of the rule that $\widehat{e}_{i+1}\widehat{r}_i = \widehat{r}_i$ to create the element $\widehat{e}_{c+1}$:

$$W_T^S \widehat{r}_i = \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+2,t_{c+2}} \widehat{e}_{c+1} \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1}$$

For the moment, we are not going to do anything more with the $\widehat{R}$ side, so let us simply call it $\widehat{R}$. Now, we note that for $k \geq c+2$, all terms in $\widehat{R}_{k,t_k}$ have index greater than $c+1$, so $\widehat{e}_{c+1}$ commutes with them. All $\widehat{e}_i$ and $\widehat{e}_j$

commute with one another, so let us commute $\widehat{e}_{c+1}$ all the way to $\widehat{L}^S$:

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{e}_{c+1} \widehat{E}_j \widehat{R} \\ &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} \widehat{L}_{s_{c+1},c+1} \cdots \widehat{L}_{s_j,j} \widehat{e}_{c+1} \widehat{E}_j \widehat{R} \end{aligned}$$

Now, we note that for $k > c+1$, all of the terms in $\widehat{L}_{s_k,k}$ have indices greater than $c+1$, so $\widehat{e}_{c+1}$ commutes with them:

$$W_T^S \widehat{r}_i = \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} (\widehat{\ell}_{s_{c+1}-1} \cdots \widehat{\ell}_{c+1}) \widehat{e}_{c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R}$$

We make use of the rule $\widehat{\ell}_i \widehat{e}_i = \widehat{e}_i \widehat{e}_{i+1}$ repeatedly:

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} \widehat{\ell}_{s_{c+1}-1} \cdots \widehat{\ell}_{c+2} \widehat{e}_{c+2} \widehat{e}_{c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R} \\ &\vdots \\ &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} \widehat{e}_{s_{c+1}-1} \cdots \widehat{e}_{c+2} \widehat{e}_{c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R} \end{aligned}$$

For $s_{c+1} - 1 \leq k \leq c+1$, we commute $\widehat{e}_k$ past as many of the $\widehat{\ell}_a$ as we can, which means that we will allow each of the $\widehat{e}_k$ to commute until they reach either a $\widehat{\ell}_k$ or a $\widehat{\ell}_{k-1}$, or until it commutes past all of the $\widehat{\ell}$. Since $\widehat{L}_{s_{c+2},c+2} = \widehat{\ell}_{s_{c+2}-1} \widehat{\ell}_{s_{c+2}-2} \cdots \widehat{\ell}_{c+2}$, all of the $\widehat{e}_k$ except for $\widehat{e}_{c+1}$ will run into an $\widehat{\ell}_k$ in $\widehat{L}_{s_{c+2},c+2}$; the indices get higher to the right of $\widehat{L}_{s_{c+2},c+2}$, so $\widehat{e}_{c+1}$ does not run into an $\widehat{\ell}_{c+1}$ or an $\widehat{\ell}_c$. Therefore, by $\widehat{e}_i \widehat{\ell}_i = \widehat{\ell}_i$, we get:

$$W_T^S \widehat{r}_i = \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} \cdots \widehat{L}_{s_j,j} \widehat{e}_{c+1} \widehat{E}_j \widehat{R}$$

Now, we turn $\widehat{e}_{c+1}$ into $\widehat{\ell}_{c+1} \widehat{r}_{c+1}$; we move $\widehat{\ell}_{c+1}$ left until it is just right of $\widehat{L}_{s_{c+2},c+2}$ and move $\widehat{r}_{c+1}$ right, to just left of $\widehat{R}_{c+2,t_{c+2}}$. We can do this because all of the indices in between are strictly greater than $c+2$, and so the $\widehat{\ell}_{c+1}$ and the $\widehat{r}_{c+1}$ commute with them. Then we simplify:

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} \cdots \widehat{L}_{s_j,j} \widehat{\ell}_{c+1} \\ &\quad \widehat{r}_{c+1} \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+2,t_{c+2}} \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_{c+2},c+2} \widehat{\ell}_{c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \\ &\quad \widehat{R}_{j,t_j} \cdots \widehat{r}_{c+1} \widehat{R}_{c+2,t_{c+2}} \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_{c+2},c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,t_{c+2}} \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1} \end{aligned}$$

Now, the gap is between $c+1$ and $c+3$ instead of between c and $c+2$. We use the rule $\widehat{\ell}_i = \widehat{\ell}_i \widehat{e}_{i+1}$ on the $\widehat{\ell}_{c+1}$ that we just moved in order to make an $\widehat{e}_{c+2}$. Then, as wedid with the $\widehat{e}_{c+1}$, we commute $\widehat{e}_{c+2}$ to the middle, turn it into an $\widehat{\ell}_{c+2}$ and an $\widehat{r}_{c+2}$, and commute them until they stop just to the right of $\widehat{L}_{s_{c+3},c+3}$ and just left of $\widehat{R}_{c+3,t_{c+3}}$. We simplify, and the gap has moved another space to the right, so it is now between $c+2$ and $c+4$. We continue in this manner, moving the gap 1 space to the right each time, until we reach $j-1$ and use if to create an $\widehat{e}_j$:

$$
\begin{aligned}
W_T^S \widehat{r}_i &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_{c+2},c+1} \widehat{e}_{c+2} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,t_{c+2}} \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1} \\
&\vdots \\
&= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_j,j-1} \widehat{E}_j \widehat{R}_{j-1,t_j} \cdots \widehat{R}_{1,t_1} \\
&= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_j,j-1} \widehat{e}_j \widehat{E}_j \widehat{R}_{j-1,t_j} \cdots \widehat{R}_{1,t_1}
\end{aligned}
$$

Now we finish simplifying, according to the formal definition of $W_{T'}^{S'}$:

$$
\begin{aligned}
W_T^S \widehat{r}_i &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_j,j-1} \widehat{E}_{j-1} \widehat{R}_{j-1,t_j} \cdots \widehat{R}_{1,t_1} \\
&= W_{T'}^{S'}
\end{aligned}
$$

where S', T' are defined as in Lemma 5.

The next case is when $i \in T$ and $i+1 \notin T$. Then we let $t_c = i$ and note that we must have $t_{c+1} > i+1$

$$
\begin{aligned}
W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c,i} \cdots \widehat{R}_{1,t_1} \widehat{r}_i \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_c \cdots \widehat{r}_{i-2} \widehat{r}_{i-1}) \widehat{r}_i \cdots \widehat{R}_{1,t_1} \\
&= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1} \\
&= W_{T'}^S
\end{aligned}
$$

where $T' = T \setminus \{i\}$.

The final case is when $i+1 \in T$ and $i \notin T$. This case is very much like (ii); we cancel out an entire $\widehat{L}_{x,y}$ (in the case of (ii) it was $\widehat{L}_{c+1,i+1}$, and in this case it will be $\widehat{L}_{c,i+1}$) and then are left with a gap which we will need to fill in.

We let $t_c = i+1$. Then $t_{c-1} < i$, so $\widehat{r}_i$ commutes with all the terms of $\widehat{R}_{c-1,t_{c-1}}$ and all elements further to the right. Therefore,

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c,i+1} \cdots \widehat{R}_{1,t_1} \widehat{r}_i \\ &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots (\widehat{r}_c \cdots \widehat{r}_{i-1} \widehat{r}_i) \widehat{r}_i \cdots \widehat{R}_{1,t_1} \end{aligned}$$

As we did in case (ii), we turn $\widehat{R}_{c,i+1}$ into a product of $\widehat{e}_a$ by applying the rule ${\widehat{r}_i}^2 = \widehat{e}_c \cdots \widehat{e}_{i+1}$ and the rule $\widehat{r}_i \widehat{e}_{i+1} = \widehat{e}_i \widehat{e}_{i+1}$ repeatedly:

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{r}_c \cdots \widehat{r}_{i-1} \widehat{e}_i \widehat{e}_{i+1} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{r}_c \cdots \widehat{r}_{i-2} \widehat{e}_{i-1} \widehat{e}_i \widehat{e}_{i+1} \cdots \widehat{R}_{1,t_1} \\ &\vdots \\ &= \widehat{L}^S \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,t_{c+1}} \widehat{e}_c \cdots \widehat{e}_{i+1} \widehat{R}_{c-1,t_{c-1}} \cdots \widehat{R}_{1,t_1} \end{aligned}$$

All of the $\widehat{e}_a$ cancel except $\widehat{e}_c$, which commutes left all the way to $\widehat{L}_{s_c,c}$, where we cancel again by the rule $\widehat{\ell}_i \widehat{e}_i = \widehat{e}_{i+1} \widehat{e}_i$:

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_c,c} \widehat{L}_{s_{c+1},c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{e}_c \widehat{R}_{j,t_j} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1} \cdots (\widehat{\ell}_{s_c-1} \cdots \widehat{\ell}_c) \widehat{e}_c \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1} \cdots \widehat{\ell}_{s_c-1} \cdots \widehat{\ell}_{c+1} \widehat{e}_{c+1} \widehat{e}_c \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{1,t_1} \\ &\vdots \\ &= \widehat{L}_{s_1,1} \cdots \widehat{e}_{s_c} \cdots \widehat{e}_{c+1} \widehat{e}_c \widehat{L}_{s_{c+1},c+1} \cdots \widehat{L}_{s_j,j} \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{1,t_1} \end{aligned}$$

Again, all of the $\widehat{e}_a$ cancel except for $\widehat{e}_c$, which commutes all the way back to just left of the $\widehat{E}_j$. Then we split it into $\widehat{\ell}_c \widehat{r}_c$ and commute $\widehat{\ell}_c$ left as far as it will go and $\widehat{r}_c$ right as far as it will go, then simplify:

$$\begin{aligned} W_T^S \widehat{r}_i &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_{c-1},c-1} \widehat{L}_{s_{c+1},c+1} \cdots \widehat{L}_{s_j,j} \\ &\quad \widehat{e}_c \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,t_{c+1}} \widehat{R}_{c-1,t_{c-1}} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_{c-1},c-1} \widehat{L}_{s_{c+1},c+1} \cdots \widehat{L}_{s_j,j} \widehat{\ell}_c \\ &\quad \widehat{r}_c \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{R}_{c+1,t_{c+1}} \widehat{R}_{c-1,t_{c-1}} \cdots \widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1} \cdots \widehat{L}_{s_{c-1},c-1} (\widehat{\ell}_{s_{c+1}-1} \cdots \widehat{\ell}_{c+1}) \widehat{\ell}_c \cdots \widehat{L}_{s_j,j} \\ &\quad \widehat{E}_j \widehat{R}_{j,t_j} \cdots \widehat{r}_c (\widehat{r}_{c+1} \cdots \widehat{r}_{t_{c+1}-1}) \cdots \widehat{R}_{1,t_1} \end{aligned}$$

Using the rule $\widehat{\ell}_c = \widehat{\ell}_c\widehat{e}_{c+1}$, we create an $\widehat{e}_{c+1}$ and move it to the middle, then do as we did in case (ii), where we pulled the middles of strands to the left. Each step, we create an $\widehat{e}_a$, then move it to the middle, make it into an $\widehat{\ell}_a\widehat{r}_a$, commute the $\widehat{\ell}_a$ as far left as we can and the $\widehat{r}_a$ as far right as we can. This moves the gap in the middle one spot farther right each step. This gives us

$$\begin{aligned} W_T^S\widehat{r}_i &= \widehat{L}_{s_1,1}\cdots\widehat{L}_{s_{c+1},c}\cdots\widehat{L}_{s_j,j}\widehat{E}_j\widehat{R}_{j,t_j}\cdots\widehat{R}_{c,t_{c+1}}\cdots\widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1}\cdots\widehat{L}_{s_{c+1},c}\widehat{e}_{c+1}\cdots\widehat{L}_{s_j,j}\widehat{E}_j\widehat{R}_{j,t_j}\cdots\widehat{R}_{c,t_{c+1}}\cdots\widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1}\cdots\widehat{L}_{s_{c+1},c}\cdots\widehat{L}_{s_j,j}\widehat{E}_j\widehat{e}_{c+1}\widehat{R}_{j,t_j}\cdots\widehat{R}_{c,t_{c+1}}\cdots\widehat{R}_{1,t_1} \\ &\vdots \\ &= \widehat{L}_{s_1,1}\cdots\widehat{L}_{s_j,j-1}\widehat{E}_j\widehat{R}_{j-1,t_j}\cdots\widehat{R}_{1,t_1} \end{aligned}$$

where the product of the $\widehat{L}_{x,y}$ is a legitimate $\widehat{L}_{S'}$ and the product of the $\widehat{R}_{x,y}$ is a legitimate $\widehat{R}_{T'}$, $|S'| = |T'|$. NOw we use the rule $\widehat{\ell}_{j-1} = \widehat{\ell}_{j-1}\widehat{e}_j$ once more to give

$$\begin{aligned} W_T^S\widehat{r}_i &= \widehat{L}_{s_1,1}\cdots\widehat{L}_{s_j,j-1}\widehat{e}_j\widehat{E}_j\widehat{R}_{j-1,t_j}\cdots\widehat{R}_{1,t_1} \\ &= \widehat{L}_{s_1,1}\cdots\widehat{L}_{s_j,j-1}\widehat{E}_{j-1}\widehat{R}_{j-1,t_j}\cdots\widehat{R}_{1,t_1} \\ &= W_{T'}^{S'} \end{aligned}$$

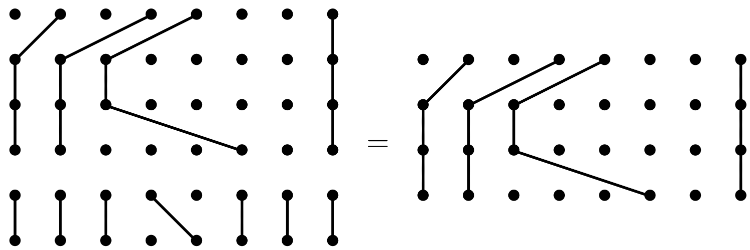
where $i+1$ has been removed from T to make T' and the corresponding element of S has been removed to make S'.

□

Corollary 6. *Every word in $\widehat{PR}_k$ reduces to a standard word, and therefore*

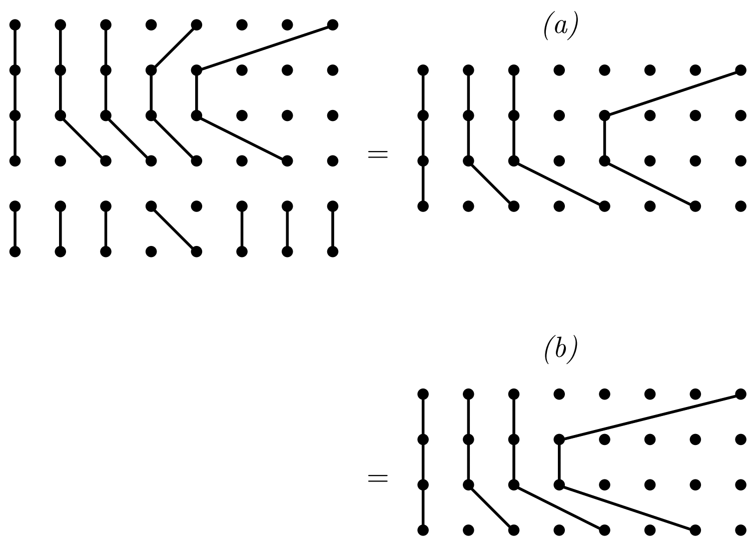
$$|\widehat{PR}_k| \le \binom{2k}{k} = |PR_k|.$$

From Theorem 3 and Theorem 4, we have an isomorphism of monoids, $PR_k \cong \widehat{PR}_k$, and thus the relations in Theorem 2 are a presentation for PR_k.



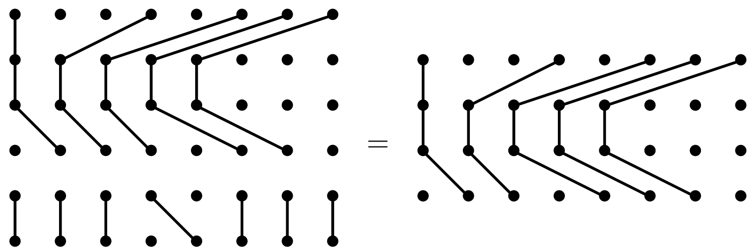
$$w^{\{2,4,5,8\}}_{\{1,2,6,8\}}\widehat{r}_4 \quad = \quad w^{\{2,4,5,8\}}_{\{1,2,6,8\}}$$

Figure 3.1: Case (i): neither i nor $i+1$ is in T



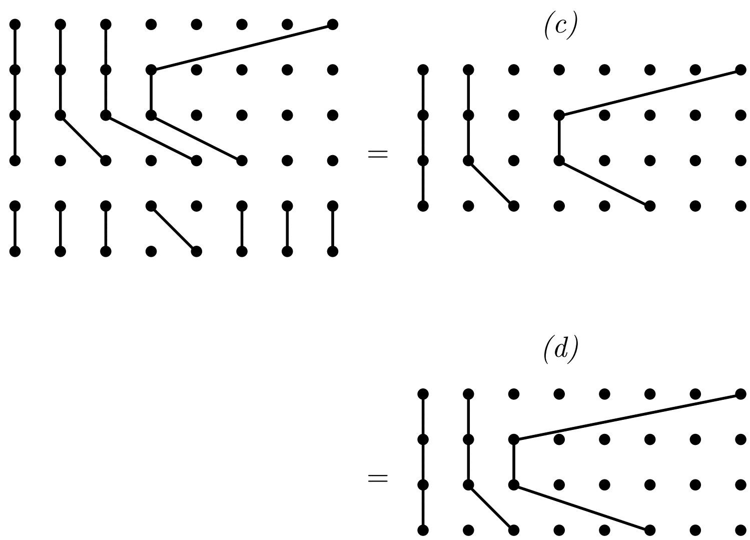
$$w^{\{1,2,3,5,8\}}_{\{1,3,4,5,7\}}\widehat{r}_4 \quad = \quad w^{\{1,2,3,8\}}_{\{1,3,5,7\}}$$

Figure 3.2: Case (ii): both i and $i+1$ are in T; from (a) to (b) we "pull" the middle of the fourth strand to the left



$$w^{\{1,4,6,7,8\}}_{\{2,3,4,6,7\}}\widehat{r}_4 \quad = \quad w^{\{1,3,6,7,8\}}_{\{2,3,5,6,7\}}$$

Figure 3.3: Case (iii): i is in T, $i+1$ is not



$$w^{\{1,2,3,8\}}_{\{1,3,5,6\}}\widehat{r}_4 \quad = \quad w^{\{1,2,8\}}_{\{1,3,6\}}$$

Figure 3.4: Case (iv): $i+1$ is in T, i is not; from (c) to (d), we "pull" the middle of the third strand left as we did between (a) and (b) in case (ii)

Chapter 4

Representations and Modules for PR_n

4.1 A Representation on Subsets

In this section, we make a vector space V^n with basis indexed by subsets of $\{1, 2, \ldots, n\}$. Then we have $\dim(V^n) = 2^n$. We will give an action of PR_n on V^n, making it a PR_n module, which we will decompose into submodules V^n_k, where $\dim(V^n_k) = \binom{n}{k}$. We will show that:

1. V^n_k is irreducible
2. Each V^n_k appears exactly once in V^n

which gives us

$$V^n \cong \bigoplus_{k=0}^{n} V^n_k$$

This, in turn, implies that

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

which is the counterpart to equation 1.1 in Young's Lattice.

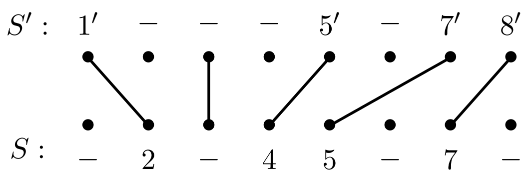
Given a subset $S \subseteq \{1, 2, \ldots, n\}$, we make a vector $\mathbf{v}_S$ labeled by S and define the following action on $\mathbf{v}_S$ by PR_n: if $d \in PR_n$ and $S \subseteq \{1, 2, \ldots, n\}$,

$$d \cdot \mathbf{v}_S = \begin{cases} \mathbf{v}_{S'}, & \text{if } S \subseteq \text{domain}(d) \\ 0, & \text{otherwise} \end{cases}$$

where if $S = \{s_1, s_2, \dots, s_j\}$

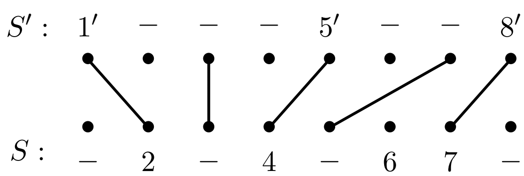
$$S' = \{d(s_1), \dots, d(s_j)\}.$$

Here is an example in PR_8:



$$\Rightarrow \quad d \cdot \mathbf{v}_{\{2,4,5,7\}} = \mathbf{v}_{\{1,5,7,8\}}$$

In this second example, S is not a subset of d's domain (note the 6), so $d \cdot \mathbf{v}_S = 0$.



$$\Rightarrow \quad d \cdot \mathbf{v}_{\{2,4,6,7\}} = 0$$

If we let

$$V^n = \operatorname{span}\big\{\mathbf{v}_S \bigm| S \subseteq \{1, \dots, n\}\ \big\}$$

then V^n is a PR_n module. We then let the operation distribute in the intuitive way:

$$\left(\sum_{i=1}^{x} \alpha_i d_i\right) \cdot \mathbf{v}_S = \sum_{i=1}^{x} \alpha_i d_i \cdot \mathbf{v}_S$$

and so V^n is also a $\mathbb{C}PR_n$ module.

Now, we define

$$V_k^n = \operatorname{span}\big\{\mathbf{v}_S \bigm| S \subseteq \{1, \dots, n\},\ |S| = k\big\}$$

Proposition 7. *V_k^n is an irreducible submodule of V^n.*

Proof. V_k^n is a submodule of V^n:

Our action on subsets cannot change the size of the subset: if $d \cdot \mathbf{v}_S = \mathbf{v}_{S'}$ every element of S' must be the image of some element of S, so that S' cannot be larger than S. If some element of S is not in the domain of d, then $d \cdot \mathbf{v}_S = 0$ which is in the span of any vector that we choose. If $d \cdot \mathbf{v}_S = \mathbf{v}_{S'}$, then each element of S' corresponds to an element of S, and vice versa, so S and S' must be the same size, and if $d\mathbf{v}_S 0$, it is in the span of $\mathbf{v}_S$, and S is clearly the same size as S. Therefore, our arguments tell us that if $\mathbf{v} \in V_k^n$, then $d\mathbf{v} \in V_k^n$.

V_k^n is irreducible:

Let S be any subset of $\{1, 2, \ldots, n\}$ of size k, and consider S', where $|S'| = k$ also. There is at least one $d \in PR_n$ such that $d \cdot \mathbf{v}_S = \mathbf{v}_{S'}$; in particular, we can take that d with domain S and range S'. Therefore, we have that

$$\mathbb{C}PR_n \mathbf{v}_S = V_k^n$$

Next, let us take any $\mathbf{v} \in V_k^n$; say $\mathbf{v} = \sum_{|T|=k} \lambda_T \mathbf{v}_T$, and choose some subset of size k, S, such that $\lambda_S \neq 0$. We let d be the element of PR_n with domain and range both equal to S. Then for $T \neq S$, $|T| \neq k$ there must be some element of T which is not an element of S, and which is therefore not in the domain of d. Therefore, we have that for $T \neq S$, $|T| = k$, $dT = 0$, and so we must have $d\mathbf{v} = \lambda_S \mathbf{v}_S$, which, as we have shown, generates all of V_n^k.

Therefore, V_n^k can have no proper submodule, and so is irreducible. □

Corollary 8.

$$V^n = \bigoplus_{k=0}^{n} V_k^n, \tag{4.1}$$

and

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \tag{4.2}$$

Proof. It is clear that there can be no other irreducible submodules of V^n, as every one of the basis elements of V^n is in one or another of these irreducibles. Therefore, V^n is a direct sum of these modules. Calculating the dimension of V_k^n, we see that it is given by the number of subsets of $\{1, 2, \cdots, n\}$ of size k (as each corresponds to a basis element), $\binom{n}{k}$. V^n has dimension 2^n, as its basis elements are indexed by the set of all subsets of $\{1, 2, \cdots, n\}$. We take the corresponding dimensions and substitute these into the formula, and we have the second of the two equalities, the formula for the row sum across Pascal's triangle. This equality corresponds to Equation 1.1 for the Symmetric Group. □

4.2 The Regular Representation

In this section, we make a vector space indexed by PR_n, which is in fact $\mathbb{C}PR_n$. Then $\dim(\mathbb{C}PR_n) = \binom{2n}{n}$. We give an action by PR_n to make this vector space a module. We then give 2 different bases for $\mathbb{C}PR_n$, and use the second to decompose the module into submodules B_k^n, with $\dim(B_k^n) = \binom{n}{k}^2$.

We show that each B_k^n appears exactly once in $\mathbb{C}PR_n$, and that they are disjoint, to get the equation

$$\mathbb{C}PR_n \cong \bigoplus_{k=0}^{n} B_k^n$$

This implies

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

which is equivalent to equation 1.2 in S_n. Then we break B_k^n down further, to get

$$B_k^n \cong \binom{n}{k} V_k^n$$

and give the decomposition of $\mathbb{C}PR_n$ into irreducibles.

We let $\mathbb{C}PR_n$ act on $\mathbb{C}PR_n$, giving us the regular representation. First, we will introduce some new notation. To denote a particular element of PR_n, say that S and T are subsets of $\{1, 2, \ldots, n\}$, $|S| = |T|$. Then let $d_{S,T}$ be the diagram with domain T and range S (that is, $d_{S,T}$ is the diagram with the vertices of S on top connected to the vertices of T on bottom). Then

$$\mathbb{C}PR_n = \text{span}\{d_{S,T} | S, T \subseteq \{1, 2, \ldots, n\}, |S| = |T|\}$$

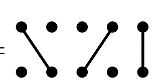
The action of the monoid algebra $\mathbb{C}PR_n$ on the module $\mathbb{C}PR_n$ is to multiply from the left, distributing linearly and then multiplying as in the monoid.

We would like to explicitly decompose this representation into its isotypic components; to do so, we want to find an appropriate basis (one which clearly shows the irreducible submodules). This basis should be chosen in such a way that given a diagram d and a basis element b, the product $d \cdot b$ is another basis element of the same rank as b, if the rank of the product is the rank of b, and zero in any other case.

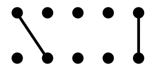
In order to find such a basis, we must first define a new term. We say that $d_{S',T'}$ is a *subdiagram* of $d_{S,T}$, and write

$$d_{S',T'} \subseteq d_{S,T}$$

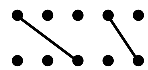
if the edges of $d_{S',T'}$ are a subset of the edges of $d_{S,T}$. For example, if

$d_{S,T} =$ 

then



is a subdiagram of $d_{S,T}$, but

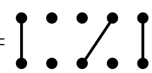


is *not* a subdiagram of $d_{S,T}$ even though its domain and ranges are both subsets of S and T, because its edges are not edges in $d_{S,T}$.

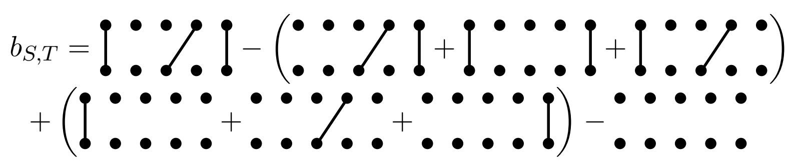
Now, we define

$$b_{S,T} = \sum_{d_{S',T'} \subseteq d_{S,T}} (-1)^{|S|-|S'|} d_{S',T'}$$

That is, we take $d_{S,T}$ and we subtract all of those subdiagrams created by removing exactly one edge, add back in those created by removing exactly two edges, and so on until we get to the empty diagram. For example, if

$d_{S,T} =$ 

then

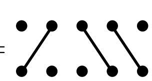


We call B the set of $b_{S,T}$. We note that D is a basis for $\mathbb{C}PR_n$ by definition. If we order the $b_{S,T}$ in the same way as the $d_{S,T}$, there are the same number; if we write the linear transformation which takes us from D to B, then it is triangular with ones on the diagonal, and so it is invertible. Therefore, B is also a basis for $\mathbb{C}PR_n$.

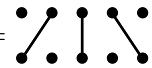
Proposition 9. *If $d \cdot d_{S,T} = d_{M,N}$, then*

$$d \cdot b_{S,T} = \begin{cases} b_{M,N} & \textit{if } \operatorname{rank}(d_{M,N}) = \operatorname{rank}(d_{S,T}) \\ 0 & \textit{if } \operatorname{rank}(d_{M,N}) < \operatorname{rank}(d_{S,T}) \end{cases}$$

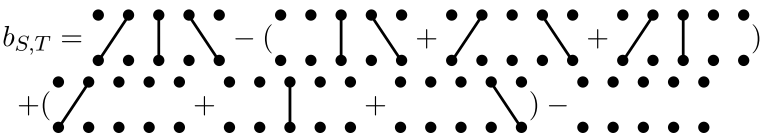
Before we prove this, here is an example to illustrate:
Let $b_{S,T}$ be as above. Then if we let

$d =$ 

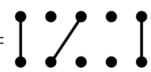
then

$$d_{M,N} = d \cdot d_{S,T} =$$ 

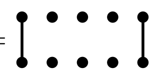
has the same rank as $d_{S,T}$, and

$$d \cdot b_{S,T} =$$ 

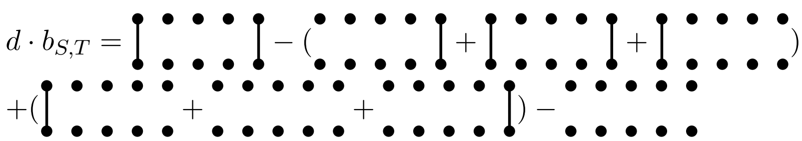
which is in fact $b_{M,N}$. If, on the other hand, we let

$$d =$$ 

then

$$d \cdot d_{S,T} =$$ 

has rank less than the rank of $d_{S,T}$, and

$$d \cdot b_{S,T} =$$ 

where all of the elements pair up and cancel out, giving $d \cdot b_{S,T} = 0$.

Proof. For convenience, let us call the elements of M, N, S, T the sets of m_i, n_i, s_i, t_i, respectively, where $t_1 < t_2 < t_3 \cdots$ and so on for the other sets. First, say that $\text{rank}(d_{M,N}) = \text{rank}(d_{S,T})$. Then T, the image of $d_{S,T}$ is a subset of the domain of d, so that for every t_i, $d(t_i)$ exists. Further, since d is one-to-one, each $d(t_i)$ is distinct. If we have a subset $S' \subseteq S$, then let us call $d_{S,T}(S') = T'$; then we have that for $s \in S'$, $d_{S',T'}(s) = d_{S,T}(s)$ by definition of T' and the fact that there is exactly one way to connect two sets of points as a planar rook diagram. If, on the other hand, $s \notin S'$, s has no image in $d_{S',T'}$. Therefore, every edge in the diagram of $d_{S',T'}$ is an edge in the diagram of $d_{S,T}$, and so $d_{S',T'} \subseteq d_{S,T}$. Then we have, for $s \in S'$,

$$d \cdot d_{S',T'}(s) = d \cdot d_{S,T}(s).$$

In particular, if $s_i \in S$, then $d \cdot d_{S',T'}(s_i) = d \cdot d_{S,T}(s_i) = d(t_i)$ (since $d_{S,T}$ must be strictly increasing). Further, the set $d \cdot d_{S',T'}(S')$ gives us the entire image of $d \cdot d_{S',T'}$, since S' is the entire domain of the function: there is nothing in the image of $d \cdot d_{S',T'}$ which is not a member of the set $d(T')$. This tells us that if a subdiagram $d_{S',T'}$ is the subdiagram containing the $i_1, i_2, \cdots i_k$ edges of $d_{S,T}$, then $d \cdot d_{S',T'}$ is the $i_1, i_2, \cdots i_k$ edges of $d \cdot d_{S,T}$.

We think back to the way that $b_{S,T}$ is constructed. We begin with $d_{S,T}$, then subtract off all of those subsets with one edge less, add back in those with two edges less, and so on. We recall once again that we defined $d \cdot d_{S,T} = d_{M,N}$. Our results above tell us the following:

$$\begin{aligned} d \cdot b_{S,T} &= d \cdot \textstyle\sum_{d_{S',T'} \subseteq d_{S,T}} (-1)^{|S|-|S'|} d_{S',T'} \\ &= \textstyle\sum_{d_{S',T'} \subseteq d_{S,T}} (-1)^{|S|-|S'|} d \cdot d_{S',T'} \\ &= \textstyle\sum_{d_{U,V} \subseteq d \cdot d_{S,T}} (-1)^{|S|-|U|} f_{U,V} \\ &= \textstyle\sum_{d_{M',N'} \subseteq d_{M,N}} (-1)^{|M|-|M'|} d_{M',N'} \\ &= b_{M,N} \end{aligned}$$

The last steps can be done because multiplying by d does not change the size of any of the subdiagrams, and so $|M| - |M'| = |S| - |S'|$, which then tells us that we have all of the subdiagrams of $d_{M,N}$, and that the given sum has the correct sign on all of them for $b_{M,N}$. Therefore, we have the case in which the rank of $d_{M,N}$ is equal to the rank of $d_{S,T}$.

Now, we consider the case in which $\text{rank}(d_{M,N}) < \text{rank}(d_{S,T})$. In this case, there must be at least one element in the image of $d_{S,T}(T)$ which is not in the domain of d. Say that the ath element of T, which corresponds to the ath edge of $d_{S,T}$, is not in the domain of d. Then if $d_{S',T'} \subseteq d_{S,T}$ contains the ath edge of $d_{S,T}$, we have that $s_a \in S'$, $t_a \in T'$, and so $d_{S'-s_a,T'-t_a}$ is a subdiagram of $d_{S,T}$ which is distinct from $d_{S',T'}$, has one fewer edge, and satisfies

$$d \cdot d_{S',T'} = d \cdot d_{S'-s_a,T'-t_a}$$

as the only edge that is missing on the right side is one which does has no image under d. Since there is exactly one edge of difference between $d_{S',T'}$ and $d_{S'-s_a,T'-t_a}$, they have opposite signs as summands in $b_{S,T}$. Therefore, when we take the product with d, we have

$$\begin{aligned} d \cdot b_{S,T} &= d \cdot \textstyle\sum_{d_{S',T'} \subseteq d_{S,T}} (-1)^{|S|-|S'|} d_{S',T'} \\ &= \textstyle\sum_{d_{S',T'} \subseteq d_{S,T}} (-1)^{|S|-|S'|} d \cdot d_{S',T'} \\ &= \textstyle\sum_{d_{S',T'} \subseteq d_{S,T}, t_a \in T} (-1)^{|S|-|S'|} d \cdot d_{S',T'} + (-1)^{|S|-|S'-t_a|} d \cdot d_{S'-t_a,T'-t_a} \\ &= \textstyle\sum_{d_{S',T'} \subseteq d_{S,T}, t_a \in T} d \cdot d_{S',T'} - d \cdot d_{S',T'} \\ &= \textstyle\sum 0 \\ &= 0 \end{aligned}$$

where, from the third line to the fourth, we simply turn all $S' - s_a$ into S' and $T' - t_a$ into T', changing signs as appropriate. $\square$

Now, the decomposition of the regular representation into submodules is quite clear. We define $B_k^n = \text{span}(b_{S,T})| \, |S| = k)$; then for $0 \leq k \leq n$, B_k^n is a submodule of $\mathbb{C}PR_n$ of size $\binom{n}{k}^2$ (since we choose S and T both of size k). These submodules are clearly disjoint, and equally clearly use up all of our basis elements. We have not yet decomposed $\mathbb{C}PR_n$ into irreducibles, but it is the direct sum of these submodules: that is,

$$\mathbb{C}PR_n = \bigoplus_{k=0}^{n} B_k^n \tag{4.3}$$

Corollary 10. $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Proof. Given that

$$\mathbb{C}PR_n = \bigoplus_{k=0}^{n} B_k^n$$

as stated in equation 4.3 above, we note as well that $\dim(\mathbb{C}PR_n) = \binom{2n}{n}$ and $\dim(B_k^n) = \binom{n}{k}^2$. Therefore we have

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

another famous identity. This is the parallel of Equation 1.2 for the Symmetric Group. $\square$

We have broken $\mathbb{C}PR_n$ down into B_k^n as a module with action by the other copy of the group on the left; now we would like to break the B_k^n into irreducibles. For ease of notation, we will first introduce some more terminology. Given $b_{S,T}$, we will call $d_{S,T}$ the *main diagram* of $b_{S,T}$, $\text{rank}(d_{S,T})$ the rank, and S the domain of $b_{S,T}$.

We consider the group action, and realize that if $b_{S,T} \in B_k^n$, then when multiplied by any element $d \in \mathbb{C}PR_n$, if $\text{rank}(db_{S,T}) = \text{rank}(b_{S,T})$, then the domain of $db_{S,T}$ is the same as the domain of $b_{S,T}$. If the rank would be lowered, then $db_{S,T} = 0$, which can be interpreted in $\mathbb{C}PR_n$ as $0b_{S,T}$, so that the domain still remains unchanged. Therefore, the set

$$B_T^n = \text{span}\{b_{S,T} | T \text{fixed}\}$$

is a submodule of B^n_k for every T with $|T| = k$; therefore, there are $\binom{n}{k}$ such submodules, since there are $\binom{n}{k}$ such sets T, and between them they use all of the basis elements of B^n_k. Therefore,

$$B^n_k = \bigoplus_{|T|=k} B^n_T.$$

Proposition 11. *$B^n_T \cong V^n_{|T|}$ as $\mathbb{C}PR_n$ modules.*

Proof. The mapping $b_{S,T} \mapsto \mathbf{v}_S$ is a bijection since T is fixed; each element of B^n_T corresponds to exactly one element of $V^n_{|T|}$, and vice versa. Extend this action linearly. We recall the action on $\mathbf{v}_S$ by $\mathbb{C}PR_n$, and the action on $b_{S,T}$ by $\mathbb{C}PR_n$. If S is a subset of the domain of $d \in \mathbb{C}PR_n$, then we have

$$d \cdot b_{S,T} = b_{d(S),T} \mapsto \mathbf{v}_{d(S)},$$

and on the other hand in $V^n_{|T|}$ we have

$$d \cdot \mathbf{v}_S = \mathbf{v}_{d(S)}$$

If S is not a subset of the domain of d, then we have

$$d \cdot b_{S,T} = 0 \mapsto 0,$$

and

$$d \cdot \mathbf{v}_S = 0.$$

Therefore, this mapping is a bijection which preserves action by $\mathbb{C}PR_n$, and so is a $\mathbb{C}PR_n$-module isomorphism. □

Corollary 12. *$\mathbb{C}PR_n$ decomposes into irreducible submodules as*

$$\mathbb{C}PR_n \cong \bigoplus_{T\subseteq\{1,\cdots,n\}} B^n_T \cong \bigoplus_{k=0}^{n} \binom{n}{k} V^n_k \tag{4.4}$$

Proof. The decomposition is clear; the isomorphism and the fact that these are the irreducibles both follow from the above Proposition. □

4.3 Restriction of PR_n irreducibles to PR_{n-1}.

We consider an element of PR_{n-1} to be an element of PR_n which fixes n; then PR_{n-1} can be viewed as a submonoid of PR_n, and we can restrict V^n_k to PR_{n-1}. As a PR_{n-1} module, V^n_k may be reducible: If, on the one hand,

$\mathbf{v}_S \in V_k^n$ such that $n \notin S$ then $d\mathbf{v}_S = 0$ or $\mathbf{v}_T$, where $\mathbf{v}_T \in V_k^n$ and $n \notin T$. Then the span of such $\mathbf{v}_S$,

$$\text{span}\{\mathbf{v}_S | \mathbf{v}_S \in V_k^n, n \notin S\},$$

is closed under action by the monoid, and is a submodule. On the other hand, if $\mathbf{v}_S \in V_k^n$ such that $n \in S$ and $d \in PR_{n-1}$, then either $d\mathbf{v}_S = 0$ or $\mathbf{v}_T$, where $\mathbf{v}_T \in V_k^n$ and $n \in T$ (since d must fix n). Therefore, the span of such $\mathbf{v}_S$,

$$\text{span}\{\mathbf{v}_S | \mathbf{v}_S \in V_k^n, n \in S\},$$

is also a submodule; V_k^n is the direct sum of these two submodules.

The first of these submodules can be seen directly to be a submodule of V^{n-1}: any subset of $\{1, \ldots, n-1, n\}$ of size k which does not contain n is in fact a subset of $\{1, \ldots, n-1\}$ of size k. Therefore, there is an obvious mapping between this module and V_k^{n-1}: the identity mapping, which is clearly an isomorphism.

The second of the submodules requires little more effort to view as a submodule of V^{n-1}. We note that removing n from each of the subsets, and treating the elements of PR_{n-1} as we would normally has no effect on the action. Removing an element of the subset makes all of our subsets size $k-1$ instead of k; therefore, we have a correspondence between this submodule and V_{k-1}^{n-1}.

Therefore, we have

$$V_k^n \downarrow_{PR_{n-1}}^{PR_n} \cong V_{k-1}^{n-1} \oplus V_k^{n-1}.$$

When we substitute dimensions into this formula, we have the famous recursion of Pascal's Triangle:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

4.4 Constructing the Bratteli diagram for PR_n

In this section, we will construct a picture similar to Young's Lattice (figure 1.1) for PR_n. Let's remember, from section 2.3, how we went about building Young's Lattice from the information about the representation theory of S_n. The elements in the nth row correspond to the irreducible modules of S_n. Edges are drawn between μ in the $n-1$st row and λ in the nth row if V^μ, the irreducible module corresponding to μ, is a component of Res $\downarrow_{S_{n-1}}^{S_n} (V^\lambda)$.

Therefore, let us determine what the elements and the edges of the Bratteli diagram for PR_n should be.

Elements correspond to irreducible modules. We note that, by equation 4.4, the irreducibles for PR_n are labeled by $0, 1, \ldots, n$. Therefore, the elements in the nth row of the Bratteli lattice for PR_n are subsets of size k, $0 \leq k \leq n$.

Edges are drawn by the rule

$$V_k^n \downarrow_{PR_{n-1}}^{PR_n} \cong V_{k-1}^{n-1} \oplus V_k^{n-1}.$$

That is to say, we draw edges between the subset of size k in the nth row and the subsets of size k and $k-1$ in the $(n-1)$st row. Again, we will put in the dimensions of the irreducible modules (which we have calculated) to the right of the elements. Using these rules, here are the first 6 rows of the Bratteli diagram for PR_n.

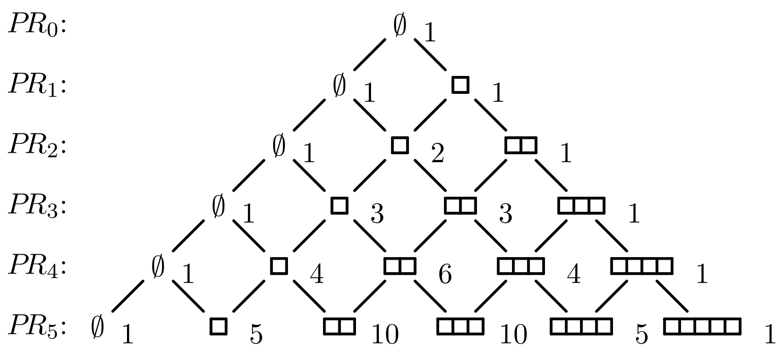


Figure 4.1: The Bratteli Diagram for PR_n

We can see that these first six rows of the Bratteli lattice are exactly the first six rows of Pascal's Triangle. Further, since the edges in the Bratteli lattice are, in every row, exactly the same as the edges in Pascal's Triangle, and the recursion is the same, the Bratteli lattice for PR_n is, in fact, exactly Pascal's Triangle. In summary, here is the same list we wrote in the introduction for Young's Lattice and the Symmetric Group, now written for Pascal's Triangle and the Planar Rook Monoid.

- The vertices in the nth row are indexed by subsets of size k of n. The irreducible modules of PR_n are also labeled by subsets of size k and we denote them by V_k^n.

- The edges between the nth row and the $(n-1)$st row of the lattice correspond to the restriction rules from PR_n down to PR_{n-1}. That is, if we view the PR_n module as a module for PR_{n-1}, it decomposes as

$$V_k^n \cong V_{k-1}^{n-1} \oplus V_k^{n-1}$$

  where there are edges exactly between these vertices.

- The dimension of V_k^n, which is $\binom{n}{k}$, (written next to k in the nth row of the lattice) is also the number of shortest paths from the 0th row to k in the nth row.

- The sum of $\binom{n}{k}$ across the row is 2^n; that is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{4.5}$$

  This comes from the representation on subsets, which has dimension 2^n such that every V_k^n appears exactly once.

- The sum of $\binom{n}{k}^2$ across the nth row is $\binom{2n}{n}$; that is

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{4.6}$$

  This comes from a representation of PR_n on a vector space of dimension $\binom{2n}{n}$ (in fact, on itself) such that each V_k^n appears $\dim V_k^n = \binom{n}{k}$ times.

Chapter 5

Conjugacy and Characters in PR_n

5.1 Conjugacy Classes

5.1.1 Conjugacy in Monoids

Recall that in a group, we say that a and b are conjugate if there is some x such that $xax^{-1} = b$; the important fact about conjugacy is that if a and b are conjugate, then $\mathrm{Tr}(a) = \mathrm{Tr}(b)$ in every representation of the group. In a monoid, some elements may be invertible, but not all are, or it would be a group, so conjugation cannot work in quite the normal fashion. In the case of the Planar Rook Monoid, x^{-1} does not exist unless x is the identity, and if we can only conjugate with the identity, we get each element in a class by itself; this is not a very useful notion of conjugacy. We would like some notion of conjugacy which is workable in a monoid which will keep the trace constant on conjugacy classes, and if possible we would like to find a notion of conjugacy which also gives us conjugacy classes of maximal size.

We will define a relation $a \approx b$ if there are $d, d' \in PR_n$ with $dd' = a$ and $d'd = b$. We will extend this relation by saying that if there are $x_1, x_2 \ldots, x_k \in PR_n$ such that $a \approx x_1, x_1 \approx x_2, \ldots, x_k \approx b$, then $a \approx_k b$.

From there, we will turn this into an equivalence relation, by letting $\sim$ be the transitive closure of $\approx$: that is, if there is some finite k such that $a \approx_k b$, then $a \sim b$.

If we let d be the identity and $d' = a \in PR_n$ for some a, then we have that $a \sim a$. Now, consider a, b such that $a \sim b$. Then there is some k such that $a \approx_k b$, so there is some chain $a \approx x_1,\ x_1 \approx x_2, \cdots,\ x_k \approx b$. Now, $\approx$ is symmetric: if $x \approx y$, then there are some elements $d, d' \in PR_n$ such that $dd' = x$ and $d'd = y$; then d', d are elements such that $d'd = y$ and $dd' = x$, so $x \approx y$. By the above reasoning, we have $b \approx x_k, \cdots, x_1 \approx a$, and this tells us that $b \sim a$. In our definition of $\sim$, we have forced transitivity: if $a \sim b$ and $b \sim c$, then we have j and k such that $a \approx_j b$ and $b \approx_k c$; then we can combine the two chains $a \approx x_1,\ x_1 \approx x_2, \cdots,\ x_j \approx b$ and $b \approx x_1,\ x_1 \approx x_2, \cdots,\ x_k \approx c$ to get that $a \approx_{j+k+1} c$; then $a \sim c$, so $\sim$ is transitive. Therefore, $\sim$ is an equivalence relation; then we just want to show that $\sim$ preserves trace.

Trace has the property that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all A and B. Therefore, if $a = dd'$ and $b = d'd$, then

$$\mathrm{Tr}([a]) = \mathrm{Tr}([d][d']) = \mathrm{Tr}([d'][d]) = \mathrm{Tr}([b])$$

for all representations. If there is a chain $a \approx x_1,\ x_1 \approx x_2, \cdots,\ x_k \approx b$ which makes $a \approx b$, where each equivalence in the chain is by the first part of the definition, then $\mathrm{Tr}([a]) = \mathrm{Tr}([b])$ because equality is transitive. This tells us that the trace is constant on our equivalence classes, and so our equivalence relation is a viable definition of conjugacy; at worst, it may give us too many conjugacy classes.

5.1.2 Conjugacy Classes in PR_n

Proposition 13. *All diagrams with exactly k vertical edges, and no other edges, are conjugate.*

Proof. Any two diagrams with exactly k vertical edges and no other edges must be of the form $d_{S,S}$ and $d_{T,T}$ where $|S| = |T| = k$. Then we have

$$d_{S,T} \cdot d_{T,S} = d_{S,S}$$

and

$$d_{T,S} \cdot d_{S,T} = d_{T,T}$$

Therefore, $d_{S,S} \approx d_{T,T}$. □

Proposition 14. *Any diagram with k vertical edges, and perhaps some diagonal edges, is conjugate to the same diagram with only the k vertical edges.*

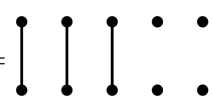
Proof. We shall consider only the case where $d(s_i) > s_i$; the other case follows from symmetry.

Call a set of points $R = \{x, x+1, \ldots, x+y\}$ a *right-leaning ladder* if, for every $s \in R$ which is in the domain of d, $d(s) > s$, there is some $s \in R$ with $d(s) = x+y$, and for all $s \in R$, $d(s) \in R$. We note that $x+y$ cannot be in the domain of d¡ since we cannot have both $d(x+y) > x+y$ and $d(x+y) \in R$.

Say that for some x and y, d contains the right-leaning ladder that contains the points from x to $x+y$. Let d' be d with the diagonal which goes to $x+y$ erased. Then $de_{x+y} = d$, since for $x \neq x+y$, $d(e_{x+y}(s)) = d(s)$, and neither $d(e_{x+y}(x+y))$ nor $d(x+y)$ exists. On the other hand, $e_{x+y}d = d'$: for $s \neq d^{-1}(x+y)$ (the pre-image, not the inverse), we have $d(s) \neq x+y$, and so $e_{x+y}(d(s)) = d(s)$, and $e_{x+y}(d(d^{-1}(x+y))) = e_{x+y}(x+y)$ does not exist. This gives us exactly the domain and range of d', and so we must have $e_{x+y}d = d'$. Therefore, d is conjugate to d'.

We note that d' is the same as d, except that the second-rightmost edge in the right-leaning ladder which begins with x in d is now the rightmost edge in the corresponding ladder in d'. We can continue this process of removing the rightmost edge of a right-leaning ladder; every right-leaning diagonal edge is in some right-leaning ladder and so will be removed eventually, which tells us that d is conjugate to the same diagram with every right-leaning edge removed. We can then repeat the process with our left-leaning edges, and we have, as desired, that every diagram with exactly k vertical edges, and possibly some diagonal edges, is conjugate to the diagram obtained when we remove all of the diagonal edges. □

Now, let id_k be the diagram with vertical edges on the leftmost k pairs of vertices, and all others empty; that is, $id_k = d_{\{1,\cdots,k\},\{1,\cdots,k\}}$. For instance, in PR_5,

$id_3 =$ 

Then as a direct consequence of the above propositions, we have

Corollary 15. *The set of id_k, $0 \leq k \leq n$ is a set of conjugacy class representatives in PR_n.*

5.2 Computing the Irreducible Characters of PR_n

Given the conjugacy classes and the irreducible modules, we compute the irreducible characters of PR_n. Show that the columns of the character table are rows of Pascal's Triangle.

We recall that the irreducible submodules are V_k^n for all k. We will write the character on V_k^n as χ_k^n. Now we consider the values of $\chi_k^n(id_j)$.

We consider that if $S \subseteq \{1, 2, \cdots, j\}$, then id_j fixes $\mathbf{v}_S$, which will give us a 1 on the diagonal. If $S \not\subseteq \{1, 2, \cdots, j\}$, then $id_j \cdot \mathbf{v}_S = 0$, which will give us a 0 on the diagonal. Therefore, there are exactly as many 1s on the diagonal of id_j in V_k^n as there are subsets of size k in $\{1, 2, \cdots, j\}$. This number is $\binom{j}{k}$, where we define (as usual) $\binom{j}{k} = 0,\ j < k$. Then we must have, since all other diagonal entries are zero, that

$$\chi_k^n(id_j) = \binom{j}{k}.$$

Then if we arrange our character table so that id_0 is the leftmost conjugacy class and id_n is the rightmost conjugacy class, χ_0^n is the topmost character and χ_n^n is the bottom-most character, then the rows of Pascal's Triangle are the columns of the character table. For instance, below is the character table for PR_5:

	id_0	id_1	id_2	id_3	id_4	id_5
χ_0^5	1	1	1	1	1	1
χ_1^5	0	1	2	3	4	5
χ_2^5	0	0	1	3	6	10
χ_3^5	0	0	0	1	4	10
χ_4^5	0	0	0	0	1	5
χ_5^5	0	0	0	0	0	1

Chapter 6

Conclusion

In this paper, we set out to present the Planar Rook Monoid, and then argue that it has a connection to Pascal's Triangle which is analogous to the Symmetric Group's connection to Young's Lattice. The ties between the Symmetric Group and Young's Lattice are highlighted when we study the representation theory of S_n; that is, we treat the permutations as linear transformations acting on a vector space. Then we build a lattice which encodes the representation-theoretic information, and we find Young's Lattice. Similarly, we have studied key points in the representation theory of the Planar Rook Monoid, and when we build the analogous lattice, we find that it is Pascal's Triangle. For reference, Figure 6.1 gives Young's Lattice, and Figure 6.2 gives Pascal's Triangle.

From Y, we got the following information, which corresponds to information about S_n:

- The vertices in the nth row are indexed by integer partitions λ of n. The irreducible modules of S_n are also labeled by $\lambda \vdash n$ and we denote them by V^λ.

- The edges between the nth row and the $(n-1)$st row of the lattice correspond to the restriction rules from S_n down to S_{n-1}. That is, if we view the S_n module as a module for S_{n-1}, it decomposes as

$$V^\lambda = \bigoplus_\mu V^\mu$$

  where the μ runs over the partitions connected to λ by an edge.

- The dimension of V^λ, denoted f_λ, (written next to λ in the lattice) is also the number of shortest paths from $\emptyset$ to λ.

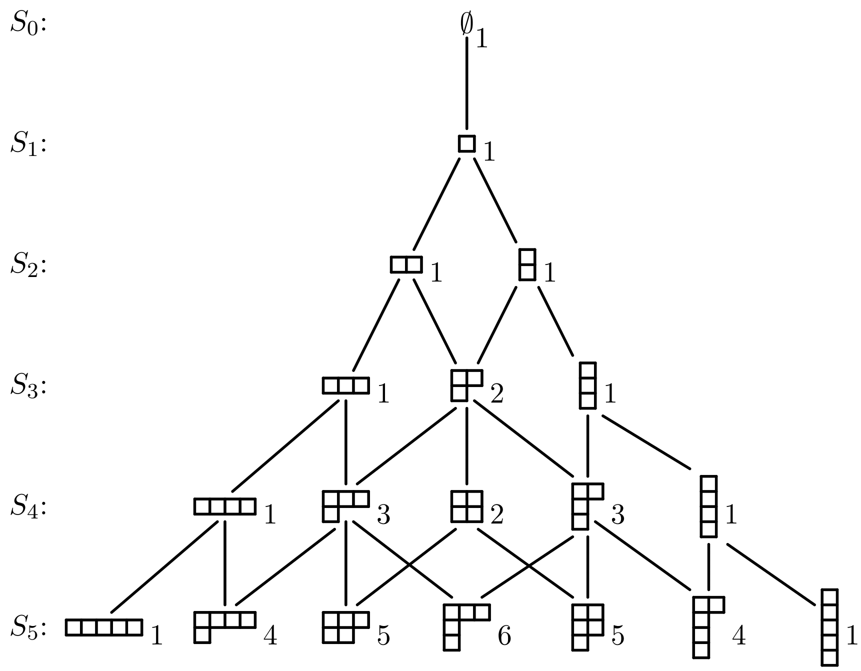


Figure 6.1: Young's Lattice, Y

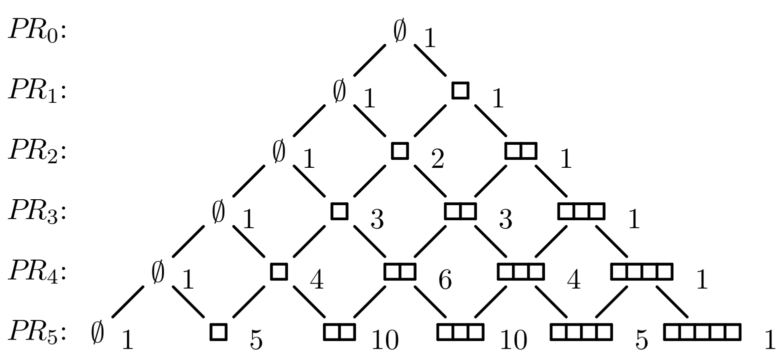


Figure 6.2: The Bratteli Diagram for PR_n, P

- The sum across the nth row of the f_λ is the number of permutations of $\{1, 2, \ldots, n\}$ which are their own inverses. This is the set of involutions I_n in S_n. This is to say
$$\sum_{\lambda \vdash n} f_\lambda = |I_n| \tag{6.1}$$
This comes from a representation of S_n on a vector space of dimension $|I_n|$ such that each V^λ appears exactly once.

- The sum of ${f_\lambda}^2$ across the nth row is $n!$; that is
$$\sum_{\lambda \vdash n} {f_\lambda}^2 = n! \tag{6.2}$$
This comes from the regular representation of S_n: a representation of S_n on itself, such that each V^λ appears $\dim(V^\lambda)$ times.

We get equivalent information about P, corresponding to information about PR_n:

- The vertices in the nth row are indexed by subsets of size k of n. The irreducible modules of PR_n are also labeled by subsets of size k and we denote them by V_k^n.

- The edges between the nth row and the $(n-1)$st row of the lattice correspond to the restriction rules from PR_n down to PR_{n-1}. That is, if we view the PR_n module as a module for PR_{n-1}, it decomposes as
$$V_k^n \cong V_{k-1}^{n-1} \oplus V_k^{n-1}$$
where there are edges exactly between these vertices.

- The dimension of V_k^n, which is $\binom{n}{k}$, (written next to k in the nth row of the lattice) is also the number of shortest paths from the 0th row to k in the nth row.

- The sum of $\binom{n}{k}$ across the row is 2^n; that is
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{6.3}$$
This comes from the representation on subsets, which has dimension 2^n such that every V_k^n appears exactly once.

- The sum of $\binom{n}{k}^2$ across the nth row is $\binom{2n}{n}$; that is

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{6.4}$$

This comes from the regular representation of PR_n: the representation on itself, such that each V_k^n appears $\dim V_k^n = \binom{n}{k}$ times.

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